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Derivations of Lie Algebras

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Introduction

Let L be a Lie algebra over a field of characteristic 0 and let $D(L)$ be the Lie algebra of all derivations of L. The problems concerning the structure of $D(L)$ and its relations with the structure of L have been investigated by several authors in [5], [7], [8], [9], [11], [14], [16], [17] etc. In a recent paper [10] G. Leger has studied the structural properties of Lie algebras L such that $D(L)=I(L)$, where $I(L)$ is the set of all inner derivations of L, and proved the following results:

(1) If the center of L is not (0) and if $D(L)=I(L)$, then L is not solvable and the radical of L is nilpotent.

(2) If the center of L is not (0) and if the nilpotent radical is quasi-cyclic, then $D(L)\neq I(L)$.

Here a nilpotent Lie algebra N is called quasi-cyclic provided N has a subspace U such that $N=U+[N, N]$ with $U\cap[N, N]=(0)$ and such that N is the direct sum of the subspaces U^i where $U^1=U$ and $U^i=[U, U^{i-1}]$ for $i\geq 2$.

We denote by $C(L)$ the set of all central derivations of L, that is, the set of all derivations of L mapping L into the center. It is the purpose of this paper to investigate the properties of Lie algebras L such that $C(L)\subset I(L)$, Lie algebras L such that $I(L)\subset C(L)$ and Lie algebras L such that $D(L)=I(L)+C(L)$, and to generalize Leger's results above.

There actually exist the Lie algebras satisfying each of these three conditions as shown in Remarks 1, 2 and 3.

In Section 2 we shall give the forms of the derivations which are at the same time inner and central. In Section 3 we shall study the Lie algebras whose central derivations are all inner. We shall show that if the center Z of L is not (0) and if $C(L)\subset I(L)^*$, the algebraic hull of $I(L)$, then for the radical R of L $\mathrm{ad}_L R$ contains no non-zero semisimple elements (Theorem 1), and that if $Z\neq(0)$ and if $C(L)\subset I(L)$ and $I(L)$ is splittable, then the radical R is nilpotent (Theorem 2). The essential part of (1) above is to assert the nilpotency of the radical and we shall show that this is a special case of our results above (Corollary to Theorem 2 and Remark 1).

In Section 4 we shall show that, when $Z\neq(0)$, $I(L)=C(L)$ if and only if

$L^3=(0)$, $L^2=Z$ and $\dim Z=1$ (Theorem 3).

In Section 5 we shall study the Lie algebras L such that $D(L)=I(L)+C(L)$, that is, which have as few derivations as possible. We shall prove that if $D(L)=I(L)+C(L)$, then the radical R is either non-quasi-cyclic or an abelian direct summand of L (Theorem 5). Taking account of (1), (2) is equivalent to the statement that if $D(L)=I(L)$ then R is not quasi-cyclic, and this is a special case of our result except when R is an abelian direct summand (Corollary to Theorem 5 and Remark 3).

In Section 6 we shall study the Lie algebras L whose radicals R satisfy the conditions considered in Sections 2–5. We can not generally expect that L satisfies the corresponding conditions. We shall show that if $C(R)\subset I(R)$ (resp. $C(R)\subset I(R)^*$) then $C(L)\subset I(L)$ (resp. $C(L)\subset I(L)^*$) (Theorem 6). We shall also prove that $D(R)=I(R)+C(R)$ if and only if L is the direct sum of an ideal L_1, which is the direct sum of a semisimple ideal, a characteristically solvable ideal R_1 with $D(R_1)=I(R_1)+C(R_1)$ and a central ideal, and of an ideal L_2, whose radical is abelian, whose center is (0) and such that $L_2=[L_2, L_2]$, and that in this case $D(L_1)=I(L_1)+C(L_1)$ and $D(L_2)\neq I(L_2)+C(L_2)$ (Theorem 7).

1. Preliminaries

Let L be a Lie algebra over a field K of characteristic 0 and let $D(L)$ be the derivation algebra of L, that is, the Lie algebra of all derivations of L. For any element x of L, the adjoint mapping $\operatorname{ad} x: y\to[x, y]$ is a derivation of L which is called inner. We denote by $I(L)$ the ideal of all inner derivations of L.

A derivation of L is called central provided that it maps L into the center of L. We denote by $C(L)$ the set of all central derivations of L. Then an endomorphism of L is a central derivation if and only if it maps L into the center of L and $[L, L]$ into (0). $C(L)$ is a subalgebra of $D(L)$.

