J. Sci. Hiroshima Univ. Ser. A-I
28 (1964), 159-165

Note on First Recurrence and First Passage Times for Renewal Processes

Sumiyasu Yamamoto and Takao Shohōji

(Received September 20, 1964)

1. Introduction

We shall deal with the probability generating functions of the 'first recurrence times' and the 'first passage times' for renewal processes with discrete (time) parameters, which were treated by W. Feller [3], [4] and M. S. Bartlett [1], [2]. Bartlett obtained not only some results for the 'first recurrence time' probability generating functions due to Feller from a different point of view, but also the probability generating functions of the 'first recurrence times' and the 'first passage times' when some conditions are imposed on the paths of the processes. Though Bartlett's approch seems to be ingenious at a glance, it is too complicated to apply the method to more involved cases where several conditions are imposed on the paths of the processes.

We shall generalize Feller's approch, and systematically derive the 'first recurrence time' as well as the 'first passage time' probability generating functions when several conditions are imposed on the paths of the processes. The results obtained will cover the most complicated results due to Bartlett as their special cases, and one of them agrees with a formula due to J. H. B. Kemperman [5].

2. Notations

We shall deal with a process with possible states $a, b, c, \cdots$, or $1, 2, \cdots$, which has the following two properties:

(i) The process is temporally homogeneous;

(ii) Any one of the states (or the sets of states) considered has the renewal property.

For convenience, we denote various kinds of events, their probabilities and their probability generating functions treated in this paper as follows:

$\underset{S}{\pi^{ab}}(m+n|m)$: The event of the process being in state b at time $m+n$ $(n\geqq 1)$, given that the process is in state a at time m and *no intermediate passage* to any one of the states in the set S occurs.

$\underset{S}{\lambda}^{ab}(m+n\,|\,m)$: The event of the process being in state b *for the first time* at time $m+n$ $(n\geqq 1)$, given that the process is in state a at time m and *no intermediate passage* to any one of the states in the set S occurs.

It should be added that those events concern with the passage from state a to state b when state b does not coincide with state a, and with the recurrence of state a when state b coincides with state a.

The probabilities of those events are denoted by enclosing them in the parenthesis { }. The assumption (i) shows that

$$\{\underset{S}{\pi}^{ab}(m+n\,|\,m)\} = \{\underset{S}{\pi}^{ab}(n\,|\,0)\} \equiv \{\underset{S}{\pi}^{ab}(n)\}$$

$$\{\underset{S}{\lambda}^{ab}(m+n\,|\,m)\} = \{\underset{S}{\lambda}^{ab}(n\,|\,0)\} \equiv \{\underset{S}{\lambda}^{ab}(n)\}$$

The generating functions for those probabilities are denoted as

$$\Pi_S^{ab}(z) \equiv \sum_{n=1}^{\infty} \{\underset{S}{\pi}^{ab}(n)\}\, z^n \qquad (|z|<1)$$

$$\Lambda_S^{ab}(z) \equiv \sum_{n=1}^{\infty} \{\underset{S}{\lambda}^{ab}(n)\}\, z^n \qquad (|z|\leqq 1)$$

$$\Lambda_{1,2,\dots,l}^{ab}(z) \equiv \sum_{n=1}^{\infty} \{\underset{1,2,\dots,l}{\lambda}^{ab}(n)\}\, z^n \qquad (|z|\leqq 1)$$

where z is a fixed real or complex parameter.

