J. Sci. Hiroshima Univ. Ser. A-I
28 (1964), 167-197

Some Aspects for the Composition of Relationship Algebras of Experimental Designs

Sumiyasu Yamamoto

(Received September 20, 1964)

Contents

1. Introduction and summary

In 1957, A. T. James [5] introduced first the suggestive notions of the relationships and the relationship algebra defined on a set of experimental units. He clarified the bearing of the direct decomposition of the relationship algebra on the analysis of variance for a standard experimental design, such as, for randomized block designs (RBD), for Latin square designs (LSD), and, though not sufficiently, for balanced incomplete block designs (BIBD). In 1959, R. C. Bose and D. M. Mesner [2] dealt with the association algebra generated by the association matrices of an association scheme introduced first by R. C. Bose and K. P. Nair [3]. In those days, J. Ogawa [7] dealt with the analysis of the association algebra as well as the relationship algebra of a partially balanced incomplete block design (PBIBD). H. B. Mann [6] dealt with the algebra of the general linear hypothesis. S. Yamamoto and Y. Fujii [9] treated the analysis of the relationship algebra of a PBIBD, too, and clarified the meaning of orthogonality, partially confounding and confounding of a component of treatment sum of squares to the block space.

During the course of our research for the analysis of a PBIBD, several questions and ideas suggested themselves to us. Some of them will be mentioned below.

i) What is an association scheme? What is an association algebra? Can we consider an association algebra as a kind of the relationship algebra defined among a set of treatment parameters? If so, an association scheme or an association algebra may be defined independently of the treatment-block incidence matrix of an experimental design.

ii) An association algebra defined among a set of treatment parameters determines uniquely the decomposition of the parameter sum of squares into sum of several parameter quadratic forms. In view of the fact, can we consider the algebra as an apparent structure defined among the working parameters, and as being composed of several primitive relationship algebras defined respectively among several sets of primitive parameters? If so, an association algebra is a sort of apparent parameter structures being so composed of several primitive relationship algebras that any one of the primitive parameter sums of squares may correspond faithfully to one of the component quadratic forms of the apparent parameter sum of squares.

iii) The notions of the relationships and the relationship algebra may naturally be introduced not only into a set of relevant (or treatment) parameters but also into a set of nuisance (or block) parameters. In view of the above, the nuisance parameter algebras for the so-called block designs, such as, RBD, BIBD, PBIBD, etc., are primitive, and those for the two-way elimination designs, such as, LSD, Youden square designs (YSD), etc., are factorial.

iv) The steps of constructing an experimental design may be regarded as, (1) to compose suitably the relevant (or treatment) parameter algebra from several primitive relationships, (2) to compose suitably the nuisance parameter algebra to be eliminated, and (3) to map suitably those algebras over a set of experimental units and compose them into the relationship algebra of the design in order to pick out the relevant relationships faithfully after the elimination of the nuisance relationships.

v) Mappings and compositions of the relationship algebras might play a fundamental role in the composition of a design. What are the implications of the phrases 'to map suitably' and 'to compose suitably'?

To answer those questions and to formulate those ideas, we shall provide some notions and several theorems which will be useful in the composition of the relationship algebra of an experimental design.

Some notions and results concerning the semi-simple matrix algebras are introduced in sections 2, 3 and 4. In particular, the notions of similar and partially similar mappings are defined and a fundamental theorem is presented in section 2. In section 3, the notions of confounding and partially confounding are defined in algebra-theoretic terminologies. In section 4, the notion of orthogonality is introduced in algebra-theoretic terminologies.

The remaining sections of this paper are devoted to the composition of particular relationship algebras of experimental designs. Specific features of a relationship algebra are discussed in section 5. In view of those features of the relationship algebra, the notions of G-preserving mappings and G-orthogonal compositions are introduced in section 6. In section 7, the composition of the parameters is discussed in relation to G-preserving mappings. The composition of the parameters in relation to G-orthogonal compositions is discussed in section 8.

Composition of the relationship algebras of the experimental designs is treated in section 9. This section is divided into two parts, the one devoted to the case where no nuisance parameter algebra exists in a set of experimental units, and the other devoted to the case where a nuisance parameter algebra is introduced in a set of experimental units.

2. Similar and partially similar mappings

Let $\mathfrak{R}$ be a semi-simple matrix algebra (ring without radical) over the real field [1] [8] generated by a finite number of real symmetric matrices of order m. $\mathfrak{R}$ is a set of linear transformations whose domain and range are self-dual m-dimensional real vector spaces V_m, respectively. According to the theory of semi-simple algebra, $\mathfrak{R}$ is completely reducible and can uniquely be decomposed into the direct sum of minimum two-sided ideals, say,

$$\mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \oplus \cdots \oplus \mathfrak{R}_k \tag{1}$$

apart from the order of the ideals. Each component algebra $\mathfrak{R}_i$ is isomorphic to the complete matrix algebra of order m_i, the multiplicity of which is α_i. Let E be the principal idempotent of $\mathfrak{R}$ and E_i be the principal idempotent of $\mathfrak{R}_i$ $(i=1, \cdots, k)$, then the corresponding decomposition of principal idempotent into mutually orthogonal principal idempotents of the ideals is,

$$E = E_1 + E_2 + \cdots + E_k \tag{2}$$

The ranks of those idempotents are

$$r(E_i) = \beta_i = m_i\alpha_i, \qquad r(E) = \sum_{i=1}^{k} r(E_i) = \sum_{i=1}^{k} \beta_i \leq m \tag{3}$$

Let $E_0 = I_m - E$, then $E_0^2 = E_0$, $E_0E = EE_0 = 0$ and for any E_i $(i=1, \cdots, k)$ $E_0E_i = E_iE_0 = 0$ hold. For convenience' sake, denote $r(E_0) = \beta_0 = m_0\alpha_0$.

Let F be an $n \times m$ real matrix which maps a column vector of V_m into a column vector of V_n, i.e., for any $\boldsymbol{a} \in V_m$

$$F: \quad \boldsymbol{a} \rightarrow \boldsymbol{a}^* = F\boldsymbol{a} \in V_n \tag{4}$$

Let $\mathfrak{R}^*$ be the image of $\mathfrak{R}$ induced by the following linear mapping σ of $\mathfrak{R}$ which is defined by the matrix F; i.e., for any $A \in \mathfrak{R}$

(5) $$\sigma: \quad A \to A^* = FAF'$$

or

(6) $$\sigma: \quad \mathfrak{R} \to \mathfrak{R}^* = F\mathfrak{R}F' = \sigma(\mathfrak{R})$$

where $\mathfrak{R}^* = \{A^*;\ A^* = FAF',\ A \in \mathfrak{R}\}$

DEFINITION 1. A linear mapping σ of $\mathfrak{R}$ defined by F is said to be *partially similar* if it satisfies the following two conditions

(7) $$\text{(i)} \quad \sigma(E_i)\sigma(E_j) = \delta_{ij}\, c_i\, \sigma(E_i) \qquad (c_i \geq 0,\ i, j = 0, 1, \cdots, k)$$

(8) $$\text{(ii)} \quad \text{If } c_i > 0, \text{ then } r\big(\sigma(E_i)\big) = r(E_i) \qquad (i = 0, 1, \cdots, k)$$

where $\delta_{ij}=1$ or 0 according as $i=j$ or $i \neq j$. If $c_i > 0$ for some i a partially similar mapping is said to be *proper*. In particular, σ is *similar* if $c_1 = c_2 = \cdots = c_k = c > 0$.

As to the matrix F which defines a linear mapping σ of $\mathfrak{R}$, the following theorem holds.

THEOREM 1. *A linear mapping σ of $\mathfrak{R}$ defined by F is partially similar if and only if*

(9) $$F'F = \sum_{i=0}^{k} c_i E_i \qquad (c_i \geq 0)$$

In this case, the image $\mathfrak{R}^* = \sigma(\mathfrak{R})$ of $\mathfrak{R}$ is also a semi-simple matrix algebra. As we can assume $c_1 > 0,\ c_2 > 0,\ \cdots,\ c_l > 0,\ c_{l+1} = \cdots = c_k = 0$ without loss of generality, $\mathfrak{R}^*$ can be decomposed into the direct sum of minimum two-sided ideals, such as,

(10) $$\mathfrak{R}^* = \sigma(\mathfrak{R}) = \sigma(\mathfrak{R}_1) \oplus \sigma(\mathfrak{R}_2) \oplus \cdots \oplus \sigma(\mathfrak{R}_l)$$

Any one of the component algebra $\sigma(\mathfrak{R}_i)$ is isomorphic to $\mathfrak{R}_i$ and the multiplicity of the irreducible representations is also α_i. The corresponding decomposition of the principal idempotent into mutually orthogonal principal idempotents of the ideals is,

(11) $$\tilde{E} = \tilde{E}_1 + \tilde{E}_2 + \cdots + \tilde{E}_l$$

where

$$(12) \qquad \tilde{E}_j = \frac{1}{c_j} F E_j F', \qquad \tilde{E} = \sum_{j=1}^{l} \frac{1}{c_j} F E_j F' \qquad (j=1, \cdots, l)$$

Proof. Assume first that F satisfies the condition (9), then we have $\sigma(E_i)\sigma(E_j) = FE_iF'FE_jF' = \delta_{ij}c_iFE_iF' = \delta_{ij}c_i\sigma(E_i)$, for $i, j=0, 1, \cdots, k$, and if $c_i>0$, $r(\sigma(E_i)) = r(FE_iF') = r(E_iF'FE_i) = r(c_iE_i) = r(E_i)$, for $i=0, 1, \cdots, k$. The linear mapping σ defined by F is, therefore, partially similar.

