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On a Design for Two-way Elimination of Heterogeneity and its Analysis

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1. Introduction and summary

In this paper we shall propose a new type of design for two-way elimination of heterogeneity and deal with its analysis. Many types of designs for two-way elimination of heterogeneity, such as, Latin square designs, Youden square designs and some other extended designs, have been proposed and investigated. The type of row-column incidence matrices, however, of those traditional designs is a restrictive one in that it is complete. In other words, every row block consists of all plots, each of which belongs to any one of the column blocks and vice versa.

The row-column incidence matrix of a design for two-way elimination of heterogeneity may not necessarily be complete, but any one of the plots belongs to one and only one of the row blocks and one and only one of the column blocks simultaneously. In this connection, we shall propose in section 2 a new type of row-column incidence matrix for two-way elimination of heterogeneity. The matrix is not complete but a direct arrangement of some complete type matrices. In section 3 we shall introduce a treatment-plot incidence matrix subject to some conditions, and shall define a relationship algebra of the design. Complete analysis of the relationship algebra of the design will be presented in section 4 along the line due to S. Yamamoto and Y. Fujii [3]. Analysis of variance of the design will be presented in section 5. Two examples of the design proposed in this paper will be presented in section 6.

It will be seen that, in any one of the complete portions of row-column incidence, the design is insufficient for the purpose of the analysis, but the suitable combination of these portions gives us a design for two-way elimination which is sufficient for the purpose of the analysis.

2. A new type of row-column incidence matrix

Let Ψ_1 be the row-plot incidence matrix defined as,

$$\Psi_1 = \|\psi_{1fa}\| \tag{1}$$

where $$\psi_{1fa}=\begin{cases}1 & \text{if } f\text{-th plot belongs to } a\text{-th row,}\\ 0 & \text{otherwise,}\end{cases}$$

and let Ψ_2 be the column-plot incidence matrix defined as,

(2) $$\Psi_2=\|\psi_{2fp}\|$$

where $$\psi_{2fp}=\begin{cases}1 & \text{if } f\text{-th plot belongs to } p\text{-th column,}\\ 0 & \text{otherwise.}\end{cases}$$

The elements of the row-column incidence matrix,

(3) $$M=\Psi_1'\Psi_2=\|m_{ap}\|$$

are assumed to be 1 or 0 according as the a-th row and p-th column have a common plot or not.

If we introduce the notion of connectedness between two treatments familiar in a block design into the relation between two rows (columns) of such an incomplete row-column incidence matrix M, we can divide it into, say, h connected portions. Without loss of generality, after labeling suitably the number of plots, rows and columns, we can express the matrices M, Ψ_1 and Ψ_2 as follows:

(4) $$M=\Psi_1'\Psi_2=\begin{Vmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_h \end{Vmatrix}$$

where M_i is an $x_i\times y_i$ matrix,

(5) $$\Psi_1=\begin{Vmatrix} \Psi_{11} & & & \\ & \Psi_{12} & & \\ & & \ddots & \\ & & & \Psi_{1h} \end{Vmatrix}$$

where Ψ_{1i} is an $x_iy_i\times x_i$ matrix, and

(6) $$\Psi_2=\begin{Vmatrix} \Psi_{21} & & & \\ & \Psi_{22} & & \\ & & \ddots & \\ & & & \Psi_{2h} \end{Vmatrix}$$

where Ψ_{2i} is an $x_iy_i\times y_i$ matrix, for any $i=1, 2, \cdots, h$. Let the number of rows, columns and plots be b_1, b_2 and n respectively, then we have $\sum_{i=1}^{h} x_i = b_1$, $\sum_{i=1}^{h} y_i = b_2$ and $\sum_{i=1}^{h} x_i y_i = n$.

If we denote

$$\Psi_1'\Psi_1 = D_1, \qquad \Psi_2'\Psi_2 = D_2 \tag{7}$$

the definitions of Ψ_1 and Ψ_2 show that these are diagonal matrices, all diagonal elements of which are positive integers.

Define U_1 and U_2 as

$$U_1 = \Psi_1 D_1^{-1} \Psi_1', \qquad U_2 = \Psi_2 D_2^{-1} \Psi_2'$$

then we can express these as

$$U_1 = \begin{Vmatrix} \Psi_{11} D_{11}^{-1} \Psi_{11}' & & \\ & \ddots & \\ & & \Psi_{1h} D_{1h}^{-1} \Psi_{1h}' \end{Vmatrix}$$

$$U_2 = \begin{Vmatrix} \Psi_{21} D_{21}^{-1} \Psi_{21}' & & \\ & \ddots & \\ & & \Psi_{2h} D_{2h}^{-1} \Psi_{2h}' \end{Vmatrix}$$

where $D_{1i} = \Psi_{1i}'\Psi_{1i}$ and $D_{2i} = \Psi_{2i}'\Psi_{2i}$ for $i = 1, 2, \cdots, h$.

We shall prove the following theorem.

THEOREM I. *The matrices U_1 and U_2 are commutative, if and only if the row-column incidence matrix M is expressed as*

$$M = \begin{Vmatrix} G(x_1 \times y_1) & & & \\ & G(x_2 \times y_2) & & \\ & & \ddots & \\ & & & G(x_h \times y_h) \end{Vmatrix} \tag{8}$$

where $G(x_i \times y_i)$ denotes an $x_i \times y_i$ matrix whose elements are all unity for any $i = 1, 2, \cdots, h$.

PROOF. Assume that $U_1 U_2 = U_2 U_1$. The assumption holds if and only if

$$\Psi_{1i} D_{1i}^{-1} \Psi_{1i}' \Psi_{2i} D_{2i}^{-1} \Psi_{2i}' = \Psi_{2i} D_{2i}^{-1} \Psi_{2i}' \Psi_{1i} D_{1i}^{-1} \Psi_{1i}' \tag{9}$$

holds for any one of the connected portions of M, i.e., for any $M_i=\Psi'_{1i}\Psi_{2i}$. An (f, g) element $(f\neq g)$ of the matrix $\Psi_{1i}D_{1i}^{-1}\Psi'_{1i}\Psi_{2i}D_{2i}^{-1}\Psi'_{2i}$ is non-zero if and only if there exist a row a containing f-th plot, and a column p containing g-th plot, and the row a and the column p have a common f'-th plot. Similarly the (f, g) element of the matrix $\Psi_{2i}D_{2i}^{-1}\Psi'_{2i}\Psi_{1i}D_{1i}^{-1}\Psi'_{1i}$ is non-zero, if and only if there exist a column q containing f-th plot, and a row b containing g-th plot, and the column q and the row b have a common g'-th plot.

Thus, if $m_{aq}=1$ and $m_{bp}=1$, then either of $m_{ap}=1$ and $m_{bq}=1$, or $m_{ap}=0$ and $m_{bq}=0$ holds in each of the connected portions $M_i(i=1, 2, \cdots, h)$. Moreover, since each of the M_i is connected in the row-column incidence, it can be seen that if $m_{aq}=1$ and $m_{bp}=1$ then $m_{ap}=1$ and $m_{bq}=1$ hold for any two rows a and b, and for any two columns p and q in the same portion. Thus all elements of M_i are unity, i.e., $M_i=G(x_i\times y_i)$ for all $i=1, \cdots, h$.