$D(L)$ necessarily contains $I(L)$ and $C(L)$. Therefore when $D(L)=I(L)+C(L)$, we may say that L has as few derivations as possible.

Let L be the direct sum of the ideals $L_i (i=1, 2, \cdots, n)$ and let $D(L_i, L_j)$ be the set of all derivations of L_i into L_j. Then $D(L_i, L_i)=D(L_i)$. We denote by p_i the projection of L onto L_i and identify an element D_{ij} of $D(L_i, L_j)$ with an element $D_{ij}p_i$ of $D(L)$. Thus we have $D(L_i, L_j)\subset D(L)$. It is easy to see the following fact ([17], p. 202):

(1) If L is the direct sum of the ideals $L_i (i=1, 2, \cdots, n)$, then

$$D(L)=\sum_{i,j=1}^{n} D(L_i, L_j)$$

and, for $i \neq j$, $D(L_i, L_j)$ consists of all the linear mappings of L_i into L_j which map L_i into the center of L_j and $[L_i, L_i]$ into (0).

Let V be a finite dimensional vector space over K. Let $\mathfrak{gl}(V)$ be the algebra of all endomorphisms of V and let $GL(V)$ be the group of all automorphisms of V. Following Chevalley ([2], p. 171), a Lie subalgebra L of $\mathfrak{gl}(V)$ is called algebraic provided that L is the Lie algebra of an algebraic subgroup of $GL(V)$. For an element x of $\mathfrak{gl}(V)$ let $\mathfrak{g}(x)$ be the smallest algebraic Lie subalgebra of $\mathfrak{gl}(V)$ containing x, that is, the set of all replicas of x ([2], p. 180). Then L is algebraic if and only if $\mathfrak{g}(x) \subset L$ for any element x of L ([2], p. 181). We denote by L^* the algebraic hull of L, that is, the smallest algebraic Lie subalgebra of $\mathfrak{gl}(V)$ containing L.

It is known that the derivation algebra $D(H)$ of any Lie algebra H is an algebraic Lie subalgebra of $\mathfrak{gl}(H)$ ([2], p. 179).

For an element x of $\mathfrak{gl}(V)$ such that $[x, L] \subset L$, the endomorphism of $L: y \to [x, y]$ is a derivation of L, which we denote by $\mathrm{ad}_L x$. For a subset M of $\mathfrak{gl}(V)$ such that $[M, L] \subset L$, we denote by $\mathrm{ad}_L M$ the set of all $\mathrm{ad}_L x$ with x in M. By the fact that the set of all elements y of $\mathfrak{gl}(V)$ such that $[y, L] \subset L$ is an algebraic Lie algebra, we have

$$\mathrm{ad}_L \mathfrak{g}(x) \subset D(L) \qquad \text{and} \qquad \mathrm{ad}_L L^* \subset D(L).$$

$I(L)$ is $\mathrm{ad}_L L$ which will sometimes be denoted by $\mathrm{ad}\, L$ simply.

In [15] we have shown the following facts. Let L be a Lie subalgebra of $\mathfrak{gl}(V)$. Then:

(2) For any element x of $\mathfrak{gl}(V)$ such that $[x, L] \subset L$,

$$\mathrm{ad}_L \mathfrak{g}(x) = \mathfrak{g}(\mathrm{ad}_L x)$$

([15], p. 303).

(3) L^* is the linear space spanned by all $\mathfrak{g}(x)$ with x in L ([15], p. 297).

By using these facts we can prove

Lemma 1. *Let L be a Lie subalgebra of $\mathfrak{gl}(V)$ and let H be a subalgebra of L. Then*

$$(\mathrm{ad}_L H)^* = \mathrm{ad}_L H^*.$$

Proof. By (3) $(\mathrm{ad}_L H)^*$ is spanned by all $\mathfrak{g}(\mathrm{ad}_L x)$ with x in H. It follows from (2) that $(\mathrm{ad}_L H)^*$ is spanned by all $\mathrm{ad}_L\, \mathfrak{g}(x)$ with x in H. By using (3) and the linearity of ad_L, we see that $(\mathrm{ad}_L H)^*$ is equal to $\mathrm{ad}_L H^*$.

We finally recall some properties of linear Lie algebras:

(4) For a Lie subalgebra L of $\mathfrak{gl}(V)$, the radical of L^* is the algebraic hull of the radical of L. L is algebraic if and only if the radical is algebraic ([3], p. 129).