Conversely, assume that σ is partially similar, i.e., (7) and (8) hold. Since we can assume that $c_0=c_1\cdots=c_{s-1}=0$, $c_s>0, \cdots, c_k>0$ $(0\leq s\leq k)$ without loss of generality, we have

$$(13) \quad \begin{array}{lll} \text{(a)} & FE_pF'FE_pF' = 0 & \text{for } p=0, 1, \cdots, s-1 \\ \text{(b)} & FE_qF'FE_qF' = c_qFE_qF' & \text{for } q=s, \cdots, k \\ \text{(c)} & FE_iF'FE_jF' = 0 & \text{for } i\neq j \text{ and } i, j=0, 1, \cdots, k \\ \text{(d)} & r(FE_qF') = r(E_q) & \text{for } q=s, \cdots, k \end{array}$$

Since $E_i's$ are mutually orthogonal symmetric idempotents, there exists an orthogonal matrix P which transforms all E_i diagonal simultaneously, such as

$$(14) \qquad P'E_iP = \left(\begin{array}{ccccccc} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{array} \right) \begin{array}{l} \left.\right\} \beta_0+\cdots+\beta_{i-1} \\ \\ \left.\right\} \beta_i \\ \\ \end{array} \equiv \boldsymbol{e}_i, \qquad E_i = P\boldsymbol{e}_iP'$$

where $\beta_i = r(E_i) = m_i\alpha_i$. If we denote $FP=K$ the conditions (a), (b), (c), (d) may be expressed as,

$$(13') \quad \begin{array}{lll} \text{(a}'\text{)} & K\boldsymbol{e}_pK'K\boldsymbol{e}_pK' = 0 & \text{for } p=0, 1, \cdots, s-1 \\ \text{(b}'\text{)} & K\boldsymbol{e}_qK'K\boldsymbol{e}_qK' = c_qK\boldsymbol{e}_qK' & \text{for } q=s, \cdots, k \\ \text{(c}'\text{)} & K\boldsymbol{e}_iK'K\boldsymbol{e}_jK' = 0 & \text{for } i\neq j, \ i, j=0, 1, \cdots, k \\ \text{(d}'\text{)} & r(K\boldsymbol{e}_qK') = r(\boldsymbol{e}_q) & \text{for } q=s, \cdots, k \end{array}$$

From (a′), we have $K\boldsymbol{e}_p(K\boldsymbol{e}_p)'=0$ and $K\boldsymbol{e}_p=0$ for $p=0, 1, \cdots, s-1$. The latter shows that all elements of the first $\beta_0+\beta_1+\cdots+\beta_{s-1}$ columns of K are zero. Denote the matrix which consists of the remaining $\tilde{u}=\beta_s+\cdots+\beta_k$ columns of K by $\tilde{K}$, then we have

$$K=[0:\tilde{K}],\quad KK'=\tilde{K}\tilde{K}',\quad K'K=\left(\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & \tilde{K}'\tilde{K} \end{array}\right) \tag{15}$$

Since (d′) and (c′) hold, we have

$$\begin{aligned} r(\tilde{K}'\tilde{K}) &= r(\tilde{K}\tilde{K}')=r(KK')=r\big(K(\sum_{i=0}^{k}\boldsymbol{e}_i)K'\big) \\ &= \sum_{q=s}^{k} r(K\boldsymbol{e}_qK')=\sum_{q=s}^{k}\beta_q=\tilde{u} \end{aligned} \tag{16}$$

The $\tilde{u}\times\tilde{u}$ matrix $\tilde{K}'\tilde{K}$ is, therefore, non-singular. Using (13′) and (15) we have

$$\begin{aligned} \tilde{K}\tilde{K}'\tilde{K}\tilde{K}' &= KK'KK'=K(\sum_{i=0}^{k}\boldsymbol{e}_i)K'K(\sum_{i=0}^{k}\boldsymbol{e}_i)K' \\ &= \sum_{q=s}^{k} c_qK\boldsymbol{e}_qK' \\ &= \tilde{K}\begin{pmatrix} \left.\begin{matrix} c_s & & \\ & \ddots & \\ & & c_s \end{matrix}\right\}\beta_s & & 0 \\ & \ddots & \\ 0 & & \left.\begin{matrix} c_k & & \\ & \ddots & \\ & & c_k \end{matrix}\right\}\beta_k \end{pmatrix}\tilde{K}' \end{aligned} \tag{17}$$

Multiplying (17) by $(\tilde{K}'\tilde{K})^{-1}\tilde{K}'$ from the left and by $\tilde{K}(\tilde{K}'\tilde{K})^{-1}$ from the right we have

$$\tilde{K}'\tilde{K}=\begin{pmatrix} c_s & & & & & 0 \\ & \ddots & & & & \\ & & c_s & & & \\ & & & \ddots & & \\ & & & & c_k & \\ 0 & & & & & \ddots \\ & & & & & & c_k \end{pmatrix}$$

This implies

$$K'K=\sum_{q=s}^{k} c_q\boldsymbol{e}_q,$$

and, therefore,

$$F'F=\sum_{q=s}^{k}c_qE_q$$

Thus we have

$$F'F=\sum_{i=0}^{k}c_iE_i \qquad (c_i\geq 0)$$

The remaining part of the theorem may be proved as follows.

Since σ is a linear mapping of $\mathfrak{R}$ and $\sigma(\mathfrak{R})\sigma(\mathfrak{R})\subseteq\sigma(\mathfrak{R})$ holds under (9), $\sigma(\mathfrak{R})$ is an algebra. Moreover, since for any element A^* of $\sigma(\mathfrak{R})$, there exists an element $A\in\mathfrak{R}$ which satisfies the relation $A^*=FAF'$, $\sigma(\mathfrak{R})$ may be generated by symmetric matrices of order n. Thus, $\sigma(\mathfrak{R})$ is semi-simple.

As $\mathfrak{R}_i=E_i\mathfrak{R}E_i$, $\mathfrak{R}\mathfrak{R}_i\subseteq\mathfrak{R}_i$, and $\mathfrak{R}_i\mathfrak{R}\subseteq\mathfrak{R}_i$ for $i=1,\cdots,k$, it is easy to see that

$$\sigma(\mathfrak{R})\sigma(\mathfrak{R}_i)\subseteq\sigma(\mathfrak{R}_i), \qquad \sigma(\mathfrak{R}_i)\sigma(\mathfrak{R})\subseteq\sigma(\mathfrak{R}_i)$$

for $i=1,\cdots,k$. The image $\sigma(\mathfrak{R}_i)=F\mathfrak{R}_iF'$ of any two-sided ideal $\mathfrak{R}_i$ is also a two-sided ideal of the image algebra $\sigma(\mathfrak{R})$. If $c_i=0$ for some $i\,(i=1,\cdots,k)$, we have $\sigma(\mathfrak{R}_i)=FE_i\mathfrak{R}E_iF'=0$ as $FE_i=0$. In this case, the image of the two-sided ideal $\mathfrak{R}_i$ degenerates to a null ideal of $\sigma(\mathfrak{R})$. If $c_i>0$ for some $i\,(i=1,\cdots,k)$, it can be seen that the image algebra $\sigma(\mathfrak{R}_i)$ has no proper two-sided ideal and is isomorphic to $\mathfrak{R}_i$ by the mapping $\frac{1}{c_i}\sigma$. The principal idempotent of $\sigma(\mathfrak{R}_i)$ is

$$\tilde{E}_i=\frac{1}{c_i}FE_iF'$$

Thus the proof is complete.

Corollary 1. *A linear mapping σ of $\mathfrak{R}$ defined by F is similar if and only if*

(18) $$F'F=c_0E_0+cE \qquad (c>0,\ c_0\geq 0)$$

When $\mathfrak{R}$ contains the unit matrix I_m, we say that the semi-simple matrix algebra $\mathfrak{R}$ is *full-rank*. Under the terminology, we have the following corollary to the Theorem 1.

COROLLARY 2. *If $\mathfrak{R}$ is full-rank, the condition* (9) *for σ to be partially similar is reduced to*

$$(9') \qquad F'F = \sum_{i=1}^{k} c_i E_i \in \mathfrak{R} \qquad (c_i \geq 0),$$

and the condition (18) *for σ to be similar is reduced to*

$$(18') \qquad F'F = c\, I_m \qquad (c > 0)$$

3. Mappings with confounding

Let $\mathfrak{R}$ be the semi-simple $m \times m$ matrix algebra defined in the previous section and let $\mathfrak{B}$ be a semi-simple algebra generated by a finite number of real symmetric matrices of order n. Assume that $\mathfrak{B}$ is not full-rank and its principal idempotent is $\tilde{E}_b$.

Let F be an $n \times m$ real matrix and consider a mapping σ defined by F, i.e.,

$$(19) \qquad \sigma: \quad \mathfrak{R} \to \mathfrak{R}^* = F\mathfrak{R}F' = \{A^* : A^* = FAF',\, A \in \mathfrak{R}\}$$

DEFINITION 2. Linear mapping σ of $\mathfrak{R}$ is said to be

(i) *partially confounded* with $\mathfrak{B}$, if

$$(20) \qquad \tilde{\sigma}: \quad \mathfrak{R} \to \tilde{\mathfrak{R}} = (I_n - \tilde{E}_b)\mathfrak{R}^*(I_n - \tilde{E}_b)$$
$$= \{\tilde{A} : \tilde{A} = (I_n - \tilde{E}_b)FAF'(I_n - \tilde{E}_b),\, A \in \mathfrak{R}\}$$

is a proper partially similar mapping of $\mathfrak{R}$ and

$$\tilde{E}_b \mathfrak{R}^* \tilde{E}_b \neq 0,$$

(ii) *orthogonal* to $\mathfrak{B}$, if

$$\tilde{\sigma}: \quad \mathfrak{R} \to \tilde{\mathfrak{R}} = (I_n - \tilde{E}_b)\mathfrak{R}^*(I_n - \tilde{E}_b)$$

is a proper partially similar mapping of $\mathfrak{R}$ and

$$\tilde{E}_b \mathfrak{R}^* \tilde{E}_b = 0,$$

and

(iii) *confounded* with $\mathfrak{B}$, if

$$\tilde{E}_b \mathfrak{R}^* \tilde{E}_b = \mathfrak{R}^*$$

The following theorem is an immediate consequence of the definition and Theorem 1.

THEOREM 2. *The necessary and sufficient conditions for*

(i) *σ is partially confounded with $\mathfrak{B}$, are*

$$F'(I_n - \tilde{E}_b)F = \sum_{i=0}^{k} c_i E_i \tag{21}$$

$$(c_i \geqq 0 \text{ and } c_i > 0 \text{ for some } i \neq 0)$$

and

$$\tilde{E}_b F E \neq 0$$

(ii) *σ is orthogonal to $\mathfrak{B}$, are*

$$F'(I_n - \tilde{E}_b)F = \sum_{i=0}^{k} c_i E_i \tag{21$'$}$$

$$(c_i \geqq 0 \text{ and } c_i > 0 \text{ for some } i \neq 0)$$

and

$$\tilde{E}_b F E = 0$$

and

(iii) *σ is confounded with $\mathfrak{B}$, is*

$$(I_n - \tilde{E}_b)FE = 0 \tag{21$''$}$$

PROOF. Since $\tilde{E}_b \mathfrak{R}^* \tilde{E}_b = \tilde{E}_b F \mathfrak{R} F' \tilde{E}_b = 0$ or $\neq 0$ if and only if $\tilde{E}_b F E F' \tilde{E}_b = 0$ or $\neq 0$, and if and only if $\tilde{E}_b F E = 0$ or $\neq 0$, (i) and (ii) follow immediately from Theorem 1. If $\tilde{E}_b \mathfrak{R}^* \tilde{E}_b = \mathfrak{R}^*$, it follows $(I_n - \tilde{E}_b)FEF'(I_n - \tilde{E}_b) = 0$. Then, $(I_n - \tilde{E}_b)FE = 0$. Conversely, if $(I_n - \tilde{E}_b)FE = 0$, $\tilde{E}_b \mathfrak{R}^* \tilde{E}_b = \tilde{E}_b F E \mathfrak{R} E F' \tilde{E}_b = F \mathfrak{R} F' = \mathfrak{R}^*$.