Conversely, assume that (8) holds. Since $D_{1i}=y_iI_{x_i}$, $D_{2i}=x_iI_{y_i}$, and

$$U_1U_2=\Psi_1D_1^{-1}\left\|\begin{matrix} G(x_1\times y_1) & & \\ & \ddots & \\ & & G(x_h\times y_h)\end{matrix}\right\|D_2^{-1}\Psi'_2$$

$$=\left\|\begin{matrix} \frac{1}{x_1y_1}G(x_1y_1\times x_1y_1) & & \\ & \ddots & \\ & & \frac{1}{x_hy_h}G(x_hy_h\times x_hy_h)\end{matrix}\right\|$$

the matrix U_1U_2 is symmetric. As U_1 and U_2 are symmetric, U_1 and U_2 are commutative.

3. A design and its relationship algebra

In this section we shall define a design for two-way elimination, the row-column incidence matrix of which is given by (8). Assume that each plot receives any one of the v treatments, and that among those v treatments an association of m-associate classes is defined:

(a) Any two treatments are either 1st, or 2nd, $\cdots$, or m-th associates, the relation of association being symmetrical.

(b) Each treatment α has n_i i-th associates, the number n_i being independent of α.

(c) If any two treatments α and β are i-th associates, then the number of treatments being j-th associates of α and k-th associates of β, is p^i_{jk} and is independent of the pair of i-th associates α and β.

Association matrices which are matrix representation of the scheme are,

$$A_i = \|a^{\beta}_{\alpha i}\|, \qquad i = 0, 1, 2, \cdots, m$$

where $$a^{\beta}_{\alpha i} = \begin{cases} 1 & \text{if the treatment } \alpha \text{ is } i\text{-th associate of } \beta, \\ 0 & \text{otherwise.} \end{cases}$$

A commutative algebra $\mathfrak{A}$ generated by those association matrices $A_0 = I_v$, $A_1, \cdots, A_m$ is called an association algebra. It is known that the algebra is completely reducible and its minimum two-sided ideals are linear. Let $A^{\#}_0 = \frac{1}{v} G_v$, $A^{\#}_1, \cdots, A^{\#}_m$ be their principal idempotents, then the linear closure of these idempotents also gives the association algebra, i.e.,

$$\mathfrak{A} = [A^{\#}_0, A^{\#}_1, \cdots, A^{\#}_m]$$

Let $\varPhi$ be the treatment-plot incidence matrix defined as,

(10) $$\varPhi = \|\varphi_{f\alpha}\|$$

where $$\varphi_{f\alpha} = \begin{cases} 1 & \text{if } f\text{-th plot receives } \alpha\text{-th treatment,} \\ 0 & \text{otherwise.} \end{cases}$$

$\varPhi$ may be expressed in the following form

(11) $$\varPhi = \left\| \begin{matrix} \varPhi_1 \\ \varPhi_2 \\ \vdots \\ \varPhi_h \end{matrix} \right\|$$

where $\varPhi_i$'s are $(x_i y_i \times v)$ matrices respectively for $i = 1, 2, \cdots, h$.

The treatment-row incidence matrix of the design is

(12) $$\varPhi' \varPsi_1 = N_1 = \|\varPhi'_1 \varPsi_{11} \cdots\cdots \varPhi'_h \varPsi_{1h}\|$$

The treatment-column incidence matrix of the design is

(13) $$\varPhi' \varPsi_2 = N_2 = \|\varPhi'_1 \varPsi_{21} \cdots\cdots \varPhi'_h \varPsi_{2h}\|$$

Now we assume that the design satisfies the following four assumptions:

1° $\quad U_1U_2 = U_2U_1$

2° $\quad \Phi'_i\Phi_i = r_iI_v \quad (i=1, 2, \cdots, h), \quad \sum_{i=1}^{h} r_i = r$

3° $\quad N_1D_1^{-1}N'_1 = \rho_{10}A_0^{\#} + \rho_{11}A_1^{\#} + \cdots + \rho_{1m}A_m^{\#}$

4° $\quad N_2D_2^{-1}N'_2 = \rho_{20}A_0^{\#} + \rho_{21}A_1^{\#} + \cdots + \rho_{2m}A_m^{\#}$

where $\rho_{10}, \rho_{11}, \cdots,$ and ρ_{1m} are latent roots of $N_1D_1^{-1}N'_1$ and $\rho_{20}, \rho_{21}, \cdots,$ and ρ_{2m} are latent roots of $N_2D_2^{-1}N'_2$.

Denote as,

$$T_i^{\#} = \Phi A_i^{\#}\Phi' \qquad (i=1, 2, \cdots, m)$$

(14)
$$V_1 = U_1 - U_1U_2, \qquad V_2 = U_2 - U_1U_2, \qquad W = U_1U_2$$

$$\left(\text{For } A_0^{\#},\ \Phi A_0^{\#}\Phi' = \frac{r}{n}G_n \text{ holds}\right)$$

From the assumptions 1°, 2°, 3° and 4°, the following relations may easily be verified,

$$V_1V_2 = 0, \qquad V_1W = 0, \qquad V_2W = 0$$

$$WT_i^{\#} = 0$$

(15)
$$T_i^{\#}T_j^{\#} = r\delta_{ij}T_i^{\#}$$

$$T_i^{\#}V_1T_j^{\#} = \rho_{1i}\delta_{ij}T_i^{\#}$$

$$T_i^{\#}V_2T_j^{\#} = \rho_{2i}\delta_{ij}T_i^{\#}, \qquad (i, j = 1, 2, \cdots, m)$$

An algebra $\mathfrak{R}$ generated by I, G, $T_1^{\#}, T_2^{\#}, \cdots, T_m^{\#}$, V_1, V_2 and W is called the relationship algebra of the design.

4. Analysis of the relationship algebra of the design

In order to find the ideals of the algebra $\mathfrak{R}$, the following Lemmas are useful.

LEMMA 1.

$$0 \leq \rho_{1i} \leq r, \quad 0 \leq \rho_{2i} \leq r, \quad 0 \leq \rho_{1i} + \rho_{2i} \leq r, \qquad \text{for } i = 1, 2, \cdots, m.$$

PROOF. Since $N_1D_1^{-1}N_1'$ and $N_2D_2^{-1}N_2'$ are positive semi-definite matrices, it follows $\rho_{1i}\geq 0$, $\rho_{2i}\geq 0$. Since $\{T_i^\sharp(I-V_1-V_2)\}\ \{T_i^\sharp(I-V_1-V_2)\}'T_i^\sharp=r(r-\rho_{1i}-\rho_{2i})T_i^\sharp$, it follows $r\geq\rho_{1i}+\rho_{2i}$.

LEMMA 2.

(i) $T_i^\sharp V_1=V_1T_i^\sharp=0,$ *if and only if* $\rho_{1i}=0.$

$T_i^\sharp V_1=V_1T_i^\sharp=T_i^\sharp,$ *if and only if* $\rho_{1i}=r.$

$T_i^\sharp V_1\neq V_1T_i^\sharp,$ *if and only if* $0<\rho_{1i}<r.$

(ii) $T_i^\sharp V_2=V_2T_i^\sharp=0,$ *if and only if* $\rho_{2i}=0.$

$T_i^\sharp V_2=V_2T_i^\sharp=T_i^\sharp,$ *if and only if* $\rho_{2i}=r.$

$T_i^\sharp V_2\neq V_2T_i^\sharp,$ *if and only if* $0<\rho_{2i}<r.$

(iii) $T_i^\sharp(V_1+V_2)=(V_1+V_2)T_i^\sharp=0,$

if and only if $\rho_{1i}=\rho_{2i}=0.$

$T_i^\sharp(V_1+V_2)=(V_1+V_2)T_i^\sharp=T_i^\sharp,$

if and only if $\rho_{1i}+\rho_{2i}=r.$

Lemma 2 is essentially the same with that given in [3].