(5) Let L be a solvable algebraic Lie subalgebra of $\mathfrak{gl}(V)$ and let N be the ideal consisting of all nilpotent elements of L. Then for any maximal abelian subalgebra A of semisimple elements of L,

$$L=A+N, \qquad A\cap N=(0)$$

([3], p. 130).

(6) Let L be a Lie subalgebra of $\mathfrak{gl}(V)$ with radical R. For any maximal abelian subalgebra A of semisimple elements of L, there exists a maximal semisimple subalgebra S of L such that

$$A=A\cap S+A\cap R$$

where $A\cap S$ is a Cartan subalgebra of S, $A\cap R$ is a maximal abelian subalgebra of semisimple elements of R and $[S, A\cap R]=(0)$ ([13], p. 211).

2. Derivations which are inner and central

In this section we study the derivations of a Lie algebra which are at the same time inner and central. We shall prove

LEMMA 2. *Let L be a Lie algebra over K. Let R be the radical and Z be the center of L. Then*:

(1) *$I(L)\cap C(L)=\mathrm{ad}_L Z_1$, where Z_1 is the set of all elements x of L such that $[x, L]\subset Z$.*

(2) *$I(L)\cap C(L)\subset I(L)^*\cap C(L)\subset \mathfrak{N}$, where $\mathfrak{N}$ is the ideal of all nilpotent elements of $(\mathrm{ad}_L R)^*$. In particular, if L is a Lie subalgebra of $\mathfrak{gl}(V)$, $\mathfrak{N}=\mathrm{ad}_L N$ with N the ideal of all nilpotent elements of R^*.*

PROOF. (1): If $Z=(0)$, then $C(L)=(0)$ and $Z_1=Z=(0)$. If $L=[L, L]$, then $C(L)=(0)$ and

$$[Z_1, L]=[Z_1, [L, L]]=(0),$$

that is, $Z_1=Z$. Therefore in these cases, $I(L)\cap C(L)=\mathrm{ad}_L Z_1=(0)$.

When $Z\neq(0)$ and $L\neq[L, L]$, for an element x of L $\mathrm{ad}_L x$ is in $C(L)$ if and only if $[x, L]\subset Z$, that is, x is in Z_1. Therefore $I(L)\cap C(L)=\mathrm{ad}_L Z_1$.

(2): By Ado's theorem, any Lie algebra over K has a faithful representation. Hence we may assume that L is a Lie subalgebra of $\mathfrak{gl}(V)$.

By (4) in §1, R^* is the radical of L^*. Let A be a maximal abelian subalgebra of semisimple elements of R^*. Then, by (5) and (6) in §1, there exists a maximal semisimple subalgebra S of L^* such that

$$L^* = S + R^*, \qquad R^* = A + N,$$

$$S \cap R^* = (0), \qquad A \cap N = (0), \qquad [S, A] = (0).$$

From the facts that $[L^*, L^*]=[L, L]$ and that $S=[S, S]$, it follows that S is a maximal semisimple subalgebra of L. Consequently we have $L=S+R$ and therefore

$$[L, L] = S + [L, R].$$

Now let D be any element of $I(L)^* \cap C(L)$. Since $I(L)^* = \mathrm{ad}_L L^*$ by Lemma 1, we have

$$D = \mathrm{ad}_L(s + a + n) \qquad \text{with } s \text{ in } S, a \text{ in } A \text{ and } n \text{ in } N.$$

Since D is central, $D[L, L]=(0)$ and therefore

$$DS = (0) \qquad \text{and} \qquad D[L, R] = (0).$$

It follows that

$$(\mathrm{ad}_L s)S = -\mathrm{ad}_L(a + n)S \subset S \cap R^* = (0),$$

whence $[s, S]=(0)$ and therefore $s=0$. Thus $D=\mathrm{ad}_L(a+n)$.