3. Probability generating functions of first passage and first recurrence

Let state a be the initial state, and state b be the terminal state, and S be a set of specified states. Assume that state b is not included in the set S. Consider an event $\underset{S}{\pi}^{ab}(n\,|\,0)$, then we can decompose the event into n mutually exclusive events as

$$\underset{S}{\pi}^{ab}(n\,|\,0) = \bigcup_{i=1}^{n-1} [\underset{S}{\lambda}^{ab}(i\,|\,0) \cap \underset{S}{\pi}^{bb}(n\,|\,i)] \cup \underset{S}{\lambda}^{ab}(n\,|\,0) \tag{1}$$

From the assumptions (i) and (ii), we have

$$\{\underset{S}{\pi}^{ab}(n)\} = \sum_{i=1}^{n-1} \{\underset{S}{\lambda}^{ab}(i)\}\ \{\underset{S}{\pi}^{bb}(n-i)\} + \{\underset{S}{\lambda}^{ab}(n)\} \tag{2}$$

Multiplying both sides by z^n and summing over n, we have

$$\Pi_S^{ab}(z) = \Lambda_S^{ab}(z)\Pi_S^{bb}(z) + \Lambda_S^{ab}(z) \qquad (b \notin S)$$

Thus, we have the probability generating function of the first passage times from state a to state b without intermediate passage to any one of the states in S:

$$\Lambda_S^{ab}(z) = \frac{\Pi_S^{ab}(z)}{1+\Pi_S^{bb}(z)} \qquad (b \notin S) \tag{3}$$

As stated above, if state a coincides with state b, we have the 'first recurrence time' probability generating function without intermediate passage to S:

$$\Lambda_S^{bb}(z) = \frac{\Pi_S^{bb}(z)}{1+\Pi_S^{bb}(z)} \qquad (b \notin S) \tag{4}$$

Using (3) and (4), we have

$$\Pi_S^{ab}(z) = \frac{\Lambda_S^{ab}(z)}{1-\Lambda_S^{bb}(z)} \qquad (b \notin S) \tag{5}$$

and

$$\Pi_S^{bb}(z) = \frac{\Lambda_S^{bb}(z)}{1-\Lambda_S^{bb}(z)} \qquad (b \notin S) \tag{6}$$

Now, we shall proceed to derive the recurrence formulas for the number of conditioned states in S. Consider an event $\underset{S}{\lambda^{ab}}(n)$, $(b \notin S,\ c \notin S,\ b \neq c)$. It may be divided into n mutually exclusive events as

$$\underset{S}{\lambda^{ab}}(n\,|\,0) = \bigcup_{i=1}^{n-1} \left[\underset{S\cup b}{\lambda^{ac}}(i\,|\,0) \cap \underset{S}{\lambda^{cb}}(n\,|\,i) \right] \cup \underset{S\cup c}{\lambda^{ab}}(n\,|\,0) \tag{7}$$

We have, therefore,

$$\{\underset{S}{\lambda^{ab}}(n)\} = \sum_{i=1}^{n-1} \{\underset{S\cup b}{\lambda^{ac}}(i)\}\ \{\underset{S}{\lambda^{cb}}(n-i)\} + \{\underset{S\cup c}{\lambda^{ab}}(n)\} \tag{8}$$

Taking the probability generating functions on both sides, we have

$$\Lambda_S^{ab}(z) = \Lambda_{S\cup b}^{ac}(z)\Lambda_S^{cb}(z) + \Lambda_{S\cup c}^{ab}(z) \qquad (b \notin S,\ c \notin S,\ b \neq c) \tag{9}$$

As we can replace the terminal state b with state c, we have

$$\Lambda_S^{ac}(z) = \Lambda_{S\cup c}^{ab}(z)\Lambda_S^{bc}(z) + \Lambda_{S\cup b}^{ac}(z) \qquad (b \notin S,\ c \notin S,\ b \neq c) \tag{10}$$

From (9) and (10), we have a recurrence formula

(11)
$$A_{S\cup c}^{ab}(z)=\frac{A_S^{ab}(z)-A_S^{ac}(z)\,A_S^{cb}(z)}{1-A_S^{bc}(z)\,A_S^{cb}(z)}$$