LEMMA 3. *Nine matrices* $T_i^\sharp$, $V_1T_i^\sharp$, $T_i^\sharp V_1$, $V_2T_i^\sharp$, $T_i^\sharp V_2$, $V_1T_i^\sharp V_1$, $V_1T_i^\sharp V_2$, $V_2T_i^\sharp V_1$ *and* $V_2T_i^\sharp V_2$ *are linearly independent, if and only if*

$$0<\rho_{1i}<r,\quad 0<\rho_{2i}<r\qquad\textit{and}\qquad 0<\rho_{1i}+\rho_{2i}<r,$$

or if and only if

$$V_1T_i^\sharp\neq T_i^\sharp V_1,\quad V_2T_i^\sharp\neq T_i^\sharp V_2\qquad\textit{and}\qquad (V_1+V_2)T_i^\sharp\neq T_i^\sharp(V_1+V_2).$$

PROOF. Assume that $0<\rho_{1i}<r$, $0<\rho_{2i}<r$ and $0<\rho_{1i}+\rho_{2i}<r$, and for some constants a, b, c, d, e, f, g, h and k, we have

$$(16)\qquad aT_i^\sharp+bV_1T_i^\sharp+cT_i^\sharp V_1+dV_2T_i^\sharp+eT_i^\sharp V_2+fV_1T_i^\sharp V_1$$
$$+gV_1T_i^\sharp V_2+hV_2T_i^\sharp V_1+kV_2T_i^\sharp V_2=0$$

Multiplying (16) by $V_1T_i^\sharp$ or $V_2T_i^\sharp$ from the right and by $T_i^\sharp V_1$ or $T_i^\sharp V_2$ from the left, we have

(17) $$a+b+c+f=0$$

(18) $$a+b+e+g=0$$

(19) $$a+c+d+h=0$$

(20) $$a+d+e+k=0$$

Multiplying (16) by $T_i^\sharp$ from the right and V_1 or V_2 from the left, we have

(21) $$(a+b)r+(c+f)\rho_{1i}+(e+g)\rho_{2i}=0$$

$$(a+d)r+(c+h)\rho_{1i}+(e+k)\rho_{2i}=0$$

Multiplying (16) by V_1 or V_2 from the right and $T_i^\sharp$ from the left, we have

(22) $$(a+c)r+(b+f)\rho_{1i}+(d+h)\rho_{2i}=0$$

$$(a+e)r+(b+g)\rho_{1i}+(d+k)\rho_{2i}=0$$

Using from (17) to (22), we have

(23) $$a=f=g=h=k=-b=-c=-d=-e$$

As $(I-V_1-V_2)T_i^\sharp(I-V_1-V_2)\neq 0$, we have $a=0$, and the proof is complete.

LEMMA 4. *Among nine matrices* $T_i^\sharp$, $V_1T_i^\sharp$, $T_i^\sharp V_1$, $V_2T_i^\sharp$, $T_i^\sharp V_2$, $V_1T_i^\sharp V_1$, $V_1T_i^\sharp V_2$, $V_2T_i^\sharp V_1$ *and* $V_2T_i^\sharp V_2$:

(i) *Four matrices* $T_i^\sharp$, $V_1T_i^\sharp$, $T_i^\sharp V_1$ *and* $V_1T_i^\sharp V_1$ *are linearly independent and the rest are zero matrices if and only if*

$$0<\rho_{1i}<r \quad \text{and} \quad \rho_{2i}=0$$

(ii) *Four matrices* $T_i^\sharp$, $V_2T_i^\sharp$, $T_i^\sharp V_2$ *and* $V_2T_i^\sharp V_2$ *are linearly independent and the rest are zero matrices if and only if*

$$0<\rho_{2i}<r \quad \text{and} \quad \rho_{1i}=0$$

(iii) *Four matrices* $V_1T_i^\sharp V_1$, $V_1T_i^\sharp V_2$, $V_2T_i^\sharp V_1$ *and* $V_2T_i^\sharp V_2$ *are linearly independent and the rest are linearly dependent on these if and only if*

$$\rho_{1i}+\rho_{2i}=r \quad \text{and} \quad 0<\rho_{1i} \quad \text{or} \quad \rho_{2i}<r.$$

The proof of this lemmas is analogous to that given by S. Yamamoto and Y. Fujii [3], and the proof is omitted.

Now we have the following theorem.

THEOREM II. *In a design satisfying the assumptions* 1°, 2°, 3° *and* 4°, *each component of the treatment sum of squares, corresponding respectively to each of the mutually orthogonal families of treatment contrasts determined by the association scheme, can be classified into one of the following seven cases according to the magnitude of the corresponding densities* ρ_{1i} *and* ρ_{2i} *in the spectral resolution of* $N_1D_1^{-1}N_1'$ *and* $N_2D_2^{-1}N_2'$.

(1) *The case being orthogonal to rows and columns*: $\rho_{1i}=\rho_{2i}=0$. *In this case,* $[T_i^\#]$ *is the one dimensional two-sided ideal of* $\mathfrak{R}$, *and the principal idempotent of the ideal is* $E_i^{(1)}=\frac{1}{r}T_i^\#$. *The component* S.S. (sum of squares) *of* $\alpha_i(=\mathrm{tr}(A_i^\#))$ *degrees of freedom corresponding to* $A_i^\#$ *and being defined by* $\frac{1}{r}T_i^\#$, *is orthogonal to the row-block space and the column-block space.*

(2) *The case being confounded with columns*: $\rho_{1i}=0$ *and* $\rho_{2i}=r$. *In this case,* $[T_i^\#]$ *is the one dimensional two-sided ideal of* $\mathfrak{R}$, *and the principal idempotent of the ideal is* $E_i^{(1)}=\frac{1}{r}T_i^\#$. *The component* S.S. *of* α_i *degrees of freedom corresponding to* $A_i^\#$ *and defined by* $\frac{1}{r}T_i^\#$, *is orthogonal to the row-block space but confounded with the column-block space.*

(3) *The case being confounded with rows*: $\rho_{2i}=0$ *and* $\rho_{1i}=r$. *This is similar to the case* (2).

(4) *The case being orthogonal to rows with partial confounding to columns*: $\rho_{1i}=0$ *and* $0<\rho_{2i}<r$. *In this case,* $[T_i^\#, V_2T_i^\#, T_i^\#V_2, V_2T_i^\#V_2]$ *is the four dimensional two-sided ideal of* $\mathfrak{R}$, *and the principal idempotent of the ideal is*

$$E_i^{(2)}=\frac{1}{r-\rho_{2i}}\left(T_i^\#-V_2T_i^\#-T_i^\#V_2+\frac{r}{\rho_{2i}}\,V_2T_i^\#V_2\right)$$

The component S.S. *of* $2\alpha_i$ *degrees of freedom corresponding to* $A_i^\#$ *and being defined by* $E_i^{(2)}$, *is orthogonal to row-block space and partially confounded with column-block space. The non-principal idempotent of the ideal being orthogonal to both row and column-block spaces, is*

$$F_i^{(1)}=E_i^{(2)}(I-V_2)=\frac{1}{r-\rho_{2i}}(I-V_2)T_i^\#(I-V_2)$$

The residual idempotent of the ideal being orthogonal to $F_i^{(1)}$ *and confounded* *with the column-block space, is*

$$E^{(1)}_{V_2 i} = E^{(2)}_i V_2 = \frac{1}{\rho_{2i}} V_2 T^{\sharp}_i V_2$$

The degrees of freedom of these compoents defined by $F^{(1)}_i$ *and* $E^{(1)}_{V_2 i}$ *are* α_i.