4. Orthogonal compositions

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be semi-simple matrix algebras of order u and v, respectively. Assume that an $n \times u$ real matrix F_1 and an $n \times v$ real matrix F_2 define respectively a partially similar mapping of $\mathfrak{A}_1$ and that of $\mathfrak{A}_2$. Assume, further, two image matrix algebras $\mathfrak{A}_1^*=\sigma_1(\mathfrak{A}_1)$ and $\mathfrak{A}_2^*=\sigma_2(\mathfrak{A}_2)$ have a two-sided ideal $\mathfrak{B}$ in common.

DEFINITION 3. If two difference algebras $\mathfrak{A}_1^*-\mathfrak{B}$ and $\mathfrak{A}_2^*-\mathfrak{B}$ are mutually orthogonal, we say $\mathfrak{A}_1^*$ *is orthogonal to* $\mathfrak{A}_2^*$ *modulo* $\mathfrak{B}$ and vice versa. We say that $\mathfrak{R}=\mathfrak{A}_1^*\cup\mathfrak{A}_2^*$, the smallest algebra containing $\mathfrak{A}_1^*$ and $\mathfrak{A}_2^*$, is the algebra composed by the *orthogonal composition of* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *modulo* $\mathfrak{B}$.

In order to express more concretely, assume that the decompositions of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ to their minimum two-sided ideals be

$$\mathfrak{A}_1=\mathfrak{A}_{11}\oplus\mathfrak{A}_{12}\oplus\cdots\oplus\mathfrak{A}_{1k}, \tag{22}$$

$$\mathfrak{A}_2=\mathfrak{A}_{21}\oplus\mathfrak{A}_{22}\oplus\cdots\oplus\mathfrak{A}_{2l}, \tag{23}$$

and the corresponding decompositions of the principal idempotents be

$$E_1=E_{11}+E_{12}+\cdots+E_{1k}, \tag{24}$$

$$E_2=E_{21}+E_{22}+\cdots+E_{2l}, \tag{25}$$

and let $E_{10}=I_u-E_1$, and $E_{20}=I_v-E_2$.

Assume that

$$F_1'F_1=c_{10}E_{10}+\sum_{i=1}^{s}c_{1i}E_{1i}, \qquad c_{10}\geq 0,\ c_{1i}>0 \quad \text{for} \quad i=1,\cdots,s\leq k \tag{26}$$

$$F_2'F_2=c_{20}E_{20}+\sum_{j=1}^{t}c_{2j}E_{2j} \qquad c_{20}\geq 0,\ c_{2j}>0 \quad \text{for} \quad j=1,\cdots,t\leq l \tag{27}$$

and

$$\sigma_1:\ \mathfrak{A}_1\to\mathfrak{A}_1^*=F_1\mathfrak{A}_1F_1'=\sigma_1(\mathfrak{A}_1) \tag{28}$$

$$\sigma_2:\ \mathfrak{A}_2\to\mathfrak{A}_2^*=F_2\mathfrak{A}_2F_2'=\sigma_2(\mathfrak{A}_2) \tag{29}$$

Since σ_1 and σ_2 are partially similar mappings of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, the decompositions of $\mathfrak{A}_1^*$ and $\mathfrak{A}_2^*$ to their minimum two-sided ideals are

$$\mathfrak{A}_1^*=\sigma_1(\mathfrak{A}_1)=\sigma_1(\mathfrak{A}_{11})\oplus\sigma_1(\mathfrak{A}_{12})\oplus\cdots\oplus\sigma_1(\mathfrak{A}_{1s}) \tag{30}$$

$$\mathfrak{A}_2^*=\sigma_2(\mathfrak{A}_2)=\sigma_2(\mathfrak{A}_{21})\oplus\sigma_2(\mathfrak{A}_{22})\oplus\cdots\oplus\sigma_2(\mathfrak{A}_{2t}) \tag{31}$$

respectively. The corresponding decompositions of the idempotents are

$$\tilde{E}_1 = \tilde{E}_{11} + \tilde{E}_{12} + \cdots + \tilde{E}_{1s} \tag{32}$$

$$\tilde{E}_2 = \tilde{E}_{21} + \tilde{E}_{22} + \cdots + \tilde{E}_{2t} \tag{33}$$

where $\tilde{E}_{1i} = \frac{1}{c_{1i}} F_1 E_{1i} F_1'$ and $\tilde{E}_{2j} = \frac{1}{c_{2j}} F_2 E_{2j} F_2'$ for $i=1, \cdots, s$ and $j=1, \cdots, t$.

If $\mathfrak{A}_1^*$ and $\mathfrak{A}_2^*$ have the same two-sided ideal $\mathfrak{B}$, the uniqueness of the decomposition shows that it can be expressed, without loss of generality, as

$$\begin{aligned} \mathfrak{B} &= \sigma_1(\mathfrak{A}_{11}) \oplus \cdots \oplus \sigma_1(\mathfrak{A}_{1p}) \\ &= \sigma_2(\mathfrak{A}_{21}) \oplus \cdots \oplus \sigma_2(\mathfrak{A}_{2p}) \qquad 0 \leq p \leq \min(s, t) \end{aligned} \tag{34}$$

The principal idempotent of the ideal is,

$$\begin{aligned} \tilde{E}_b &= \sum_{i=1}^{p} \tilde{E}_{1i} = \sum_{i=1}^{p} \frac{1}{c_{1i}} F_1 E_{1i} F_1' \\ &= \sum_{i=1}^{p} \tilde{E}_{2i} = \sum_{i=1}^{p} \frac{1}{c_{2i}} F_2 E_{2i} F_2' \end{aligned} \tag{35}$$

Let $E_{1b} = E_{11} + \cdots + E_{1p}$ and $E_{2b} = E_{21} + \cdots + E_{2p}$, then we have the following theorem.

THEOREM 3. *Let σ_1 and σ_2 defined by* (28) *and* (29) *be the partially similar mappings of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, and assume that the image algebras $\mathfrak{A}_1^*$ and $\mathfrak{A}_2^*$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have a common two-sided ideal $\mathfrak{B}$ defined by* (34). *The image algebras $\mathfrak{A}_1^*$ and $\mathfrak{A}_2^*$ are mutually orthogonal modulo $\mathfrak{B}$ and $\mathfrak{A}_1^* \cup \mathfrak{A}_2^*$ is an algebra composed by the orthogonal composition of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ if and only if*

$$(E_1 - E_{1b}) F_1' F_2 (E_2 - E_{2b}) = 0 \tag{36}$$

If $\mathfrak{A}_1^*$ and $\mathfrak{A}_2^*$ have no common two-sided ideal, i.e., $\mathfrak{B}$ is a null algebra, the condition may be simplified as

$$E_1 F_1' F_2 E_2 = 0 \tag{36'}$$

and, moreover, if both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are full-rank matrix algebras, the condition may be simplified further as

$$F_1' F_2 = 0 \tag{36''}$$

When the condition (36) *holds, the direct decomposition of* $\mathfrak{R}=\mathfrak{A}_1^* \cup \mathfrak{A}_2^*$ *into minimum two-sided ideals is*

$$
(37) \qquad \begin{aligned} \mathfrak{R} = {} & \sigma_1(\mathfrak{A}_{11}) \oplus \cdots \oplus \sigma_1(\mathfrak{A}_{1p}) \\ & + \sigma_1(\mathfrak{A}_{1p+1}) \oplus \cdots \oplus \sigma_1(\mathfrak{A}_{1s}) \\ & + \sigma_2(\mathfrak{A}_{2p+1}) \oplus \cdots \oplus \sigma_2(\mathfrak{A}_{2t}), \end{aligned}
$$

and the corresponding decomposition of the principal idempotent of $\mathfrak{R}$ *is*

$$
(38) \qquad \tilde{E}_R = \tilde{E}_{11} + \cdots + \tilde{E}_{1p} + \tilde{E}_{1p+1} + \cdots + \tilde{E}_{1s} + \tilde{E}_{2p+1} + \cdots + \tilde{E}_{2t}
$$

or

$$
= \sum_{i=1}^{s} \frac{1}{c_{1i}} F_1 E_{1i} F_1' + \sum_{j=p+1}^{t} \frac{1}{c_{2j}} F_2 E_{2j} F_2'
$$

PROOF. If

$$
(39) \qquad (\mathfrak{A}_1^* - \mathfrak{B})(\mathfrak{A}_2^* - \mathfrak{B}) = (\mathfrak{A}_2^* - \mathfrak{B})(\mathfrak{A}_1^* - \mathfrak{B}) = 0 \quad \mathrm{mod}\ \mathfrak{B}
$$

it follows

$$
(40) \qquad \Big(\sum_{i=p+1}^{s} \frac{1}{c_{1i}} F_1 E_{1i} F_1' \Big) \Big(\sum_{j=p+1}^{t} \frac{1}{c_{2j}} F_2 E_{2j} F_2' \Big) = 0
$$

Multiplying (40) by F_1' from the left and by F_2 from the right, we have,

$$
(41) \qquad \Big(\sum_{i=p+1}^{s} E_{1i} \Big) F_1' F_2 \Big(\sum_{j=p+1}^{t} E_{2j} \Big) = 0,
$$

or

$$
(E_1 - E_{1b}) F_1' F_2 (E_2 - E_{2b}) = 0
$$

Conversely, if (36) holds, then (41) follows. Multiplying (41) by E_{1i} from the left and by E_{2j} from the right, we have

$$
(42) \qquad E_{1i} F_1' F_2 E_{2j} = 0
$$

for any $i=p+1, \cdots, s$ and $j=p+1, \cdots,$ t. Thus we have

$$
(43) \qquad \frac{1}{c_{1i}} F_1 E_{1i} F_1' \cdot \frac{1}{c_{2j}} F_2 E_{2j} F_2' = 0
$$

for any $i=p+1, \cdots, s$ and $j=p+1, \cdots, t$. This implies (40) and, therefore, (39).

The rest of the theorem is obvious and the proof is omitted.

5. Specific features of the relationship algebras

We shall define, after A. T. James [5], a *basic relationship* and a basic relationship matrix among a set of objects.