Since the set of all elements x of $\mathfrak{gl}(V)$ such that $[x, R] \subset [L, R]$ is an algebraic Lie subalgebra of $\mathfrak{gl}(V)$, we have $[L^*, R]=[L, R]$. Hence for any element y of L^*

$$(\mathrm{ad}_L y)[L, R] \subset [L^*, R] = [L, R].$$

Thus we see that $[L, R]$ is stable under $\mathrm{ad}_L a$ and $\mathrm{ad}_L n$. Since $D[L, R]=(0)$, if follows that

$$\mathrm{ad}_{[L, R]} a = -\mathrm{ad}_{[L, R]} n.$$

It is known that $\mathrm{ad}_{[L,R]} a$ is semisimple with a and that $\mathrm{ad}_{[L,R]} n$ is nilpotent with n. Consequently we have $\mathrm{ad}_{[L,R]} a=0$. It follows that

$$(\mathrm{ad}_L a)^2 L \subset (\mathrm{ad}_L a)\,[L, L] = [a, S + [L, R]] = (0).$$

But $\mathrm{ad}_L a$ is semisimple with a and therefore $\mathrm{ad}_L a = 0$. It follows that $D = \mathrm{ad}_L n$. Thus $I(L)^* \cap C(L) \subset \mathrm{ad}_L N$.

By Lemma 1, we see that $(\mathrm{ad}_L R)^* = \mathrm{ad}_L R^*$. Suppose that $\mathrm{ad}_L x$ with x in R^* is nilpotent. x is decomposed into the Jordan sum, that is, x is uniquely expressed in such a way that

$$x = x_s + x_n, \qquad [x_s, x_n] = 0$$

where x_s is semisimple and x_n is nilpotent ([2], p. 71). The components x_s and x_n of x are contained in $\mathfrak{g}(x)$ ([2], p. 184) and therefore in R^*. It is evident that

$$\mathrm{ad}_L x = \mathrm{ad}_L x_s + \mathrm{ad}_L x_n$$

is the Jordan sum decomposition of $\mathrm{ad}_L x$. Therefore

$$\mathrm{ad}_L x = \mathrm{ad}_L x_n \subset \mathrm{ad}_L N.$$

Hence $\mathfrak{N} \subset \mathrm{ad}_L N$. Since for any element n of N $\mathrm{ad}_L n$ is nilpotent with n, we have $\mathrm{ad}_L N \subset \mathfrak{N}$, whence $\mathfrak{N} = \mathrm{ad}_L N$.

Thus the proof is complete.

As a consequence of the lemma we have

COROLLARY. *Let L be a non-abelian nilpotent Lie algebra. Then $I(L) \cap C(L) \neq (0)$.*

PROOF. For a non-abelian nilpotent Lie algebra L, we have $Z_1 \neq Z$ with the set Z_1 defined in the lemma above. Hence the assertion of Corollary follows from the lemma.

3. Lie algebras whose central derivations are inner

In this section we shall study the Lie algebras L such that $C(L) \subset I(L)$ and more generally the Lie algebras L such that $C(L) \subset I(L)^*$.

We start with

LEMMA 3. *Let L be a Lie algebra whose radical is abelian. If $C(L) \subset I(L)^*$, then either the center of L is (0) or $L = [L, L]$.*

PROOF. Any Lie algebra over K has a faithful representation and therefore we may assume that L is a Lie subalgebra of $\mathfrak{gl}(V)$. Let R be the radical of L and let S be a maximal semisimple subalgebra of L. Then, since S is algebraic, by (4) in §1 we have

$$L=S+R \qquad \text{and} \qquad L^*=S+R^*.$$

Suppose that the center Z of L is not (0) and $L\neq[L, L]$. Since $[L, L]=S+[L, R]$, it follows that $R\neq[L, R]$. Choose a subspace U of R such that

$$R=U+[L, R], \qquad U\cap[L, R]=(0).$$

Define a non-zero endomorphism D of L such that

$$DU\subset Z \qquad \text{and} \qquad D(S+[L, R])=(0).$$

Then D is a central derivation of L.

By Lemma 1 and our assumption, we have $C(L)\subset \mathrm{ad}_L L^*$. Consequently we have

$$D=\mathrm{ad}_L(s+r) \qquad \text{with } s \text{ in } S \text{ and } r \text{ in } R^*.$$

Since $DS=(0)$, it immediately follows that $s=0$. Therefore $D=\mathrm{ad}_L r$ and we have

$$DU\subset[R^*, R]=[R, R]=(0),$$

which contradicts our definition of D.

Therefore we see that $Z=(0)$ or $L=[L, L]$. Thus the proof of the lemma is complete.