(5) *The case being orthogonal to columns with partial confounding to rows*: $0<\rho_{1i}<r$ *and* $\rho_{2i}=0$. *This is similar to the case* (4).

(6) *The case being confounded with both rows and columns*: $0<\rho_{1i}<r$, $0<\rho_{2i}<r$ *and* $\rho_{1i}+\rho_{2i}=r$. *In this case,* $[V_1T^{\sharp}_iV_1, V_1T^{\sharp}_iV_2, V_2T^{\sharp}_iV_1, V_2T^{\sharp}_iV_2]$ *is the four dimensional two-sided ideal of* $\mathfrak{R}$, *and the principal idempotent of the ideal is*

$$E^{(2)}_i = \frac{1}{\rho_{1i}} V_1 T^{\sharp}_i V_1 + \frac{1}{\rho_{2i}} V_2 T^{\sharp}_i V_2$$

The component S.S. *of* $2\alpha_i$ *degrees of freedom corresponding to* $A^{\sharp}_i$ *and defined by* $E^{(2)}_i$ *is totally confounded with row and column-block spaces. The non-principal idempotent of the ideal being orthogonal to the column-block space and confounded with the row-block space is* $E^{(1)}_{V_1 i} = \frac{1}{\rho_{1i}} V_1 T^{\sharp}_i V_1$. *The non-principal idempotent of the ideal being orthogonal to the row-block space and confounded with the column-block space is* $E^{(1)}_{V_2 i} = \frac{1}{\rho_{2i}} V_2 T^{\sharp}_i V_2$. *The degrees of freedom of these components are* α_i.

(7) *The case being partially confounded with both rows and columns*: $0<\rho_{1i}<r$ *and* $0<\rho_{2i}<r$ *and* $0<\rho_{1i}+\rho_{2i}<r$. *In this case,* $[T^{\sharp}_i, V_1T^{\sharp}_i, V_2T^{\sharp}_i, T^{\sharp}_iV_1, V_1T^{\sharp}_iV_1, V_2T^{\sharp}_iV_1, T^{\sharp}_iV_2, V_1T^{\sharp}_iV_2, V_2T^{\sharp}_iV_2]$ *is the nine dimensional two-sided ideal of* $\mathfrak{R}$, *and the principal idempotent of the ideal is*

$$E^{(3)}_i = \frac{1}{r-\rho_{1i}-\rho_{2i}} \Big(T^{\sharp}_i - V_1T^{\sharp}_i - V_2T^{\sharp}_i - T^{\sharp}_iV_1 - T^{\sharp}_iV_2 + V_2T^{\sharp}_iV_1 + V_1T^{\sharp}_iV_2 + \frac{r-\rho_{2i}}{\rho_{1i}} V_1T^{\sharp}_iV_1 + \frac{r-\rho_{1i}}{\rho_{2i}} V_2T^{\sharp}_iV_2 \Big)$$

The component S.S. *of* $3\alpha_i$ *degrees of freedom corresponding to* $A^{\sharp}_i$ *and defined by* $E^{(3)}_i$ *is partially confounded with the row-block space and column-block space. The non-principal idempotent of the ideal being orthogonal to both of the row-block space and column-block space is*

$$F_i^{(1)} = E_i^{(3)}(I - V_1 - V_2)$$

$$= \frac{1}{r - \rho_{1i} - \rho_{2i}}(I - V_1 - V_2)T_i^{\sharp}(I - V_1 - V_2)$$

The non-principal idempotent of the ideal being confounded with column-block space is

$$E_{V_2 i}^{(1)} = E_i^{(3)} V_2 = \frac{1}{\rho_{2i}} V_2 T_i^{\sharp} V_2$$

The residual idempotent of the ideal being confounded with row-block space is

$$E_{V_1 i}^{(1)} = E_i^{(3)} V_1 = \frac{1}{\rho_{1i}} V_1 T_i^{\sharp} V_1$$

The degrees of freedom of these components defined by $F_i^{(1)}$, $E_{V_2 i}^{(1)}$ and $E_{V_1 i}^{(1)}$ are α_i.

The proofs of the case (1) to (5) are (formally) the same as S. Yamamoto and Y. Fujii [3].

Proof of case (6): From Lemma 3 (iii), it follows that

$$I(\mathfrak{J}) = (\mathfrak{J})\left\|\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right\|$$

$$G(\mathfrak{J}) = (\mathfrak{J})\left\|\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right\|$$

$$V_1(\mathfrak{J}) = (\mathfrak{J})\left\|\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right\|$$

$$V_2(\mathfrak{J}) = (\mathfrak{J})\left\|\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right\|$$

$$T_j^{\sharp}(\mathfrak{J}) = (\mathfrak{J})\delta_{ij}\left\|\begin{matrix} \rho_{1i} & \rho_{2i} & & \\ \rho_{1i} & \rho_{2i} & & 0 \\ & & \rho_{1i} & \rho_{2i} \\ 0 & & \rho_{1i} & \rho_{2i} \end{matrix}\right\|$$

$$W(\mathfrak{J})=(\mathfrak{J})\begin{Vmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{Vmatrix}$$

where $(\mathfrak{J})=(V_1T_i^{\#}V_1,\ V_2T_i^{\#}V_1,\ V_1T_i^{\#}V_2,\ V_2T_i^{\#}V_2)$. Thus the sub-algebra $[V_1T_i^{\#}V_1,\ V_2T_i^{\#}V_1,\ V_1T_i^{\#}V_2,\ V_2T_i^{\#}V_2]$ is a four-dimensional two-sided ideal of $\mathfrak{R}$. The ideal is irreducible because we can find the following irreducible representation:

$$I\to\begin{Vmatrix} 1&0\\ 0&1 \end{Vmatrix},\qquad G\to\begin{Vmatrix} 0&0\\ 0&0 \end{Vmatrix},\qquad V_1\to\begin{Vmatrix} 1&0\\ 0&0 \end{Vmatrix}$$

$$V_2\to\begin{Vmatrix} 0&0\\ 0&1 \end{Vmatrix},\qquad W\to\begin{Vmatrix} 0&0\\ 0&0 \end{Vmatrix},\qquad T_j^{\#}\to\delta_{ij}\begin{Vmatrix} \rho_{1i}&\rho_{2i}\\ \rho_{1i}&\rho_{2i} \end{Vmatrix}$$

Thus the principal idempotent of the ideal is $E_i^{(2)}$, and its trace is $2\alpha_i$.