Consider a set of objects and assume a basic relationship R between objects as a set of ordered pairs (i, j) of them. If the ordered pair (i, j) of objects belongs to R, we say that i and j are in the relation R. A basic relationship R among a set of n objects can be expressed as an $n\times n$ basic *relationship matrix* of 0's and 1's:

$$(44) \qquad R=\|r_{ij}\|, \qquad r_{ij}=\begin{cases}1 & \text{if } i \text{ is related to } j \text{ by the relationship } R,\\ 0 & \text{otherwise.}\end{cases}$$

In this paper we shall assume further, as James has done, that each of the basic relationships is symmetrical for any pair of objects (i, j). Thus all of the basic relationship matrices are assumed to be symmetric matrices of 0's and 1's.

Under the operations of matrix multiplication, matrix addition and scalar multiplication, a family of basic relationship matrices generates a semi-simple matrix algebra, which we call the relationship algebra defined among a set of the objects.

If a basic relationship R among a set of n objects belongs to a family of basic relationships, it is natural to assume that the family contains another relationship R^c, the negation of R, too. The negation of R can be expressed by the matrix of 0's and 1's, as

$$(45) \qquad R^c=G-R$$

where G is an $n\times n$ matrix, all elements of which are unity. Hence, we may naturally assume that a family of basic relationship matrices as well as a relationship algebra contain the *universal relationship* matrix G.

Another special basic relationship, which James has assumed to be included in a family of basic relationships is the *identity relationship* of each object itself. The assumption of the existence of identity relationship implies that the differentiation from one object to another in the set is always possible. The assumption, however, seems to be too restrictive and we shall assume that the family of basic relationship matrices as well as the relationship algebra contain not necessarily the unit matrix.

In this paper, the symbols G and I are used exclusively for the universal relationship matrix and the identity relationship matrix, respectively. An algebra generated by the linear closure of a set of matrices $A, B, \cdots, M$, is denoted by $[A, B, \cdots, M]$.

Although a relationship algebra always contains $[G]$ as its subalgebra, it contains not always $[G]$ as its two-sided ideal. Those algebras defined for the association schemes and the plot relationship algebras of the standard experimental designs eventually have $[G]$ as their one-dimensional two-sided ideal. In this connection, we have the following theorem.

THEOREM 4. *A relationship algebra contains $[G]$ as its two-sided ideal if and only if row (or column) sums of each basic relationship matrix are constant. In other words, for each basic relationship R, the number of objects related by R to a fixed object is independent of the object.*

Proof of the theorem is easy and will be ommited.

An algebra generated by G only, i.e., $[G]$, is the most *trivial* algebra defined among a set of objects; the only basic relationship defined among them is that they belong to the same set of objects. An algebra generated by I_m and G_m, i.e., $[I_m, G_m]$ is the most important *primitive* relationship algebra defined among a set of m objects, say $\tau_1, \cdots, \tau_m$. This can uniquely be decomposed into direct sum of two-sided ideals as

$$[I_m, G_m] = [G_m] \oplus \left[I_m - \frac{1}{m} G_m\right] \tag{46}$$

The corresponding decomposition of the principal idempotent into the sum of component principal idempotents is

$$I_m = \frac{1}{m} G_m + \left(I_m - \frac{1}{m} G_m\right) \tag{47}$$

The algebra determines uniquely the decomposition of sum of squares as

$$\boldsymbol{\tau}' I_m \boldsymbol{\tau} = \frac{1}{m} \boldsymbol{\tau}' G_m \boldsymbol{\tau} + \boldsymbol{\tau}' \left(I_m - \frac{1}{m} G_m\right) \boldsymbol{\tau} \tag{48}$$

or

$$\sum_{i=1}^{m} \tau_i^2 = m\bar{\tau}^2 + \sum_{i=1}^{m} (\tau_i - \bar{\tau})^2 \tag{49}$$

where $\boldsymbol{\tau}' = (\tau_1, \tau_2, \cdots, \tau_m)$ and $\bar{\tau} = \frac{1}{m} \sum_{i=1}^{m} \tau_i$.

Thus we are interested in a relationship algebra containing $[G]$ as its two-sided ideals. The (partially) similar mapping of such an algebra which maps $[G]$ to $[G]$, and the orthogonal composition modulo $[G]$ of two or more algebras of such a type are of interest.

6. G-preserving mappings and G-orthogonal compositions

In this section we shall deal with some special types of partially similar mappings treated in section 2. A special type of orthogonal compositions treated in section 4 will also be discussed.

Let $\mathfrak{R}$ be a relationship algebra defined over a set of m objects, and F be an $n \times m$ real matrix which defines a partially similar mapping σ of $\mathfrak{R}$. Assume, further, $\mathfrak{R}$ has $[G_m]$ as its two-sided ideal.

DEFINITION 4. A partially similar mapping σ of $\mathfrak{R}$ defined by F is said to be

(i) G-*preserving*, if the relation

$$\sigma([G_m]) = [G_n] \tag{50}$$

holds, where $[G_n]$ is an ideal of the image algebra $\mathfrak{R}^* = \sigma(\mathfrak{R})$, and

(ii) G-*eliminating*, if the relation

$$\sigma([G_m]) = 0 \tag{51}$$

holds.

In connection with the definition, we have the following theorem.

THEOREM 5.

(i) *If and only if*

$$nFG_m = mG_nF \neq 0 \tag{52}$$

holds, a (partially) similar mapping σ defined by F is G-preserving. The condition (52) *may be interpreted as that each row sum as well as the column sum of the elements of F is constant.*

(ii) *If and only if*

$$FG_m = 0 \tag{53}$$

holds, a partially similar mapping σ defined by F is G-eliminating. The *condition* (53) *may be interpreted as that each row sum of the elements of* F *is gero.*

PROOF.

(i) If σ is G-preserving (partially) similar, there exists a positive constant g, such as,

$$\left(\frac{1}{gm}FG_mF'\right)^2=\frac{1}{gm}FG_mF'=\frac{1}{n}G_n \tag{54}$$

Multiplying by mnF from the right, we obtain

$$nFG_m=mG_nF(\neq 0) \tag{55}$$

In this case, if we denote the element of F in the i-th row and the k-th column as f_{ij}, and put $f_{i\cdot}=\sum_{j=1}^{m}f_{ij}$, $f_{\cdot j}=\sum_{i=1}^{n}f_{ij}$, we have

$$n\begin{pmatrix} f_{1\cdot} & f_{1\cdot} & \cdots & f_{1\cdot} \\ f_{2\cdot} & f_{2\cdot} & \cdots & f_{2\cdot} \\ \vdots & \vdots & & \vdots \\ f_{u\cdot} & f_{n\cdot} & \cdots & f_{n\cdot} \end{pmatrix}=m\begin{pmatrix} f_{\cdot 1} & f_{\cdot 2} & \cdots & f_{\cdot m} \\ f_{\cdot 1} & f_{\cdot 2} & \cdots & f_{\cdot m} \\ \vdots & \vdots & & \vdots \\ f_{\cdot 1} & f_{\cdot 2} & \cdots & f_{\cdot m} \end{pmatrix} \tag{56}$$

Thus we have $f_{1\cdot}=f_{2\cdot}=\cdots=f_{n\cdot}$ $(=a$, say) and $f_{\cdot 1}=f_{\cdot 2}=\cdots=f_{\cdot m}(=b=\frac{n}{m}a$, say).

Conversely, if (55) holds for a (partially) similar mapping σ, we have

$$\frac{1}{m}FG_mF'=\frac{a^2}{m}G_n=ab\cdot\frac{1}{n}G_n \qquad (ab>0) \tag{57}$$

and, therefore,

$$\sigma([G_m])=[G_n]$$

(ii) If σ is G-eliminating partially similar, we have

$$\frac{1}{m}FG_mF'=0 \tag{58}$$

The condition is equivalent to

$$FG_m=0$$

The converse is obvious.

DEFINITION 5. When a (partially) similar mapping σ of a relationship algebra $\mathfrak{R}$ is G-preserving, the image algebra $\mathfrak{R}^*=\sigma(\mathfrak{R})$ is said to be *the relationship algebra composed by G-preserving (partially) similar mapping* of $\mathfrak{R}$. The relationship algebra $\sigma(\mathfrak{R})^{\cup}[I_n, G_n]=\sigma(\mathfrak{R})^{\cup}[I_n]$ is said to be *the full-rank relationship algebra composed by G-preserving (partially) similar mapping* of $\mathfrak{R}$.

Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be relationship algebras defined respectively over the set of s and t objects. Assume that both $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are full-rank and they have $[G_s]$ and $[G_t]$ as their two-sided ideals, respectively. Assume that a $v\times s$ matrix $\boldsymbol{\Phi}_1$ and a $v\times t$ matrix $\boldsymbol{\Phi}_2$ define G-preserving (partially) similar mapping σ_1 and σ_2 of $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

DEFINITION 6. When σ_1 and σ_2 give an orthogonal composition of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ modulo $[G_v]$, we say σ_1 and σ_2 are G-*orthogonal. The algebra* $\sigma_1(\mathfrak{R}_1)^{\cup}\sigma_2(\mathfrak{R}_2)$ is said to be *the relationship algebra composed by* G-*orthogonal composition* of $\mathfrak{R}_1$ and $\mathfrak{R}_2$. The algebra $\sigma_1(\mathfrak{R}_1)^{\cup}\sigma_2(\mathfrak{R}_2)^{\cup}[I_v, G_v]=\sigma_1(\mathfrak{R}_1)^{\cup}\sigma_2(\mathfrak{R}_2)^{\cup}[I_v]$ is said to be *the full-rank relationship algebra composed by* G-*orthogonal composition* of $\mathfrak{R}_1$ and $\mathfrak{R}_2$.

The following theorem is an immediate consequence of Theorems 3 and 5.

THEOREM 6. *Two* G-*preserving (partially) similar mappings* σ_1 *and* σ_2 *of* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *defined respectively by* $\boldsymbol{\Phi}_1$ *and* $\boldsymbol{\Phi}_2$ *give a* G-*orthogonal composition of* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *if and only if*

$$\boldsymbol{\Phi}_1'\boldsymbol{\Phi}_2 = c\,G_{s\times t}, \qquad c=\frac{v}{st}c_1c_2, \tag{59}$$

where $G_{s\times t}$ *is an* $s\times t$ *matrix whose elements are all unity.* c_1 *and* c_2 *are the row sums of the elements of* $\boldsymbol{\Phi}_1$ *and* $\boldsymbol{\Phi}_2$, *respectively.*

7. G-preserving composition of parameter algebras

Consider a vector of s parameters $\boldsymbol{\tau}'=(\tau_1, \tau_1, \cdots, \tau_s)$ and assume that among those parameters a relationship algebra $\mathfrak{A}$ has been composed. Assume, further, $\mathfrak{A}$ is full-rank and has $[G_s]$ as its one dimensional two-sided ideals.