We can now prove the following

THEOREM 1. *Let L be a Lie algebra over a field of characteristic* 0. *If the center of L is not* (0) *and if* $C(L)\subset I(L)^*$, *then the radical R of L contains no elements x such that* $\mathrm{ad}_L x$ *is semisimple and non-zero. If furthermore* $C(L)\neq(0)$, *then R is not abelian.*

PROOF. If $L=[L, L]$, then $C(L)=(0)$ and $R=[L, R]$. From the fact that all derivations of L map R into the nilpotent radical, it follows that R is nilpotent. For any element x of R, $\mathrm{ad}_L x$ is nilpotent, for $\mathrm{ad}_R x$ is nilpotent and

$$(\mathrm{ad}_L x)^n L\subset(\mathrm{ad}_R x)^{n-1} R.$$

Therefore the assertion of the theorem is true in this case.

Now assume that $L \neq [L, L]$. Since L has a faithful representation, we may assume that L is a Lie subalgebra of $\mathfrak{gl}(V)$. By Lemma 1 we have $I(L)^* = \mathrm{ad}_L L^*$.

Suppose that there exists an element x of R such that $\mathrm{ad}_L x$ is not 0 and is semisimple. Decompose x into the Jordan sum $x = x_s + x_n$. Then x_s and x_n are contained in R^*. It is evident that

$$\mathrm{ad}_L x = \mathrm{ad}_L x_s + \mathrm{ad}_L x_n$$

is the Jordan sum decomposition of $\mathrm{ad}_L x$. Since $\mathrm{ad}_L x$ is semisimple by our supposition, it follows that $\mathrm{ad}_L x_n = 0$, which means that $[x_n, L] = (0)$. By the fact that the set of all elements y of $\mathfrak{gl}(V)$ such that $[(x_n), y] = (0)$ is an algebraic Lie subalgebra of $\mathfrak{gl}(V)$ containing L, we see that

$$[x_n, L^*] = (0).$$

Take a maximal abelian subalgebra A of semisimple elements of the radical R^* of L^* containing x_s. Then by (5) and (6) in §1, there exists a semisimple subalgebra S of L^* such that

$$L^* = S + R^*, \qquad R^* = A + N,$$

$$S \cap R^* = (0), \qquad A \cap N = (0), \qquad [S, A] = (0),$$

where N is the ideal of all nilpotent elements of R^*. S is really a subalgebra of L since $[L^*, L^*] = [L, L]$ and $S = [S, S]$. It follows that

$$\mathrm{ad}_L(S + A)x = [S + A, x_s] + [S + A, x_n] = (0).$$

x is not contained in $[L, R]$, since for any element y of $[L, R]$ $\mathrm{ad}_L y$ is nilpotent. $\mathrm{ad}_L(S + A)$ is completely reducible and maps respectively R and $[L, R]$ into themselves. Therefore there exists a subspace R_1 of R containing $[L, R]$ such that

$$R = (x) + R_1, \qquad (x) \cap R_1 = (0), \qquad \mathrm{ad}_L(S + A)R_1 \subset R_1.$$

R_1 is obviously a subalgebra of R.

Choosing a non-zero element z of the center Z of L, we define an endomorphism D of L in the following way:

$$Dx = z, \qquad D(S + R_1) = (0).$$

From the facts that $L=S+R$ and that $[L, L]\subset S+R_1$, it follows that D is a central derivation of L. Taking account of the assumption that $C(L)\subset I(L)^*$, by Lemma 2 we see that

$$D=\mathrm{ad}_L n \qquad \text{with } n \text{ in } N.$$

Let $\tilde{Z}$ be the center of L^*. Then $Z\subset\tilde{Z}$ and $N\cap\tilde{Z}$ is stable under $\mathrm{ad}_{L^*}A$. Since $\mathrm{ad}_{L^*}A$ is completely reducible, there exists a subspace U of N such that

$$N=U+(N\cap\tilde{Z}), \qquad U\cap(N\cap\tilde{Z})=(0), \qquad (\mathrm{ad}_{L^*}A)U\subset U.$$

It follows that

$$n=u+z \qquad \text{with } u \text{ in } U \text{ and } z \text{ in } N\cap\tilde{Z}.$$

We now have on the one hand

$$Dx=[n, x]=[n, x_s]\in[A, N]\cap Z\subset N\cap\tilde{Z}$$

and on the other hand

$$Dx=[n, x_s]=[u+z, x_s]=[u, x_s]\in U.$$

Consequently $Dx=0$, which contradicts the definition of D.

Thus we conclude that R contains no elements x such that $\mathrm{ad}_L x$ is not 0 and is semisimple.