(11′)
$$=\frac{\begin{vmatrix} A_S^{ab}(z) & A_S^{ac}(z) \\ A_S^{cb}(z) & 1 \end{vmatrix}}{\begin{vmatrix} 1 & A_S^{bc}(z) \\ A_S^{cb}(z) & 1 \end{vmatrix}} \qquad (b\notin S,\ c\notin S,\ b\neq c)$$

We have, similarly,

(12)
$$A_S^{cb}(z)=A_{S\cup b}^{ca}(z)\,A_S^{ab}(z)+A_{S\cup a}^{cb}(z) \qquad (a\notin S,\ b\notin S,\ a\neq b)$$

Using (9) and (12), we have a recurrence formula corresponding to the reverse direction of (11):

(13)
$$A_S^{ab}(z)=\frac{A_{S\cup c}^{ab}(z)+A_{S\cup b}^{ac}(z)\,A_{S\cup a}^{cb}(z)}{1-A_{S\cup b}^{ac}(z)\,A_{S\cup b}^{ca}(z)}$$

(13′)
$$=-\frac{\begin{vmatrix} A_{S\cup c}^{ab}(z) & A_{S\cup b}^{ac}(z) \\ A_{S\cup a}^{cb}(z) & -1 \end{vmatrix}}{\begin{vmatrix} -1 & A_{S\cup b}^{ac}(z) \\ A_{S\cup b}^{ca}(z) & -1 \end{vmatrix}} \qquad (a\notin S,\ b\notin S,\ c\notin S,\ a\neq b\neq c\neq a)$$

The formula (11) or (11′) is the recurrence formula expressing a more conditioned probability generating function by less conditioned ones. On the other hand, the formula (13) or (13′) is that expressing less conditioned one by more conditioned ones.

Next, we shall apply the recurrence formula (11′) to some concrete cases. Let the initial state be state 0, the terminal state be state 1, and those states to which no intermediate passage occurs, be 2, 3, …, and k. From (11′), we have

(14)
$$A_{2,3,\dots,k}^{01}(z)=\frac{\begin{vmatrix} A_{2,3,\dots,k-1}^{01}(z) & A_{2,3,\dots,k-1}^{0k}(z) \\ A_{2,3,\dots,k-1}^{k1}(z) & 1 \end{vmatrix}}{\begin{vmatrix} 1 & A_{2,3,\dots,k-1}^{1k}(z) \\ A_{2,3,\dots,k-1}^{k1}(z) & 1 \end{vmatrix}}$$

Using the formula (11) repeatedly, we have, after some calculation,

$$(15) \qquad A^{01}_{2,3,\dots,k}(z) = \frac{\begin{vmatrix} A^{01}(z) & A^{02}(z) & A^{03}(z) \cdots A^{0k}(z) \\ A^{21}(z) & 1 & A^{23}(z) \cdots A^{2k}(z) \\ A^{31}(z) & A^{32}(z) & 1 \quad \cdots A^{3k}(z) \\ \vdots & \vdots & \vdots \qquad \vdots \\ A^{k1}(z) & A^{k2}(z) & A^{k3}(z) \cdots \quad 1 \end{vmatrix}}{\begin{vmatrix} 1 & A^{12}(z) & A^{13}(z) \cdots A^{1k}(z) \\ A^{21}(z) & 1 & A^{23}(z) \cdots A^{2k}(z) \\ A^{31}(z) & A^{32}(z) & 1 \quad \cdots A^{3k}(z) \\ \vdots & \vdots & \vdots \qquad \vdots \\ A^{k1}(z) & A^{k2}(z) & A^{k3}(z) \cdots \quad 1 \end{vmatrix}}$$

Substituting (3) into (15), we have

$$(16) \qquad A^{01}_{2,3,\dots,k}(z) = \frac{\det \|\delta_{ij} + \Pi^{ij}(z)\|}{\det \|\delta_{pq} + \Pi^{pq}(z)\|} \qquad \begin{pmatrix} i = 0, 2, 3, \cdots, k \\ j, p, q = 1, 2, \cdots, k \end{pmatrix}$$

The formula (15) is that expressing the 'conditional first passage time' probabillity generating function by the 'unconditional first passage time' probability generating functions. The formula (16) is that expressed by the 'unconditional passage time' and 'unconditional recurrence time' probability generating funnctions.