Let the direct decomposition of $\mathfrak{A}$ to its minimum two-sided ideals be

$$\mathfrak{A}=\mathfrak{A}_0\oplus\mathfrak{A}_1\oplus\cdots\oplus\mathfrak{A}_k \tag{60}$$

where $\mathfrak{A}_0=[G_s]$. Let the corresponding decomposition of the unit matrix to mutually orthogonal principal idempotents of the ideals be

(61) $$I_s = E_0 + E_1 + \cdots + E_k$$

where $E_0 = s^{-1}G_s$.

As the decomposition (61) is unique apart from the order of the component idempotents, the relationship algebra determines uniquely the decomposition of parameter sum of squares into its mutually orthogonal components, i.e.,

(62) $$\sum_{i=1}^{s} \tau_i^2 = \boldsymbol{\tau}' I_s \boldsymbol{\tau} = \boldsymbol{\tau}' E_0 \boldsymbol{\tau} + \boldsymbol{\tau}' E_1 \boldsymbol{\tau} + \cdots + \boldsymbol{\tau}' E_k \boldsymbol{\tau}$$

Let $\boldsymbol{\Phi}$ be a $v \times s$ real matrix and assume that $\boldsymbol{\Phi}$ defines a G-preserving (partially) similar mapping σ of $\mathfrak{A}$,

(63) $$\sigma: \quad \mathfrak{A} \ni A \rightarrow A^* = \boldsymbol{\Phi} A \boldsymbol{\Phi}',$$

satisfying

(64) $$\boldsymbol{\Phi}'\boldsymbol{\Phi} = \sum_{i=0}^{k} c_i E_i \qquad (c_0 > 0,\ c_i \geqq 0 \quad \text{for} \quad i = 1, \cdots, k)$$

Since the matrix $\boldsymbol{\Phi}$ is a linear mapping from V_s to V_v, $\boldsymbol{\tau}$ is mapped as

(65) $$\boldsymbol{\Phi}: \quad \boldsymbol{\tau} \rightarrow \boldsymbol{\tau}^* = \boldsymbol{\Phi}\boldsymbol{\tau}$$

DEFINITION 7. A v-dimensional parameter vector $\boldsymbol{\mu}' = (\mu_1, \mu_2, \cdots, \mu_v)$ is said to be *a parameter vector composed of* $\boldsymbol{\tau}$ *by* G-*preserving composition* if it consists of the two components: a component $\boldsymbol{\tau}^*$, the image of $\boldsymbol{\tau}$; and a component $\boldsymbol{\delta}$ which may or may not be 0 and is orthogonal to $\boldsymbol{\tau}^*$ for any $\boldsymbol{\tau}$, i.e.,

(66) $$\boldsymbol{\mu} = \boldsymbol{\Phi}\boldsymbol{\tau} + \boldsymbol{\delta}$$

where $\boldsymbol{\Phi}'\boldsymbol{\delta} = 0$.

Among the v parameters, it is natural to assume that there exists the primitive relationship algebra $[I_v\ G_v]$. Thus, the relationship algebra among the v parameters is composed of $\mathfrak{A}^*$, the image of $\mathfrak{A}$, and $[I_v, G_v]$, i.e.,

$$\mathfrak{R} = \mathfrak{A}^{*\cup}[I_v, G_v] = \mathfrak{A}^{*\cup}[I_v]$$

$\mathfrak{R}$ is the full-rank relationship algebra composed of $\mathfrak{A}$ by G-preserving composition.

We may assume, without loss of generality, for (64) that $c_i > 0$ for $i = 1, \cdots, m$ and $c_j = 0$ for $j = m+1, \cdots, k\,(1 \leq m \leq k)$ except a trivial case of all $c_i = 0$ for $i = 1, \cdots, k$. In this case, Theorem 1 shows that $\mathfrak{R}$ is decomposed into its minimum two-sided ideals as

$$\mathfrak{R} = \mathfrak{A}^{*\cup}[I_v] = \mathfrak{A}_0^* \oplus \mathfrak{A}_1^* \oplus \cdots \oplus \mathfrak{A}_m^* \oplus \mathfrak{A}_e^*$$

and the corresponding decomposition of the principal idempotent is

$$I_v = \frac{1}{c_0}\Phi E_0 \Phi' + \frac{1}{c_1}\Phi E_1 \Phi' + \cdots + \frac{1}{c_m}\Phi E_m \Phi' + E_e^* \tag{67}$$

where

$$E_e^* = I_v - \sum_{i=0}^{m} \frac{1}{c_i}\Phi E_i \Phi', \qquad \frac{1}{c_0}\Phi E_0 \Phi' = \frac{1}{v} G_v$$

The unique decomposition of parameter sum of squares for $\boldsymbol{\mu}$ into mutually orthogonal components is, after some calculation,

$$\begin{aligned} \sum_{\alpha=1}^{v} \mu_\alpha^2 &= \boldsymbol{\mu}' I_v \boldsymbol{\mu} \\ &= c_0 \boldsymbol{\tau}' E_0 \boldsymbol{\tau} + c_1 \boldsymbol{\tau}' E_1 \boldsymbol{\tau} + \cdots + c_m \boldsymbol{\tau}' E_m \boldsymbol{\tau} + \boldsymbol{\delta}'\boldsymbol{\delta} \end{aligned} \tag{68}$$

If, in particular, Φ defines a similar mapping of $\mathfrak{A}$,

$$\sum_{\alpha=1}^{v} \mu_\alpha^2 = \boldsymbol{\mu}' I_v \boldsymbol{\mu} = c(\boldsymbol{\tau}' E_0 \boldsymbol{\tau} + \cdots + \boldsymbol{\tau}' E_k \boldsymbol{\tau}) + \boldsymbol{\delta}'\boldsymbol{\delta} \tag{69}$$

It can be seen that a (partially) similar mapping of a parameter relationship algebra will give a sort of weighted faithful mapping of each original parameter sum of squares into sum of squares of the composed parameters.

A simple example of G-preserving composition is given below.

Example 1. $[I, G] \rightarrow$ *Group divisible association algebra*

Suppose that a primitive relationship algebra

$$\mathfrak{A} = [I_m, G_m] = \left[\frac{1}{m} G_m\right] \oplus \left[I_m - \frac{1}{m} G_m\right] \tag{70}$$

is defined over a parameter vector $\boldsymbol{\alpha}' = (\alpha_1, \cdots, \alpha_m)$.

The algebra determines a unique decomposition of the unit matrix;

$$I_m = \frac{1}{m} G_m + \left(I_m - \frac{1}{m} G_m\right)$$

and corresponding decomposition of the parameter sum of squares,

(71) $$\boldsymbol{\alpha}' I_m \boldsymbol{\alpha} = \frac{1}{m} \boldsymbol{\alpha}' G_m \boldsymbol{\alpha} + \boldsymbol{\alpha}' \left(I_m - \frac{1}{m} G_m\right) \boldsymbol{\alpha}$$

or $$\sum_{i=1}^{m} \alpha_i^2 = m\bar{\alpha}^2 + \sum_{i=1}^{m} (\alpha_i - \bar{\alpha})^2$$

where $$\bar{\alpha} = \frac{1}{m} \sum_{i=1}^{m} \alpha_i$$

Consider a linear mapping σ of $\mathfrak{A}$ defined by

(72) $$\boldsymbol{\Phi} = I_m \otimes \boldsymbol{j}_n \qquad (\boldsymbol{j}_n' = (1, 1, \cdots, 1))$$

Since $\boldsymbol{\Phi} \boldsymbol{j}_m = \boldsymbol{j}_{mn}$, $\boldsymbol{\Phi}' \boldsymbol{j}_{mn} = n \boldsymbol{j}_m$ and $\boldsymbol{\Phi}' \boldsymbol{\Phi} = n I_m$ hold, σ is G-preserving similar.

It can be verified that the composed full-rank algebra $\sigma(\mathfrak{A})^{\cup}[I_{mn}]$ is a group divisible association algebra and is decomposed as

(73) $$\sigma(\mathfrak{A})^{\cup}[I_{mn}] = \boldsymbol{\Phi} \left[\frac{1}{m} G_m\right] \boldsymbol{\Phi}' \oplus \boldsymbol{\Phi} \left[I_m - \frac{1}{m} G_m\right] \boldsymbol{\Phi}' \oplus \tilde{\mathfrak{A}}_e$$

The corresponding decomposition of the unit matrix is

(74) $$I_{mn} = \frac{1}{mn} G_m \otimes G_n + \frac{1}{n} \left(I_m - \frac{1}{m} G_m\right) \otimes G_n + I_m \otimes \left(I_n - \frac{1}{n} G_n\right)$$

Let the composed parameter vector be

$$\boldsymbol{\tau} = \boldsymbol{\Phi} \boldsymbol{\alpha} + \boldsymbol{\delta}, \qquad \boldsymbol{\Phi}' \boldsymbol{\delta} = 0,$$

then we have, after (68),

(75) $$\sum_{j=1}^{mn} \tau_j^2 = mn\bar{\alpha}^2 + n \sum_{i=1}^{m} (\alpha_i - \bar{\alpha})^2 + \boldsymbol{\delta}' \boldsymbol{\delta}$$

If we denote $\{(i-1)n+j\}$-th element of $\boldsymbol{\delta}$ in a usual way as α_{ij}, the restriction for $\boldsymbol{\delta}$ may be expressed as

(76) $$\sum_{j=1}^{n} \delta_{ij} = 0 \qquad \text{for all} \quad i.$$

Each component of the sum of squares of working parameters $(\tau_1, \cdots, \tau_{mn})$ corresponds respectively to the sum of squares for general mean $\bar{\alpha}$ of the primitive parameters, to the sum of squares of α_i about the mean, and to the residual sum of squares. Those components are sums of squares for general mean, for between groups and for within groups, respectively.

Repeated application of such G-preserving similar compositions will give a series of nested type association algebras. Another series of G-preserving partially similar compositions will give a series of triangular type association algebras. Details of those and their applications will be seen in the forthcomming paper.

8. G-orthogonal composition of parameter algebras

Consider two vectors of s and t parameters $\boldsymbol{\alpha}' = (\alpha_1, \cdots, \alpha_s)$ and $\boldsymbol{\beta}' = (\beta_1, \cdots, \beta_t)$ and assume that two relationship algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have been composed among the sets of parameters, respectively. Assume further, both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are full-rank and have $[G_s]$ and $[G_t]$ as their two-sided ideals, respectively.