To prove the second assertion of the theorem, assume that the center $Z\neq(0)$ and $C(L)\subset I(L)^*$. If R is abelian, then by Lemma 3 we see that $L=[L, L]$. Therefore $C(L)=(0)$.

The proof of the theorem is complete.

A Lie subalgebra L of $\mathfrak{gl}(V)$ is called splittable provided that for any element x of L the semisimple component x_s is always contained in L [12]. Then any algebraic Lie subalgebra of $\mathfrak{gl}(V)$ is splittable ([13], p. 184).

By using Theorem 1 we can show the following

THEOREM 2. *Let L be a Lie algebra over a field of characteristic 0 such that $I(L)$ is splittable. If the center of L is not (0) and if $C(L)\subset I(L)$, the radical of L is nilpotent.*

PROOF. Let R be the radical of L. Then $\mathrm{ad}_L R$ is the radical of $I(L)$. From our assumption that $I(L)$ is splittable, it follows that $\mathrm{ad}_L R$ is also splittable ([15], p. 292). Theorem 1 tells us that $\mathrm{ad}_L R$ contains no non-zero semisimple elements. Therefore $\mathrm{ad}_L R$ consists of nilpotent elements. Thus R is nilpotent and the proof is complete.

As a consequence of Theorem 2 we have the following corollary which was proved by Leger ([10], p. 642).

COROLLARY. *If the center of L is not* (0) *and if* $D(L)=I(L)$, *then L is not solvable and the radical of L is nilpotent.*

PROOF. Since $D(L)$ is algebraic, $I(L)$ is algebraic and therefore splittable. By Theorem 2 the radical is nilpotent.

If L is solvable, then L is nilpotent and therefore it has an outer derivation by Schenkman's theorem (see [8]). Thus L is not solvable, completing the proof.

It is to be noted that, if $I(L)$ is splittable and if the center of L is not (0) and $C(L)\subset I(L)^*$, then $I(L)$ is algebraic. In fact, by using Theorem 1 and the splittability of $I(L)$, we see that $\mathrm{ad}_L R$ consists of nilpotent elements and therefore it is algebraic. From (4) in §1, it follows that $I(L)$ is algebraic.

REMARK 1. There exists a Lie algebra which satisfies the assumption in Theorem 2, but which does not satisfy the assumption in its corollary. Let L be the Lie algebra over K described in terms of a basis $x_1, x_2, \cdots, x_8$ by the following table:

$$[x_1, x_2]=x_3, \quad [x_1, x_3]=x_4, \quad [x_1, x_4]=x_5, \quad [x_1, x_5]=x_6,$$

$$[x_1, x_6]=x_8, \quad [x_1, x_7]=x_8, \quad [x_2, x_3]=x_5, \quad [x_2, x_4]=x_6,$$

$$[x_2, x_5]=x_7, \quad [x_2, x_6]=2x_8, \quad [x_3, x_4]=-x_7+x_8, \quad [x_3, x_5]=-x_8,$$

$$[x_i, x_j]=0 \quad \text{for} \quad i+j>8.$$

This was given in [1], p. 123, as an example of characteristically nilpotent Lie algebras. The center is (x_8). $I(L)$ is splittable since $\mathrm{ad}_L x_i$ is nilpotent for each i. Let D be a derivation of L and put

$$Dx_i=\sum_{j=1}^{8}\lambda_{ij}x_j \qquad (i=1, 2, \cdots, 8).$$

Then after calculation we obtain

$$\lambda_{ij}=0 \quad \text{for} \quad i\geqq j, \qquad \lambda_{12}=\lambda_{24}=\lambda_{36}=\lambda_{67}=0,$$

$$-\lambda_{13}=\lambda_{35}=\lambda_{46}=\lambda_{57}, \qquad \lambda_{14}=\lambda_{25}=\lambda_{47}=-\lambda_{48}=\lambda_{58},$$

$$\lambda_{15} = -\lambda_{37}, \qquad \lambda_{23} = \lambda_{34} = \lambda_{45} = \lambda_{56} = \lambda_{78},$$

$$\lambda_{38} = -2\lambda_{16} + \lambda_{26} + \lambda_{27}, \qquad \lambda_{68} = \lambda_{23} - 2\lambda_{13}.$$