The formula (15) is the same one obtained by Kemperman [5] through an alternative method.

Starting from (13′), we have

$$(17) \qquad A^{01}_{2,3,\dots,k-1}(z) = -\frac{\begin{vmatrix} A^{01}_{2,3,\dots,k}(z) & A^{0k}_{1,2,\dots,k-1}(z) \\ A^{k1}_{0,2,\dots,k-1}(z) & -1 \end{vmatrix}}{\begin{vmatrix} -1 & A^{0k}_{1,2,\dots,k-1}(z) \\ A^{k0}_{1,2,\dots,k-1}(z) & -1 \end{vmatrix}}$$

When the number of possible states is finite, say $k+1$, we have

$$(18) \qquad A^{01}(z) = -\frac{\begin{vmatrix} A^{01}_{*}(z) & A^{02}_{*}(z) & A^{03}_{*}(z) \cdots A^{0k}_{*}(z) \\ A^{21}_{*}(z) & -1 & A^{23}_{*}(z) \cdots A^{2k}_{*}(z) \\ A^{31}_{*}(z) & A^{32}_{*}(z) & -1 \quad \cdots A^{3k}_{*}(z) \\ \vdots & \vdots & \vdots \qquad \vdots \\ A^{k1}_{*}(z) & A^{k2}_{*}(z) & A^{k3}_{*}(z) \cdots \quad -1 \end{vmatrix}}{\begin{vmatrix} -1 & A^{02}_{*}(z) & A^{03}_{*}(z) \cdots A^{0k}_{*}(z) \\ A^{20}_{*}(z) & -1 & A^{23}_{*}(z) \cdots A^{2k}_{*}(z) \\ A^{30}_{*}(z) & A^{32}_{*}(z) & -1 \quad \cdots A^{3k}_{*}(z) \\ \vdots & \vdots & \vdots \qquad \vdots \\ A^{k0}_{*}(z) & A^{k2}_{*}(z) & A^{k3}_{*}(z) \cdots \quad -1 \end{vmatrix}}$$

where $A_*^{ij}(z)$ denotes the probability generating function of the first passage times from state i to state j conditional on *no intermediate passage to any other states.*

We shall deal with some special cases. If the terminal state 1 coincides with the initial state 0 in the formulas (14), (15) and (16), we have the formulas for the 'first recurrence time' probability generating functions. If any one of the conditioned states, say state k in (14), coincides with the initial state 0, the formula turns out two formulas for the probability generating functions of the conditional first passage times without intermediate recurrence of state 0:

$$(19)\qquad A^{01}_{2,\dots,k-1,0}(z)=\frac{A^{01}_{2,3,\dots,k-1}(z)-A^{00}_{2,3,\dots,k-1}(z)\,A^{01}_{2,\dots,k-1}(z)}{1-A^{01}_{2,3,\dots,k-1}(z)\,A^{10}_{2,3,\dots,k-1}(z)}$$

$$=\frac{A^{01}_{2,\dots,k-2,0}(z)-A^{0,k-1}_{2,\dots,k-2,0}(z)\,A^{k-1,1}_{2,\dots,k-2,0}(z)}{1-A^{1,k-1}_{2,\dots,k-2,0}(z)\,A^{k-1,1}_{2,3,\dots,k-2,0}(z)}$$