Let the direct decompositions of those algebras be

(77)
$$\mathfrak{A}_1 = \mathfrak{A}_{10} \oplus \mathfrak{A}_{11} \oplus \cdots \oplus \mathfrak{A}_{1k}, \qquad \mathfrak{A}_{10} = [G_s]$$
$$\mathfrak{A}_2 = \mathfrak{A}_{20} \oplus \mathfrak{A}_{21} \oplus \cdots \oplus \mathfrak{A}_{2l}, \qquad \mathfrak{A}_{20} = [G_t]$$

and the corresponding decompositions of the unit matrices be

(78)
$$I_s = E_{10} + E_{11} + \cdots + E_{1k}, \qquad E_{10} = \frac{1}{s} G_s$$
$$I_t = E_{20} + E_{21} + \cdots + E_{2l}, \qquad E_{20} = \frac{1}{t} G_t$$

respectively.

Those relationship algebras determine uniquely the decompositions of parameter sums of squares into mutually orthogonal components, i.e.,

(79)
$$\sum_{i=1}^{s} \alpha_i^2 = \boldsymbol{\alpha}' I_s \boldsymbol{\alpha} = \boldsymbol{\alpha}' E_{10} \boldsymbol{\alpha} + \boldsymbol{\alpha}' E_{11} \boldsymbol{\alpha} + \cdots + \boldsymbol{\alpha}' E_{1k} \boldsymbol{\alpha}$$
$$\sum_{j=1}^{t} \beta_j^2 = \boldsymbol{\beta}' I_t \boldsymbol{\beta} = \boldsymbol{\beta}' E_{20} \boldsymbol{\beta} + \boldsymbol{\beta}' E_{21} \boldsymbol{\beta} + \cdots + \boldsymbol{\beta}' E_{2l} \boldsymbol{\beta}$$

Let a $v \times s$ matrix $\boldsymbol{\Phi}_1$ and a $v \times t$ matrix $\boldsymbol{\Phi}_2$ define respectively G-preserving (partially) similar mappings σ_1 and σ_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and assume that σ_1 and σ_2 give G-orthogonal composition of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. We can assume, without loss of generality, that

$$\text{(80)} \qquad \boldsymbol{\Phi}_1'\boldsymbol{\Phi}_1 = \sum_{i=0}^{p} c_{1i} E_{1i}, \qquad c_{1i} > 0 \quad \text{for} \quad i = 0, 1, \cdots, p \leq k,$$

$$\boldsymbol{\Phi}_2'\boldsymbol{\Phi}_2 = \sum_{j=0}^{q} c_{2j} E_{2j}, \qquad c_{2j} > 0 \quad \text{for} \quad j = 0, 1, \cdots, q \leq l,$$

$$\boldsymbol{\Phi}_1'\boldsymbol{\Phi}_2 = c\, G_{s\times t}$$

The parameter vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are mapped respectively by $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ as,

$$\text{(81)} \qquad \boldsymbol{\Phi}_1: \quad \boldsymbol{\alpha} \to \boldsymbol{\alpha}^* = \boldsymbol{\Phi}_1\boldsymbol{\alpha}$$

$$\boldsymbol{\Phi}_2: \quad \boldsymbol{\beta} \to \boldsymbol{\beta}^* = \boldsymbol{\Phi}_2\boldsymbol{\beta}$$

DEFINITION 8. A v-dimensional parameter vector $\boldsymbol{\mu}' = (\mu_1, \mu_2, \cdots, \mu_v)$ is said to be *a parameter vector composed of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *by* G-*orthogonal composition* if it consists of the three components: a component $\boldsymbol{\alpha}^*$, the image of $\boldsymbol{\alpha}$; a component $\boldsymbol{\beta}^*$, the image of $\boldsymbol{\beta}$; and a component $\boldsymbol{\delta}$ which may or may not be 0 and is orthogonal to both $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$ for any $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, i.e.,

$$\text{(82)} \qquad \boldsymbol{\mu} = \boldsymbol{\Phi}_1\boldsymbol{\alpha} + \boldsymbol{\Phi}_2\boldsymbol{\beta} + \boldsymbol{\delta}$$

where $\boldsymbol{\Phi}_1'\boldsymbol{\delta} = 0$ and $\boldsymbol{\Phi}_2'\boldsymbol{\delta} = 0$.

The full-rank relationship algebra $\tilde{\mathfrak{A}}$ composed of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by σ_1 and σ_2 is decomposed into direct sum of minimum two-sided ideals as

$$\text{(83)} \qquad \tilde{\mathfrak{A}} = \sigma_1(\mathfrak{A}_1)^{\cup}\sigma_2(\mathfrak{A}_2)^{\cup}[I_v]$$

$$= [G_v] \oplus \sigma_1(\mathfrak{A}_{11}) \oplus \sigma_1(\mathfrak{A}_{12}) \oplus \cdots \oplus \sigma_1(\mathfrak{A}_{1p})$$

$$\oplus \sigma_2(\mathfrak{A}_{21}) \oplus \sigma_2(\mathfrak{A}_{22}) \oplus \cdots \oplus \sigma_2(\mathfrak{A}_{2q}) \oplus \mathfrak{A}_e$$

and the corresponding decomposition of the unit matrix is

$$\text{(84)} \qquad I_v = \tilde{E}_0 + \tilde{E}_{11} + \cdots + \tilde{E}_{1p} + \tilde{E}_{21} + \cdots + \tilde{E}_{2q} + \tilde{E}_e$$

where $$\tilde{E}_0 = \frac{1}{v} G_v$$

$$\tilde{E}_{1i} = \frac{1}{c_{1i}} \boldsymbol{\Phi}_1 E_{1i} \boldsymbol{\Phi}_1', \qquad i = 1, 2, \cdots, p$$

$$\tilde{E}_{2j} = \frac{1}{c_{2j}} \boldsymbol{\Phi}_2 E_{2j} \boldsymbol{\Phi}_2', \qquad j = 1, 2, \cdots, q$$

$$\tilde{E}_e = I_v - \frac{1}{v} G_v - \sum_{i=1}^{p} \frac{1}{c_{1i}} \boldsymbol{\Phi}_1 E_{1i} \boldsymbol{\Phi}_1' - \sum_{j=1}^{q} \frac{1}{c_{2j}} \boldsymbol{\Phi}_2 E_{2j} \boldsymbol{\Phi}_2'$$

The corresponding decomposition of the sum of squares for the composed parameters is

$$\sum_{r=1}^{v} \mu_r^2 = \boldsymbol{\mu}' I_v \boldsymbol{\mu} = \boldsymbol{\mu}' \tilde{E}_0 \boldsymbol{\mu} + \boldsymbol{\mu}' \tilde{E}_{11} \boldsymbol{\mu} + \cdots + \boldsymbol{\mu}' \tilde{E}_{1p} \boldsymbol{\mu} + \boldsymbol{\mu}' \tilde{E}_{21} \boldsymbol{\mu} + \cdots + \boldsymbol{\mu}' \tilde{E}_{2q} \boldsymbol{\mu} + \boldsymbol{\mu}' \tilde{E}_e \boldsymbol{\mu}$$

and may be expressed after some calculations as

$$\sum_{r=1}^{v} \mu_r^2 = \frac{1}{v} \boldsymbol{\mu}' G_v \boldsymbol{\mu} + c_{11} \boldsymbol{\alpha}' E_{11} \boldsymbol{\alpha} + \cdots + c_{1p} \boldsymbol{\alpha}' E_{1p} \boldsymbol{\alpha} + c_{21} \boldsymbol{\beta}' E_{21} \boldsymbol{\beta} + \cdots + c_{2q} \boldsymbol{\beta}' E_{2q} \boldsymbol{\beta} + \boldsymbol{\delta}' \boldsymbol{\delta} \tag{85}$$

This shows an important implication of the orthogonal composition.

A simple example of G-orthogonal composition is given below.

Example 2. $[I_s, G_s]$, $[I_t, G_t] \to$ *Two-way factorial association algebra.*

Suppose two primitive algebras

$$\mathfrak{A}_1 = [I_s, G_s] = \left[\frac{1}{s} G_s\right] \oplus \left[I_s - \frac{1}{s} G_s\right] \tag{86}$$

$$\mathfrak{A}_2 = [I_t, G_t] = \left[\frac{1}{t} G_t\right] \oplus \left[I_t - \frac{1}{t} G_t\right]$$

are defined over two vectors of parameters $\boldsymbol{\alpha}' = (\alpha_1, \cdots, \alpha_s)$ and $\boldsymbol{\beta}' = (\beta_1, \cdots, \beta_t)$, respectively.

Those algebras determine the decompositions of the sums of squares for parameters:

$$\boldsymbol{\alpha}' I_s \boldsymbol{\alpha} = \frac{1}{s} \boldsymbol{\alpha}' G_s \boldsymbol{\alpha} + \boldsymbol{\alpha}' \left(I_s - \frac{1}{s} G_s\right) \boldsymbol{\alpha}$$

$$\boldsymbol{\beta}' I_t \boldsymbol{\beta} = \frac{1}{t} \boldsymbol{\beta}' G_t \boldsymbol{\beta} + \boldsymbol{\beta}' \left(I_t - \frac{1}{t} G_t\right) \boldsymbol{\beta}$$

or

$$\sum_{i=1}^{s} \alpha_i^2 = s\bar{\alpha}^2 + \sum_{i=1}^{s} (\alpha_i - \bar{\alpha})^2, \tag{87}$$

$$\sum_{j=1}^{t} \beta_j^2 = t\bar{\beta}^2 + \sum_{j=1}^{t} (\beta_j - \bar{\beta})^2,$$

where $\bar{\alpha} = \frac{1}{s} \sum_{i=1}^{s} \alpha_i$, $\bar{\beta} = \frac{1}{t} \sum_{j=1}^{t} \beta_j$.