Proof of case (7); From Lemma 4, it follows that

$$I(\mathfrak{J}^*)=(\mathfrak{J}^*)\begin{Vmatrix} \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} & 0 & 0\\ 0 & \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} & 0\\ 0 & 0 & \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \end{Vmatrix}$$

$$G(\mathfrak{J}^*)=(\mathfrak{J}^*)\begin{Vmatrix} \begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix} & 0 & 0\\ 0 & \begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix} & 0\\ 0 & 0 & \begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix} \end{Vmatrix}$$

$$W(\mathfrak{J}^*)=(\mathfrak{J}^*)\begin{Vmatrix} \begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix} & 0 & 0\\ 0 & \begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix} & 0\\ 0 & 0 & \begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix} \end{Vmatrix}$$

$$V_1(\mathfrak{J}^*)=(\mathfrak{J}^*)\begin{Vmatrix} \begin{matrix}0&0&0\\1&1&0\\0&0&0\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}0&0&0\\1&1&0\\0&0&0\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}0&0&0\\1&1&0\\0&0&0\end{matrix} \end{Vmatrix}$$

$$V_2(\mathfrak{J}^*)=(\mathfrak{J}^*)\begin{Vmatrix} \begin{matrix}0&0&0\\0&0&0\\1&0&1\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}0&0&0\\0&0&0\\1&0&1\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}0&0&0\\0&0&0\\1&0&1\end{matrix} \end{Vmatrix}$$

$$T_j^{\#}(\mathfrak{J}^*)=(\mathfrak{J}^*)\delta_{ij}\begin{Vmatrix} \begin{matrix}r&\rho_{1i}&\rho_{2i}\\0&0&0\\0&0&0\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}r&\rho_{1i}&\rho_{2i}\\0&0&0\\0&0&0\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}r&\rho_{1i}&\rho_{2i}\\0&0&0\\0&0&0\end{matrix} \end{Vmatrix}$$

where $(\mathfrak{J}^*)=(T_i^{\#}, V_1T_i^{\#}, V_2T_i^{\#}, T_i^{\#}V_1, V_1T_i^{\#}V_1, V_2T_i^{\#}V_1, T_i^{\#}V_2, V_1T_i^{\#}V_2, V_2T_i^{\#}V_2)$. Thus the sub-algebra $[T_i^{\#}, V_1T_i^{\#}, V_2T_i^{\#}, T_i^{\#}V_1, V_1T_i^{\#}V_1, V_2T_i^{\#}V_1, T_i^{\#}V_2, V_1T_i^{\#}V_2, V_2T_i^{\#}V_2]$ is a nine-dimensional two-sided ideal of $\mathfrak{R}$. The ideal is irreducible because we can find the following irreducible representation

$$I\rightarrow\begin{Vmatrix}1&0&0\\0&1&0\\0&0&1\end{Vmatrix},\qquad G\rightarrow\begin{Vmatrix}0&0&0\\0&0&0\\0&0&0\end{Vmatrix}$$

$$W\rightarrow\begin{Vmatrix}0&0&0\\0&0&0\\0&0&0\end{Vmatrix},\qquad V_1\rightarrow\begin{Vmatrix}0&0&0\\1&1&0\\0&0&0\end{Vmatrix}$$

$$V_2\rightarrow\begin{Vmatrix}0&0&0\\0&0&0\\1&0&1\end{Vmatrix},\qquad T_j^{\#}\rightarrow\delta_{ij}\begin{Vmatrix}r&\rho_{1i}&\rho_{2i}\\0&0&0\\0&0&0\end{Vmatrix}$$

The principal idempotent of the ideal is $E_i^{(3)}$, and its trace is $3\alpha_i$.

The rest of the proof of Theorem II is easy, and is omitted.

In order to give the direct decomposition of the relationship algebra $\mathfrak{R}$, we shall rearrange ρ_{1i} and ρ_{2i} $(i=1, 2, \cdots, m)$ according to the magnitude of ρ_{1i} and ρ_{2i} $(i=1, 2, \cdots m)$ as follows

$$\rho_{1u}=\rho_{2u}=0 \qquad \text{for} \quad u=1, 2, \cdots, s$$

$$\rho_{1v}=0, \quad \rho_{2v}=r \qquad \text{for} \quad v=s+1, s+2, \cdots, b$$

$$\rho_{1w}=r, \quad \rho_{2w}=0 \qquad \text{for} \quad w=b+1, b+2, \cdots, c$$

$$\rho_{1x}=0, \quad 0<\rho_{2x}<r \qquad \text{for} \quad x=c+1, c+2, \cdots, d$$

$$0<\rho_{1y}<r, \quad \rho_{2y}=0 \qquad \text{for} \quad y=d+1, d+2, \cdots, e$$

$$0<\rho_{1z}<r, \quad 0<\rho_{2z}<r, \quad r=\rho_{1z}+\rho_{2z} \qquad \text{for} \quad z=e+1, e+2, \cdots, f$$

$$0<\rho_{1t}<r, \quad 0<\rho_{2t}<r, \quad 0<\rho_{1t}+\rho_{2t}<r \qquad \text{for} \quad t=f+1, f+2, \cdots, m.$$

$$(0 \leq s \leq b \leq c \leq d \leq e \leq f \leq m)$$

The principal idempotent $E_G^{(1)}$ of the one-dimensional two-sided ideal $[G]$ of $\mathfrak{R}$ is

$$E_G^{(1)}=\frac{1}{n}G$$

In order to obtain the remaining irreducible two-sided ideals of $\mathfrak{R}$ and their principal idempotents, we shall consider the difference algebra of $\mathfrak{R}$ modulo $[[G, T_i^{\sharp}; i=1, 2, \cdots, m]]$, i.e.,

$$\mathfrak{R}-[[G, T_1^{\sharp}, T_2^{\sharp}, \cdots, T_m^{\sharp}]]$$

where $[[G, T_1^{\sharp}, T_2^{\sharp}, \cdots, T_m^{\sharp}]]$ is the ideal of $\mathfrak{R}$ generated by G and $T_i^{\sharp}$ $(i=1, \cdots, m)$ and the principal idempotent of the ideal is $E_G^{(1)}+\sum_{u=1}^{s}E_u^{(1)}+\sum_{v=s+1}^{b}E_v^{(1)}+\sum_{w=b+1}^{c}E_w^{(1)}+\sum_{x=c+1}^{d}E_x^{(2)}+\sum_{y=d+1}^{e}E_y^{(2)}+\sum_{z=e+1}^{f}E_z^{(2)}+\sum_{t=f+1}^{m}E_t^{(3)}$. Generally, this difference algebra is isomorphic to the algebra $[I, V_1, V_2, W]$ generated by I, V_1, V_2 and W. The latter can be decomposed into the direct sum of four mutually orthogonal one-dimensional two-sided ideals $[I-V_1-V_2-W]$, $[V_1]$, $[V_2]$ and $[W]$, and their principal idempotents are respectively the generators themselves. In some cases, however, it may happen that the ideals corresponding to $[V_1]$, $[V_2]$ and $[W]$ degenerate to zero as is the case indicated in [3].

The principal idempotents E_e, B_{V_1}, B_{V_2} and B_W of the ideals of $\mathfrak{R}$ corresponding respectively to $I-V_1-V_2-W$, V_1, V_2, and W may be obtained by dropping the modulo G and $T_i^\#$ $(i=1, 2, \cdots, m)$ of the following:

$$\begin{array}{ll} V_1 = B_{V_1}, \qquad V_2 = B_{V_2} & \\ & \mod G,\ T_i^\#;\ i=1, \cdots, m \\ I - V_1 - V_2 - W = E_e, \qquad W = B_W & \end{array}$$