The formula analogous to (18) for the 'first recurrence time' probability generating function is somewhat different, but may be derived similarly as

$$(20)\qquad A^{00}(z)=\frac{\begin{vmatrix} A_*^{00}(z) & A^{01}_{*\cup 0}(z) & A^{02}_{*\cup 0}(z) & \cdots & A^{0k}_{*\cup 0}(z)\\ A^{10}_{*\cup 1}(z) & A_*^{11}(z)-1 & A^{12}_{*\cup 1}(z) & \cdots & A^{1k}_{*\cup 1}(z)\\ A^{20}_{*\cup 2}(z) & A^{21}_{*\cup 2}(z) & A_*^{22}(z)-1 & \cdots & A^{2k}_{*\cup 2}(z)\\ \vdots & \vdots & \vdots & & \vdots\\ A^{k0}_{*\cup k}(z) & A^{k1}_{*\cup k}(z) & A^{k2}_{*\cup k}(z) & \cdots & A_*^{kk}(z)-1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & A_*^{11}(z)-1 & A^{12}_{*\cup 1}(z) & \cdots & A^{1k}_{*\cup 1}(z)\\ 0 & A^{21}_{*\cup 2}(z) & A_*^{22}(z)-1 & \cdots & A^{2k}_{*\cup 2}(z)\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & A^{k1}_{*\cup k}(z) & A^{k2}_{*\cup k}(z) & \cdots & A_*^{kk}(z)-1 \end{vmatrix}}$$

Those formulas derived so far will be useful in obtaining various kinds of the probability generating functions under several conditions.

4. Example

Consider a discrete time Markov chain $X(t)$ $(t=0, 1, 2, \cdots)$ with finite possible states $0, 1, \cdots, k$. Let

$$p_{ij}\equiv Pr\{X(t+1)=j \mid X(t)=i\}\quad (i, j=0, 1, \cdots, k)$$

$$\sum_j p_{ij}=1\quad (i=0, 1, \cdots, k)$$

be the stationary transition probabities. In this case, the probability generating function of the first passage time from state i to state j without any intermediate passage to any other states, is

$$A_{*}^{ij}(z) = p_{ij}z + p_{ii}p_{ij}z^2 + \cdots + (p_{ii})^n p_{ij} z^{n+1} + \cdots$$
$$= \frac{p_{ij}z}{1 - p_{ii}z}$$

for any pair of i and j ($i, j = 0, 1, \cdots, k$). Thus, we have from (18)

$$A^{01}(z) = -\frac{\begin{vmatrix} p_{01}z & p_{02}z & p_{03}z & \cdots & p_{0k}z \\ p_{21}z & p_{22}z-1 & p_{23}z & \cdots & p_{2k}z \\ p_{31}z & p_{32}z & p_{33}z-1 & \cdots & p_{3k}z \\ \vdots & \vdots & \vdots & & \vdots \\ p_{k1}z & p_{k2}z & p_{k3}z & \cdots & p_{kk}z-1 \end{vmatrix}}{\begin{vmatrix} p_{00}z-1 & p_{02}z & p_{03}z & \cdots & p_{0k}z \\ p_{20}z & p_{22}z-1 & p_{23}z & \cdots & p_{2k}z \\ p_{30}z & p_{32}z & p_{33}z-1 & \cdots & p_{3k}z \\ \vdots & \vdots & \vdots & & \vdots \\ p_{k0}z & p_{k2}z & p_{k3}z & \cdots & p_{kk}z-1 \end{vmatrix}}$$

References

[1] Bartlett, M. S. (1953). Recurrence and first passage times. *Proc. Cambridge Philos. Soc.* **49** 263-275.

[2] Bartlett, M. S. (1956). *An introduction to stochastic processes with special reference to methods and applications.* Cambridge Univ. Press, Cambridge.

[3] Feller, W. (1947). Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67** 98-119.

[4] Feller, W. (1957). *An introduction to probability theory and its applications.* **1** (2nd ed.), John Wiley and Sons., New York.

[5] Kemperman, J. H. B. (1961). *The passage problem for a stationary Markov chain.* Univ. of Chicago Press, Chiago.

Department of Mathematics
Faculty of Science
Hiroshima University