Consider a linear mapping σ_1 of $\mathfrak{A}_1$ and a linear mapping σ_2 of $\mathfrak{A}_2$ defined respectively by

$$\boldsymbol{\Phi}_1 = I_s \otimes \boldsymbol{j}_t, \qquad \boldsymbol{\Phi}_2 = \boldsymbol{j}_s \otimes I_t. \tag{88}$$

Since $\boldsymbol{\Phi}_1 \boldsymbol{j}_s = \boldsymbol{j}_{st}$, $\boldsymbol{\Phi}_1' \boldsymbol{j}_{st} = t\boldsymbol{j}_s$, $\boldsymbol{\Phi}_2 \boldsymbol{j}_t = \boldsymbol{j}_{st}$, $\boldsymbol{\Phi}_2' \boldsymbol{j}_{st} = s\boldsymbol{j}_t$ and $\boldsymbol{\Phi}_1'\boldsymbol{\Phi}_1 = tI_s$, $\boldsymbol{\Phi}_2'\boldsymbol{\Phi}_2 = sI_t$, $\boldsymbol{\Phi}_1'\boldsymbol{\Phi}_2 = G_{s\times t}$ hold, both σ_1 and σ_2 are similar and give a G-orthogonal composition of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. The composed full-rank relationship algebra $\sigma_1(\mathfrak{A}_1)^{\cup}\sigma_2(\mathfrak{A}_2)^{\cup}[I_{st}]$ may be called the two-way factorial association algebra. The generators of the algebra are

$$I_s \otimes I_t, \quad (G_s - I_s) \otimes I_t, \quad I_s \otimes (G_t - I_t) \quad \text{and} \quad (G_s - I_s) \otimes (G_t - I_t) \tag{89}$$

The decomposition of the composed algebra is

$$\sigma_1(\mathfrak{A}_1)^{\cup}\sigma_2(\mathfrak{A}_2)^{\cup}[I_{st}] = \boldsymbol{\Phi}_1 \left[\frac{1}{s} G_s\right] \boldsymbol{\Phi}_1' \oplus \boldsymbol{\Phi}_1 \left[I_s - \frac{1}{s} G_s\right] \boldsymbol{\Phi}_1' \tag{90}$$

$$\oplus \boldsymbol{\Phi}_2 \left[I_t - \frac{1}{t} G_t\right] \boldsymbol{\Phi}_2' \oplus \tilde{\mathfrak{A}}_e$$

The corresponding decomposition of the unit matrix is

$$I_{st} = \frac{1}{st} G_s \otimes G_t + \frac{1}{t}\left(I_s - \frac{1}{s} G_s\right) \otimes G_t + \frac{1}{s} G_s \otimes \left(I_t - \frac{1}{t} G_t\right) \tag{91}$$

$$+ \left(I_s - \frac{1}{s} G_s\right) \otimes \left(I_t - \frac{1}{t} G_t\right)$$

Let the composed parameter vector be

$$\boldsymbol{\tau} = \boldsymbol{\Phi}_1 \boldsymbol{\alpha} + \boldsymbol{\Phi}_2 \boldsymbol{\beta} + \boldsymbol{\delta}, \qquad \boldsymbol{\Phi}_1' \boldsymbol{\delta} = 0, \quad \boldsymbol{\Phi}_2' \boldsymbol{\delta} = 0, \tag{92}$$

then we have, after some calculation,

(93) $$\sum_{k=1}^{st}\tau_k^2 = st(\bar{\alpha}+\bar{\beta})^2 + t\sum_{i=1}^{s}(\alpha_i-\bar{\alpha})^2 + s\sum_{j=1}^{t}(\beta_j-\bar{\beta})^2 + \boldsymbol{\delta}'\boldsymbol{\delta}$$

If we denote $\{(i-1)t+j\}$-th element of $\boldsymbol{\delta}$ in a traditional way as γ_{ij}, the restrictions for $\boldsymbol{\delta}$ may be expressed as

(94) $$\boldsymbol{\Phi}_1'\boldsymbol{\delta} = 0 \quad \Leftrightarrow \sum_j \gamma_{ij} = 0 \quad \text{for any } i,$$

$$\boldsymbol{\Phi}_2'\boldsymbol{\delta} = 0 \quad \Leftrightarrow \sum_i \gamma_{ij} = 0 \quad \text{for any } j.$$

Each component of the sum of squares of working parameters $\boldsymbol{\tau}'=(\tau_1, \cdots, \tau_{st})$ corresponds respectively to the sum of squares for pooled mean $\bar{\alpha}+\bar{\beta}$, the sum of squares of α_i about the mean $\bar{\alpha}$, the sum of squares of β_j about the mean $\bar{\beta}$, and the sum of squares for the residuals. Statistical meanings of those components are obvious.

The procedure may be extended to the general p-way factorial association algebra. Another example of G-orthogonal compositions is a series of the Graeco-Latin square compositions. Details will be seen in the forthcomming paper. It should also be noted that L_2 type algebra, cubic type algebra, etc., may be obtained as the subalgebras of special type factorial association algebras, respectively.

9. Composition of relationship algebras for experimental designs

Suppose that an experimenter has composed a relationship algebra $\mathfrak{A}$ among a set of v treatment parameters $\boldsymbol{\tau}'=(\tau_1, \cdots, \tau_v)$ of one or more primitive parameter relationship algebras. He naturally wishes to know something about those primitive relationships through an exprimentation. Assume that $\mathfrak{A}$ is full-rank and has $[G_v]$ as its two-sided ideal. Let the direct decomposition $\mathfrak{A}$ be

(95) $$\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_k, \qquad \mathfrak{A}_0 = [G_v]$$

and the corresponding decomposition of the unit matrix be

(96) $$I_v = E_0 + E_1 + \cdots + E_k, \qquad E_0 = \frac{1}{v}G_v$$

As was seen in section 7 and 8, the unique decomposition of the parameter sum of squares is

(97) $$\boldsymbol{\tau}'I_v\boldsymbol{\tau} = \boldsymbol{\tau}'E_0\boldsymbol{\tau} + \cdots + \boldsymbol{\tau}'E_k\boldsymbol{\tau}$$

and some of these component parameter sums of squares can be reduced to the sums of squares among the sets of primitive parameters.

Suppose that the experimenter wishes to design an experiment which will be conducted on n experimental units or plots.

(a) The case without nuisance parameter algebra.

Suppose that no nuisance parameter algebra except the trivial algebra $[G_n]$ is defined among the n plots, and suppose that a linear mapping σ of $\mathfrak{A}$ defined by $\boldsymbol{\Phi}$ (a linear mapping from the parameter vector space V_v to the observation vector space V_n) is G-preserving (partially) similar, i.e.,

$$(98)\qquad \begin{aligned} &\boldsymbol{\Phi}: \quad \boldsymbol{\tau} \to \boldsymbol{\tau}^* = \boldsymbol{\Phi}\boldsymbol{\tau} \\ &\sigma: \quad \mathfrak{A} \to \mathfrak{A}^* = \sigma(\mathfrak{A}) = \boldsymbol{\Phi}\mathfrak{A}\boldsymbol{\Phi}' \\ &\boldsymbol{\Phi}'\boldsymbol{\Phi} = c_0E_0 + c_1E_1 + \cdots + c_sE_s \quad (c_i > 0,\ i=0, 1, \cdots, s,\ 1 \leq s \leq k) \end{aligned}$$

Suppose further, the observation vector

$$(99)\qquad \boldsymbol{x}' = (x_1, x_2, \cdots, x_n)$$

is normally distributed with mean $\mathscr{E}(\boldsymbol{x}) = \boldsymbol{\Phi}\boldsymbol{\tau}$ and covariance matrix $V(\boldsymbol{x}) = \theta^2 I_n$.

The relationship algebra composed for the design is

$$(100)\qquad \mathfrak{R} = \sigma(\mathfrak{A})^{\cup}[I_n, G_n] = \sigma(\mathfrak{A})^{\cup}[I_n]$$

$\mathfrak{R}$ may be decomposed uniquely into direct sum of minimum two-sided ideals as

$$(101)\qquad \mathfrak{R} = \sigma(\mathfrak{A}_0) \oplus \sigma(\mathfrak{A}_1) \oplus \cdots \oplus \sigma(\mathfrak{A}_s) \oplus \mathfrak{R}_e$$

The corresponding decomposition of the unit matrix is

$$(102)\qquad I_n = \tilde{E}_0 + \tilde{E}_1 + \cdots + \tilde{E}_s + \tilde{E}_e$$

where

$$\tilde{E}_0 = \frac{1}{n} G_n, \qquad \tilde{E}_i = \frac{1}{c_i} \boldsymbol{\Phi} E_i \boldsymbol{\Phi}' \quad (i = 1, \cdots, s)$$

and

$$\tilde{E}_e = I_n - \sum_{i=0}^{s} \tilde{E}_i.$$

The unique decomposition of the sum of squares is

$$x'x = x'\tilde{E}_0 x + \cdots + x'\tilde{E}_s x + x'\tilde{E}_e x \tag{103}$$

and the analysis of variance is

$$x'\left(I_n - \frac{1}{n} G_n\right)x = \sum_{i=1}^{n} (x_i - \bar{x})^2 \tag{104}$$
$$= x'\tilde{E}_1 x + \cdots + x'\tilde{E}_s x + x'\tilde{E}_e x$$

The distributions of the component sums of squares are independent of each other and any one of them divided by θ^2 except the last one is distributed with non-central chi-square (χ'^2) distribution. The non-centrality parameters of those are

$$\lambda_i = \frac{1}{2\theta^2} \mathscr{E}(x')\tilde{E}_i \mathscr{E}(x) = \frac{1}{2\theta^2} c_i \tau' E_i \tau \qquad \text{for} \quad i = 1, 2, \cdots, s. \tag{105}$$

The last component divided by θ^2, $\frac{1}{\theta^2} x'\tilde{E}_e x$, is distributed with χ^2-distribution, as its non-centrality parameter λ_e is

$$\lambda_e = \frac{1}{2\theta^2} \mathscr{E}(x')\tilde{E}_e \mathscr{E}(x) = 0 \tag{106}$$

The degrees of freedom of those components are $r(E_i)$ for $i = 1, \cdots, s$, and $r(\tilde{E}_e) = n - 1 - \sum_{i=1}^{s} r(E_i)$, respectively.

Thus we have the following analysis of variance table.

Table 1. Analysis of Variance (without nuisance parameters)

Source of variation		Sum of squares	Degrees of freedom	Non-centrality parameter
Component of	1	$x'\tilde{E}_1 x$	$r(E_1)$	$c_1 \tau' E_1 \tau/(2\theta^2)$
treatment	2	$x'\tilde{E}_2 x$	$r(E_2)$	$c_2 \tau' E_2 \tau/(2\theta^2)$
sum of squares	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	s	$x'\tilde{E}_s x$	$r(E_s)$	$c_s \tau' E_s \tau/(2\theta^2)$
Error		$x'\tilde{E}_e x$	$n - 1 - \sum_{i=1}^{s} r(E_i)$	0
Total		$x'\left(I_n - \frac{1}{n} G_n\right) x$	$n-1$	

The simplest design may be composed by $r(>1)$ times replications of all working treatments. In this case, as $\boldsymbol{\Phi}=I_v\otimes \boldsymbol{j}_r$ and $\boldsymbol{\Phi}'\boldsymbol{\Phi}=rI_v$, σ is similar. A factorial experiment with replication is an example of the design. The treatment relationship algebra of the design is a factorial association algebra.