The results may be expressed as

$$V_1 = B_{V_1} + F_{V_1}, \qquad V_2 = B_{V_2} + F_{V_2}$$

$$I - V_1 - V_2 - W = E_e + F_e, \qquad W = B_W + F_W$$

where

$$F_{V_1} = \frac{1}{r} \sum_{w=b+1}^{c} V_1 T_w^\# V_1 + \sum_{y=d+1}^{e} \frac{1}{\rho_{1y}} V_1 T_y^\# V_1 + \sum_{z=e+1}^{f} \frac{1}{\rho_{1z}} V_1 T_z^\# V_1 + \sum_{t=f+1}^{m} \frac{1}{\rho_{1t}} V_1 T_t^\# V_1$$

$$F_{V_2} = \frac{1}{r} \sum_{v=s+1}^{b} V_2 T_v^\# V_2 + \sum_{x=c+1}^{d} \frac{1}{\rho_{2x}} V_2 T_x^\# V_2 + \sum_{z=e+1}^{f} \frac{1}{\rho_{2z}} V_2 T_z^\# V_2 + \sum_{t=f+1}^{m} \frac{1}{\rho_{2t}} V_2 T_t^\# V_2$$

$$F_e = \frac{1}{r} \sum_{u=1}^{s} T_u^\# + \sum_{x=e+1}^{d} F_x^{(1)} + \sum_{y=d+1}^{e} F_y^{(1)} + \sum_{t=f+1}^{m} F_t^{(1)}$$

$$F_W = E_G^{(1)}$$

since B_{V_1}, B_{V_2}, E_e, B_W, F_{V_1}, F_{V_2}, F_e and F_W must satisfy the following equations:

$$\left\| \begin{matrix} B_{V_1} \\ B_{V_2} \\ E_e \\ B_W \\ F_{V_1} \\ F_{V_2} \\ F_e \\ F_W \end{matrix} \right\| \left\| \begin{matrix} E_G^{(1)} + \sum E_u^{(1)} + \sum E_v^{(1)} + \sum E_w^{(1)} \\ \qquad + \sum E_x^{(2)} + \sum E_y^{(2)} + \sum E_z^{(2)} + \sum E_t^{(3)} \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ F_{V_1} \\ F_{V_2} \\ F_e \\ F_W \end{matrix} \right\|$$

We may summarize the results obtained so far by the following theorem.

THEOREM III. *The unique decomposition of the unit element of the relationship algebra $\mathfrak{R}$ corresponding to its direct decomposition is*

$$I = E_G^{(1)} + B_{V_1} + B_{V_2} + E_e + B_W + \sum E_u^{(1)} + \sum E_v^{(1)} + \sum E_w^{(1)}$$
$$+ \sum E_x^{(2)} + \sum E_y^{(2)} + \sum E_z^{(2)} + \sum E_t^{(3)}$$

Further decomposition in relation to the row-block space and the column-block space is

$$I = E_G^{(1)} + B_{V_1} + B_{V_2} + E_e + B_W + \sum_u \frac{1}{r} T_u^{\#} + \sum_x F_x^{(1)} + \sum_y F_y^{(1)} + \sum_t F_t^{(1)}$$
$$+ \sum_v \frac{1}{r} T_v^{\#} + \sum_x E_{V_2 x}^{(1)} + \sum_z E_{V_2 z}^{(1)} + \sum_t E_{V_2 t}^{(1)} + \sum_w \frac{1}{r} T_W^{\#}$$
$$+ \sum_y E_{V_2 y}^{(1)} + \sum_z E_{V_1 z}^{(1)} + \sum_t E_{V_1 t}^{(1)}$$

5. Analysis of variance for two-way design

We are considering a design which consists of $n = \sum_{j=1}^{h} x_j y_j$ experimental units in which the observation vector $\eta' = (\eta_1, \eta_2, \cdots, \eta_n)$ satisfies the linear model

$$\eta = \gamma \boldsymbol{j}_n + \Phi \boldsymbol{\tau} + \| \Psi_1 \Psi_2 \| \left\| \begin{matrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{matrix} \right\| + \boldsymbol{e} \tag{24}$$

where γ is the general mean, $\boldsymbol{\tau}' = (\tau_1, \cdots, \tau_v)$ is the treatment parameter vector, $\boldsymbol{\beta}_1' = (\beta_{11}, \beta_{12}, \cdots, \beta_{1b_1})$ is the row-block parameter vector and $\boldsymbol{\beta}_2' = (\beta_{21}, \beta_{22}, \cdots, \beta_{2b_2})$ is the column-block parameter vector being subjected to the restrictions

$$\sum_{\alpha=1}^{v} \tau_\alpha = 0, \qquad \sum_{a=1}^{b_1} \beta_{1a} = 0, \qquad \sum_{p=1}^{b_2} \beta_{2p} = 0 \tag{25}$$

respectively, and $\boldsymbol{e}' = (e_1, e_2, \cdots, e_n)$ is the error vector being normally distributed with mean vector zero and covariance matrix $\sigma^2 I_n$. The matrices Φ, Ψ_1 and Ψ_2 are the incidence matrices defined in (1), (2) and (10) and $\boldsymbol{j}_n' = (1, 1, \cdots, 1)$.

Denote,

(26) grand total: $\boldsymbol{G}=\boldsymbol{j}_n'\boldsymbol{\eta}$, treatment totals: $\boldsymbol{T}=\Psi'\boldsymbol{\eta}$,

row-block totals: $\boldsymbol{U}_1=\Psi_1'\boldsymbol{\eta}$, column-block totals: $\boldsymbol{U}_2=\Psi_2'\boldsymbol{\eta}$.

The normal equations for the least-square estimation are

$$ng+\boldsymbol{r}'\hat{\boldsymbol{\tau}}+\boldsymbol{k}_1'\hat{\boldsymbol{\beta}}_1+\boldsymbol{k}_2'\hat{\boldsymbol{\beta}}_2=\boldsymbol{G} \tag{27}$$

$$\boldsymbol{r}g+D_r\hat{\boldsymbol{\tau}}+N_1\hat{\boldsymbol{\beta}}_1+N_2\hat{\boldsymbol{\beta}}_2=\boldsymbol{T} \tag{28}$$

$$\left\|\begin{matrix}\boldsymbol{k}_1\\ \boldsymbol{k}_2\end{matrix}\right\| g+\left\|\begin{matrix}N_1'\\ N_2'\end{matrix}\right\|\hat{\boldsymbol{\tau}}+\left\|\begin{matrix}\Psi_1'\Psi_1 & \Psi_1'\Psi_2\\ \Psi_2'\Psi_1 & \Psi_2'\Psi_2\end{matrix}\right\|\left\|\begin{matrix}\hat{\boldsymbol{\beta}}_1\\ \hat{\boldsymbol{\beta}}_2\end{matrix}\right\|=\left\|\begin{matrix}\boldsymbol{U}_1\\ \boldsymbol{U}_2\end{matrix}\right\| \tag{29}$$

where, $\hat{\boldsymbol{\tau}}$, $\hat{\boldsymbol{\beta}}_1$ and $\hat{\boldsymbol{\beta}}_2$ are the estimates of $\boldsymbol{\tau}$, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, and $\boldsymbol{r}=\Phi'\boldsymbol{j}$, $\boldsymbol{k}_1=\Psi_1'\boldsymbol{j}$ and $\boldsymbol{k}_2=\Psi_2'\boldsymbol{j}$.

Multiplying (29) by

$$\|\Psi_1\ \Psi_2\|\left\|\begin{matrix}D_1^{-1}-D_1^{-1}MD_2^{-1}M'D_1^{-1} & 0\\ 0 & D_2^{-1}\end{matrix}\right\|$$

from the left, we have the following equation

$$\begin{aligned}\boldsymbol{j}g+(\Psi_1D_1^{-1}N_1'-\Psi_1D_1^{-1}MD_2^{-1}M'D_1^{-1}N_1'+\Psi_2D_2^{-1}N_2')\hat{\boldsymbol{\tau}}+\Psi_1\hat{\boldsymbol{\beta}}_1+\Psi_2\hat{\boldsymbol{\beta}}_2\\ =\Psi_1D_1^{-1}\boldsymbol{U}_1-\Psi_1D_1^{-1}MD_2^{-1}M'D_1^{-1}\boldsymbol{U}_1+\Psi_2D_2^{-1}\boldsymbol{U}_2\end{aligned} \tag{30}$$

and multiplying (30) by Φ' from the left and substracting it from (28), we have the adjusted normal equation for treatment;

$$\begin{aligned}(D_r-N_1D_1^{-1}N_1'-N_2D_2^{-1}N_2'+N_2D_2^{-1}M'D_1^{-1}N_1')\hat{\boldsymbol{\tau}}\\ =\boldsymbol{T}-N_1D_1^{-1}\boldsymbol{U}_1-N_2D_2^{-1}\boldsymbol{U}_2+N_2D_2^{-1}M'D_1^{-1}\boldsymbol{U}_1\end{aligned} \tag{31}$$

The complete table of the analysis of variance for the design will be given in Table I.

6. Illustration of allocation plan to the design for two-way elimination of heterogeneity

Example 1. An allocation plan for 12 treatments having two-way factorial association scheme.

Table 2 shows the association scheme for the treatments. Each treatment

Table 1. Analysis of variance for two-way design.

		Families of treatment contrasts	Idempotents	d.f.	S.S.	Expectation of mean squares
Treatments, eliminating rows and columns		Case (1) $u=1, \cdots, s$	$\frac{1}{r} T_u^\#$	α_u	$\boldsymbol{\eta}' \frac{1}{r} T_u^\# \boldsymbol{\eta}$	$\frac{r}{\alpha_u} \boldsymbol{\tau}' A_u^\# \boldsymbol{\tau} + \sigma^2$
		Case (4) $x=c+1, \cdots, d$	$F_x^{(1)}$	α_x	$\boldsymbol{\eta}' F_x^{(1)} \boldsymbol{\eta}$	$\frac{r-\rho_{2x}}{\alpha_x} \boldsymbol{\tau}' A_x^\# \boldsymbol{\tau} + \sigma^2$
		Case (5) $y=d+1, \cdots, e$	$F_y^{(1)}$	α_y	$\boldsymbol{\eta}' F_y^{(1)} \boldsymbol{\eta}$	$\frac{r-\rho_{1y}}{\alpha_y} \boldsymbol{\tau}' A_y^\# \boldsymbol{\tau} + \sigma^2$
		Case (7) $t=e+1, \cdots, m$	$F_t^{(1)}$	α_t	$\boldsymbol{\eta}' F_t^{(1)} \boldsymbol{\eta}$	$\frac{r-\rho_{1t}-\rho_{2t}}{\alpha_t} \boldsymbol{\tau}' A_t^\# \boldsymbol{\tau} + \sigma^2$
Rows, ignoring treatments	Treatment-components	Case (3) $w=b+1, \cdots, c$	$\frac{1}{r} T_w^\#$	α_w	$\frac{1}{r} \boldsymbol{\eta}' T_w^\# \boldsymbol{\eta}$	$\frac{1}{\alpha_w} \{ r\boldsymbol{\tau}' A_w^\# \boldsymbol{\tau} + 2\boldsymbol{\beta}_1' N_1' A_w^\# \boldsymbol{\tau} + \frac{1}{r} \boldsymbol{\beta}_1' N_1' A_w^\# N_1 \boldsymbol{\beta}_1 \} + \sigma^2$
		Case (5) $y=d+1, \cdots, e$	$E_{V_1 y}^{(1)}$	α_y	$\boldsymbol{\eta}' E_{V_1 y}^{(1)} \boldsymbol{\eta}$	$\frac{1}{\alpha_y \rho_{1y}} \{ \rho_{1y}^2 \boldsymbol{\tau}' A_y^\# \boldsymbol{\tau} + 2\rho_{1y} \boldsymbol{\beta}_1' N_1' A_y^\# \boldsymbol{\tau} + \boldsymbol{\beta}' N_1' A_y^\# N_1 \boldsymbol{\beta}_1 \} + \sigma^2$
		Case (6) $z=e+1, \cdots, f$	$E_{V_1 z}^{(1)}$	α_z	$\boldsymbol{\eta}' E_{V_1 z}^{(1)} \boldsymbol{\eta}$	$\frac{1}{\alpha_z \rho_{1z}} \{ \rho_{1z}^2 \boldsymbol{\tau}' A_z^\# \boldsymbol{\tau} + 2\rho_{1z} \boldsymbol{\beta}_1' N_1' A_z^\# \boldsymbol{\tau} + \boldsymbol{\beta}_1' N_1' A_z^\# N_1 \boldsymbol{\beta}_1 \} + \sigma^2$
		Case (7) $t=f+1, \cdots, m$	$E_{V_1 t}^{(1)}$	α_t	$\boldsymbol{\eta}' E_{V_1 t}^{(1)} \boldsymbol{\eta}$	$\frac{1}{\alpha_t \rho_{1t}} \{ \rho_{1t}^2 \boldsymbol{\tau}' A_t^\# \boldsymbol{\tau} + 2\rho_{1t} \boldsymbol{\beta}_1' N_1' A_t^\# \boldsymbol{\tau} + \boldsymbol{\beta}_1' N_1' A_t^\# N_1 \boldsymbol{\beta}_1 \} + \sigma^2$
	Rows, eliminating treatments and columns		B_{V_1}	α_R	by substract	$\frac{1}{\alpha_R} \{ \boldsymbol{\beta}_1' (D_1 - M D_2^{-1} M') \boldsymbol{\beta}_1 - \boldsymbol{\beta}_1' N_1' (\sum \frac{1}{r} A_w^\# + \sum \frac{1}{\rho_{1y}} A_y^\# + \sum \frac{1}{\rho_{1z}} A_z^\# + \sum \frac{1}{\rho_{1t}} A_t^\#) N_1 \boldsymbol{\beta}_1 \} + \sigma^2$
			V_1	$b_1 - h$	$\boldsymbol{\eta}' V_1 \boldsymbol{\eta}$	$\frac{1}{b_1 - h} \{ \boldsymbol{\tau}' (\sum \rho_w A_w^\# + \sum \rho_{1y} A_y^\# + \sum \rho_{1z} A_z^\# + \sum \rho_{1t} A_t^\#) \boldsymbol{\tau} + 2\boldsymbol{\tau}' (N_1 - N_2 D_2^{-1} M') \boldsymbol{\beta}_1 + \boldsymbol{\beta}_1' (D_1 - M D_2^{-1} M') \boldsymbol{\beta}_1 \} + \sigma^2$