It should be added that when $\boldsymbol{\delta}$ in (82) of section 8 is assumed to be normally distributed with mean vector 0 and covariance matrix $\theta^2 I_{st}$, we have a design for two-way classification.

(b) The case with nuisance parameter algebra.

Suppose that there has been composed a nuisance parameter relationship algebra $\mathfrak{B}$ among a set of n plots. $\mathfrak{B}$ has been composed by mappings and compositions of several sets of nuisance parameters among which some primitive nuisance algebras are defined. Suppose that $\mathfrak{B}$ is not full-rank and the principal idempotent of $\mathfrak{B}$ is $\tilde{E}_b$. Suppose, further, $G_n \in \mathfrak{B}$.

Suppose that there exists an $n\times v$ matrix $\boldsymbol{\Phi}$ which defines a partially confounded mapping of $\mathfrak{A}$, i.e., $\boldsymbol{\Phi}$ satisfies,

$$\boldsymbol{\Phi}'(I_n-\tilde{E}_b)\boldsymbol{\Phi}=\sum_{i=1}^{s} c_i E_i \qquad (c_i>0 \text{ for } i=1, \cdots, s\leq k) \tag{107}$$

$$\tilde{E}_b\boldsymbol{\Phi}\neq 0$$

The full-rank relationship algebra induced among the plots is

$$\mathfrak{R}=\mathfrak{A}^{*\cup}\mathfrak{B}^{\cup}[I_n], \qquad \mathfrak{A}^*=\boldsymbol{\Phi}\mathfrak{A}\boldsymbol{\Phi}' \tag{108}$$

The component algebra of $\mathfrak{A}^*$ orthogonal to $\mathfrak{B}$ and defined by

$$\tilde{\mathfrak{A}}=(I_n-\tilde{E}_b)\mathfrak{A}^*(I_n-\tilde{E}_b)=(I_n-\tilde{E}_b)\boldsymbol{\Phi}\mathfrak{A}\boldsymbol{\Phi}'(I_n-\tilde{E}_b) \tag{109}$$

is the image of $\mathfrak{A}$ of a partially similar mapping $\tilde{\sigma}$ defined by $n\times v$ matrix $F=(I_n-\tilde{E}_b)\boldsymbol{\Phi}$, because $F'F=\boldsymbol{\Phi}'(I_n-\tilde{E}_b)\boldsymbol{\Phi}$. Thus $\tilde{\mathfrak{A}}$ may be decomposed as

$$\tilde{\mathfrak{A}}=F\mathfrak{A}_1F'\oplus F\mathfrak{A}_2F'\oplus\cdots\oplus F\mathfrak{A}_sF' \tag{110}$$

and the corresponding decomposition of the idempotent is

$$\tilde{E}=\frac{1}{c_1}FE_1F'+\cdots+\frac{1}{c_s}FE_sF' \tag{111}$$

The unique decomposition of the principal idempotent $I_n-\tilde{E}_b$ of $\tilde{\mathfrak{A}}^{\cup}[I_n-\tilde{E}_b]$ is

$$I_n-\tilde{E}_b=\frac{1}{c_1}FE_1F'+\cdots+\frac{1}{c_s}FE_sF'+\tilde{E}_e \tag{112}$$

where

$$\tilde{E}_e = I_n - \tilde{E}_b - \sum_{i=1}^{s} \frac{1}{c_i} FE_iF'.$$

Suppose further the observation vector

$$\boldsymbol{x}' = (x_1, \cdots, x_n) \tag{113}$$

is normally distributed with mean

$$\mathscr{E}(\boldsymbol{x}) = \boldsymbol{\Phi\tau} + \boldsymbol{\beta} \tag{114}$$

and covariance matrix

$$V(\boldsymbol{x}) = \theta^2 I_n \tag{115}$$

where $\boldsymbol{\beta}$ is the nuisance parameter vector and subjects to the restriction $\tilde{E}_b\boldsymbol{\beta} = \boldsymbol{\beta}$.

The analysis of variance corresponding to the decomposition (112) is

$$\boldsymbol{x}'(I_n - \tilde{E}_b)\boldsymbol{x} = \frac{1}{c_1}\boldsymbol{x}'FE_1F'\boldsymbol{x} + \cdots + \frac{1}{c_s}\boldsymbol{x}'FE_sF'\boldsymbol{x} + \boldsymbol{x}'\tilde{E}_e\boldsymbol{x} \tag{116}$$

The distributions of the component sums of squares are independent of each other and any one of them divided by θ_2, except the last one, is distributed with non-central chi-square (χ'^2) distribution. The non-centrality parameters of them are

$$\begin{aligned}\lambda_i &= \frac{1}{2\theta^2}\frac{1}{c_i}(\boldsymbol{\tau}'\boldsymbol{\Phi}' + \boldsymbol{\beta}')FE_iF'(\boldsymbol{\Phi\tau} + \boldsymbol{\beta}) \\ &= \frac{1}{2\theta^2} c_i\boldsymbol{\tau}'E_i\boldsymbol{\tau}\end{aligned} \tag{117}$$

for $i=1, 2, \cdots, s$. The last component divided by θ^2 is distributed with χ^2-distribution, as its non-centrality parameter λ_e is

$$\lambda_e = \frac{1}{2\theta^2}\mathscr{E}(\boldsymbol{x}')\tilde{E}_e\mathscr{E}(\boldsymbol{x}) = 0 \tag{118}$$

The degrees of freedom of those components are $r(E_i)$ for $i=1, \cdots, s$ and

$r(\tilde{E}_e)=n-r(\tilde{E}_b)-\sum_{i=1}^{s}r(E_i)$, respectively.

The sum of squares $\boldsymbol{x}'\tilde{E}_b\boldsymbol{x}/\theta^2$ is also distributed with χ'^2 distribution, the non-centrality parameter λ_B of which is

$$\lambda_B=\frac{1}{2\theta^2}(\boldsymbol{\tau}'\boldsymbol{\Phi}'+\boldsymbol{\beta}')\tilde{E}_b(\boldsymbol{\Phi}\boldsymbol{\tau}+\boldsymbol{\beta})$$

$$=\frac{1}{2\theta^2}(\boldsymbol{\tau}'H'H\boldsymbol{\tau}+2\boldsymbol{\tau}'H'\boldsymbol{\beta}+\boldsymbol{\beta}'\boldsymbol{\beta})$$

where $H=\tilde{E}_b\boldsymbol{\Phi}$.

The analysis of variance is given in Table 2.

Table 2. Analysis of Variance (with nuisance parameters)

Source of Variation		Sum of squares	Degrees of freedom	Non-centrality parameter
Component of	1	$\boldsymbol{x}'FE_1F'\boldsymbol{x}/c_1$	$r(E_1)$	$c_1\boldsymbol{\tau}'E_1\boldsymbol{\tau}/(2\theta^2)$
treatment sum	2	$\boldsymbol{x}'FE_2F'\boldsymbol{x}/c_2$	$r(E_2)$	$c_2\boldsymbol{\tau}'E_2\boldsymbol{\tau}/(2\theta^2)$
of squares	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	s	$\boldsymbol{x}'FE_sF'\boldsymbol{x}/c_s$	$r(E_s)$	$c_s\boldsymbol{\tau}'E_s\boldsymbol{\tau}/(2\theta^2)$
Error		$\boldsymbol{x}'\tilde{E}_e\boldsymbol{x}$	$n-r(\tilde{E}_b)-\sum_{j=1}^{s}r(E_i)$	0
Sub-total		$\boldsymbol{x}'(I_n-\tilde{E}_b)\boldsymbol{x}$	$n-r(\tilde{E}_b)$	
Nuisance parameters ignoring treatments		$\boldsymbol{x}'\tilde{E}_b\boldsymbol{x}$	$r(\tilde{E}_b)$	$\{\boldsymbol{\tau}'H'H\boldsymbol{\tau}+2\boldsymbol{\tau}'H'\boldsymbol{\beta}+\boldsymbol{\beta}'\boldsymbol{\beta}\}/(2\theta^2)$
Total		$\boldsymbol{x}'\boldsymbol{x}$	n	

The way to compose an experimental design so as to preserve the similarity or partial similarity of the plot relationship algebra to the treatment relationship algebra after the elimination of the nuisance parameters, is to seek for an incidence matrix $\boldsymbol{\Phi}$, satisfying (107), from treatment parameter space V_v to the plot space V_n.

The simplest design of this type is an RBD. The most typical design of this type is a PBIBD. In a PBIBD, $\mathfrak{B}$ is a similar image of $[I_b, G_b]$ defined among b block parameters. A PBIBD has a treatment association algebra defined by the treatment-block incidence matrix. As far as the composed treatment relationship algebra contains the association algebra of the PBIBD as its subalgebra, the treatment-plot incidence matrix satisfies (107) and the (partial) similarity of the treatment relationships may be preserved. As the association algebra of a BIBD is $[I_v, G_v]$, all full-rank relationship algebras for treatments containing an ideal $[G_v]$ have $[I_v, G_v]$ as their subalgebra. In this sense, a BIBD is the best in such incomplete block designs.

The nuisance parameter algebra of a standard design for two-way elimination of heterogeneity, such as an LSD, a YSD, etc., is a factorial type association with no interactions.

There may exist many possibilities of composing new designs for the elimination of nuisance parameters.

References

[1] Albert, A. A. (1939). *Structure of algebras.* Amer. Math. Soc. Coll. Publ. 24, New York.

[2] Bose, R. C. and Mesner, D. M. (1959). On linear associative algebras corresponding to association schemes of partially balanced designs. *Ann. Math. Statist.* **30** 21-38.

[3] Bose, R. C. and Nair, K. R. (1939). Partially balanced incomplete block designs. *Sankhyā.* **4** 337-372.

[4] Graybill, F. A. and Marsaglia, G. (1957). Idempotent matrices and quadratic forms in the general linear hypothesis. *Ann. Math. Statist.* **28** 678-686.

[5] James, A. T. (1957). The relationship algebra of an experimental design. *Ann. Math. Statist.* **28** 993-1002.

[6] Mann, H. B. (1960). The algebra of a linear hypothesis. *Ann. Math. Statist.* **31** 1-15.

[7] Ogawa, J. (1959). *The theory of the association algebra and the relationship algebra of a partially balanced incomplete block design.* Inst. Statist. mimeo. series 224, Chapel Hill, N. C.

[8] van der Waerden, B. L. (1959). *Algebra II*, (4th ed.) Springer, Berlin.

[9] Yamamoto, S. and Fujii, Y. (1963). Analysis of partially balanced incomplete block designs. *J. Sci. Hiroshima Univ. Ser. A-I.* **27** 119-135.

Department of Mathematics
Faculty of Science
Hiroshima University