continued

			Idempotents	d.f.	S.S.	Expectation of mean squares
Columns, ignoring treatments	Treatment-components	Case (2) $v=s+1, \cdots, b$	$\frac{1}{r}T_v^{\#}$	α_v	$\frac{1}{r}\eta' T_v^{\#}\eta$	$\frac{1}{\alpha_v} r\tau' A_v^{\#}\tau+2\beta_2' N_2 A_v^{\#}\tau+\frac{1}{r}\beta_2' N_2' A_v^{\#} N_2\beta_2\}+\sigma^2$
		Case (4) $x=c+1, \cdots, d$	$E_{V_2x}^{(1)}$	α_x	$\eta' E_{V_2x}\eta$	$\frac{1}{\alpha_x\rho_{2x}}\{\rho_{2x}^2\tau' A_x^{\#}\tau+2\rho_{2x}\beta_2' N_2' A_x^{\#}\tau+\beta_2' N_2' A_x^{\#} N_2\beta_2\}+\sigma^2$
		Case (6) $z=e+1, \cdot\cdot, f$	$E_{V_2z}^{(1)}$	α_z	$\eta' E_{V_2z}\eta$	$\frac{1}{\alpha_z\rho_{2z}}\{\rho_{2z}^2\tau' A_z^{\#}\tau+2\rho_{2z}\beta_2' N_2' A_z^{\#}\tau+\beta_2' N_2' A_z^{\#} N_2\beta_2\}+\sigma^2$
		Case (7) $t=f+1, \cdots, m$	$E_{V_2t}^{(1)}$	α_t	$\eta' E_{V_2t}\eta$	$\frac{1}{\alpha_t\rho_{2t}}\{\rho_{2t}^2\tau' A_t^{\#}\tau+2\rho_{2t}\beta_2' N_2' A_t^{\#}\tau+\beta_2' N_2' A_t^{\#} N_2\beta_2\}+\sigma^2$
	Columns, eliminating treatments and rows		B_{V_2}	α_C	by substract	$\frac{1}{\alpha_C}\{\beta_2'(D_2-M'D_1^{-1}M)\beta_2-\beta_2' N_2'(\sum\frac{1}{r}A_v^{\#}+\sum\frac{1}{\rho_{2x}}A_x^{\#}$ $+\sum\frac{1}{\rho_{2z}}A_z^{\#}+\sum\frac{1}{\rho_{2t}}A_t^{\#})N_2\beta_2\}+\sigma^2$
			V_2	b_2-h	$\eta' V_2\eta$	$\frac{1}{b_2-h}\{\tau'(\sum\rho_{2v}A_v^{\#}+\sum\rho_{2x}A_x^{\#}+\sum\rho_{2z}A_z^{\#}+\sum\rho_t A_t^{\#})\tau$ $+2\tau'(N_2-N_1D_1^{-1}M)\beta_2+\beta_2'(D_2-M'D_1^{-1}M)\beta_2\}+\sigma^2$
Part effect			B_W	$h-1$	$\eta' B_W\eta$	$\frac{1}{h-1}\{2\tau'(N_2D_2^{-1}M'\beta_1+N_1D_1^{-1}M\beta_2)+\beta_1' MD_2^{-1}M'\beta_1$ $+2\beta_1' M\beta_2+\beta_2' M' D_1^{-1}M\beta_2\}+\sigma^2$
Error			E_e	α_E	by substract	σ^2
Total			$I-E_G^{(1)}$	$n-1$	$\eta'(I-E_G^{(1)})\eta$	$\frac{1}{n-1}\{r\tau'\tau+2\tau' N_1\beta_1+2\tau' N_2\beta_2+\beta_1' D_1\beta_1$ $+2\beta_1' M\beta_2+\beta_2' D_2\beta_2\}+\sigma^2$

$\alpha_R=b_1-h-\sum\alpha_w-\sum\alpha_y-\sum\alpha_z-\sum\alpha_t$

is the first associate of the other in the same row, the second associate of the other in the same column and third associate of the rest. Table 3 shows an allocation of those treatments as a design for two-way elimination.

Table 2

a	*b*	*c*	*d*
e	*f*	*g*	*h*
i	*j*	*k*	*l*

Table 3

rows \ columns	1	2	3	4	5	6	7	8	9	10	11
1	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*			
2	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*			
3	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*			
4									*a*	*e*	*i*
5									*b*	*f*	*j*
6									*c*	*g*	*k*
7									*d*	*h*	*l*

As the mutually orthogonal idempotents of the two-way factorial association scheme are

$$A_0^{\#}=\frac{1}{12}G_{12},\qquad A_1^{\#}=\frac{1}{3}G_3\otimes I_4-\frac{1}{12}G_3\otimes G_4$$

$$A_2^{\#}=\frac{1}{4}I_3\otimes G_4-\frac{1}{12}G_3\otimes G_4$$

$$A_3^{\#}=I_3\otimes I_4-\frac{1}{4}I_3\otimes G_4-\frac{1}{3}G_3\otimes I_4+\frac{1}{12}G_3\otimes G_4$$

and

$$U_1=\left\|\begin{matrix}\frac{1}{8}I_3\otimes G_8 & \\ & \frac{1}{3}I_4\otimes G_3\end{matrix}\right\|,\qquad U_2=\left\|\begin{matrix}\frac{1}{3}G_3\otimes I_8 & \\ & \frac{1}{4}G_4\otimes I_3\end{matrix}\right\|$$

$$U_1U_2=\left\|\begin{matrix}\frac{1}{24}G_{24} & \\ & \frac{1}{12}G_{12}\end{matrix}\right\|$$

$$N_1D_1^{-1}N_1'=3A_0^{\#}+A_1^{\#}+\frac{1}{2}A_2^{\#}$$

$$N_2D_2^{-1}N_2'=3A_0^{\#}+2A_1^{\#}+A_2^{\#}$$

the above design satisfies the assumptions 1°, 2°, 3° and 4°.

Example 2. An allocation plan for 16 treatments having L_2 association scheme.

Table 4 shows the association scheme for the treatments. Each treatment is the first associate of the other in the same row or in the same column and the second associate of the rest. Table 5 shows an allocation plan of those treatments as a design for two-way elimination

Table 4

a	*b*	*c*	*d*
e	*f*	*g*	*h*
p	*q*	*r*	*s*
u	*v*	*x*	*y*

Table 5

row \ column	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*p*	*q*	*r*	*s*												
2	*p*	*q*	*r*	*s*	*u*	*v*	*x*	*y*	*a*	*b*	*c*	*d*												
3	*e*	*f*	*g*	*h*	*p*	*q*	*r*	*s*	*u*	*v*	*x*	*y*												
4	*u*	*v*	*x*	*y*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*												
5													*a*	*e*	*p*	*u*	*b*	*f*	*q*	*v*	*c*	*g*	*r*	*x*
6													*b*	*f*	*q*	*v*	*c*	*g*	*r*	*x*	*d*	*h*	*s*	*y*
7													*c*	*g*	*r*	*x*	*d*	*h*	*s*	*y*	*a*	*e*	*p*	*u*
8													*d*	*h*	*s*	*y*	*a*	*e*	*p*	*u*	*b*	*f*	*q*	*v*

As the mutually orthogonal idempotents of the L_2 association scheme are

$$A_0^{\#} = \frac{1}{16} G_{16}$$

$$A_1^{\#} = \frac{1}{4} (I_4 \otimes G_4 + G_4 \otimes I_4) - \frac{1}{8} G_{16}$$

$$A_2^{\#} = I_{16} - \frac{1}{4} (I_4 \otimes G_4 + G_4 \otimes I_4) + \frac{1}{16} G_{16}$$

it can be seen that only for the first portion of row-column incidence as well as for the second portion, the design does not satisfy the assumptions 3° and 4°. The whole design, however, satisfies those assumptions 1°, 2°, 3° and 4°, as

$$U_1 = \left\| \begin{matrix} \frac{1}{12} I_4 \otimes G_{12} & 0 \\ 0 & \frac{1}{12} I_4 \otimes G_{12} \end{matrix} \right\| \qquad U_2 = \left\| \begin{matrix} \frac{1}{4} G_4 \otimes I_{12} & 0 \\ 0 & \frac{1}{4} G_4 \otimes I_{12} \end{matrix} \right\|$$

$$U_1 U_2 = \left\| \begin{matrix} \frac{1}{48} G_4 \otimes G_{12} & 0 \\ 0 & \frac{1}{48} G_4 \otimes G_{12} \end{matrix} \right\|$$

$$N_1 D_1^{-1} N_1' = 6A_0^{\#} + \frac{1}{3} A_1^{\#}$$

$$N_2 D_2^{-1} N_2' = 6A_0^{\#} + 3A_1^{\#}$$

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