Therefore the matrix of D is

$$\begin{pmatrix}
0 & 0 & \lambda_{13} & \lambda_{14} & \lambda_{15} & \lambda_{16} & \lambda_{17} & \lambda_{18} \\
 & 0 & \lambda_{23} & 0 & \lambda_{14} & \lambda_{26} & \lambda_{27} & \lambda_{28} \\
 & & 0 & \lambda_{23} & -\lambda_{13} & 0 & -\lambda_{15} & -2\lambda_{16}+\lambda_{26}+\lambda_{27} \\
 & & & 0 & \lambda_{23} & -\lambda_{13} & \lambda_{14} & -\lambda_{14} \\
 & & & & 0 & \lambda_{23} & -\lambda_{13} & \lambda_{14} \\
 & 0 & & & & 0 & 0 & \lambda_{23}-2\lambda_{13} \\
 & & & & & & 0 & \lambda_{23} \\
 & & & & & & & 0
\end{pmatrix}$$

From this the matrix of an inner derivation is obtained by putting

$$\lambda_{26} = \lambda_{15}, \qquad \lambda_{27} = \lambda_{16}, \qquad \lambda_{17} = 0$$

and the matrix of a central derivation is obtained by putting

$$\text{all } \lambda_{ij} = 0 \qquad \text{except } \lambda_{18} \text{ and } \lambda_{28}.$$

Thus $D(L)$ is 10 dimensional, $I(L)$ is 7 dimensional, and $C(L)$ is 2 dimensional and contained in $I(L)$.

4. Lie algebras whose inner derivations are central

In this section we shall study the Lie algebras whose inner derivations are all inner. We shall prove the following

THEOREM 3. *Let L be a Lie algebra over a field of characteristic* 0. *Then*:

(1) $I(L) \subset C(L)$ *if and only if* $I(L)^* \subset C(L)$, *and if and only if* $L^3 = (0)$.

(2) *Assume that the center Z of L is not* (0). *Then* $I(L) = C(L)$ *if and only if* $I(L)^* = C(L)$, *and if and only if* $L^2 = Z$ *and* $\dim Z = 1$.

(3) $D(L) = C(L)$ *if and only if L is abelian.*

PROOF. (1): $I(L) \subset C(L)$ means that $L^2 \subset Z$, that is, that $L^3 = (0)$. If

$L^3=(0)$, $I(L)$ consists of nilpotent elements and therefore $I(L)$ is algebraic, whence $I(L)^* \subset C(L)$.

(2): If $I(L)=C(L)$ (resp. $I(L)^*=C(L)$), then by (1) $L^3=(0)$. It follows that $I(L)$ is algebraic. Hence the first two conditions are equivalent.

Now suppose that $I(L)=C(L)$. Then $L^2 \subset Z$. If $L^2 \neq Z$, then L is the direct sum of a non-zero central ideal Z_1 and an ideal L_1 containing L^2. The identity mapping of Z_1 can be trivially extended to the derivation of L which we denote by D. Then D is central, but not inner. This contradicts our supposition. Thus we see that $L^2=Z$. By the facts that

$$\dim I(L) = \dim L/Z \qquad \text{and} \qquad \dim C(L) = \dim L/L^2 \times \dim Z,$$

we have $\dim Z=1$.

Conversely suppose that $L^2=Z$ and $\dim Z=1$. Then $I(L) \subset C(L)$. By using the formulas given above on the dimensions of $I(L)$ and $C(L)$, it is immediate that $\dim I(L)=\dim C(L)$. Therefore we have $I(L)=C(L)$.

(3): Suppose that $D(L)=C(L)$. If $Z=(0)$, we have $C(L)=(0)$ and therefore $D(L)=(0)$, whence $L=(0)$. Therefore we may assume that $Z\neq(0)$. Then by (1) it follows that $L^3=(0)$. If $L^2\neq(0)$, we take a subspace $U\neq(0)$ such that

$$L = U + L^2, \qquad U \cap L^2 = (0).$$

The identity mapping of U can be extended to an endomorphism D of L. Then D is a non-central derivation of L, which contradicts our supposition. Therefore $L^2=(0)$. The converse is evident.

Thus the proof of the theorem is complete.

Remark 2. There exists a Lie algebra satisfying the condition in the statement (2) of Theorem 3. The three dimensional Lie algebra over K with a basis x_1, x_2, x_3 such that

$$[x_1, x_2] = x_3, \qquad [x_1, x_3] = [x_2, x_3] = 0$$

is an example of such a Lie algebra.

5. Lie algebras with as few derivations as possible

In this section we are concerned with the Lie algebras which have as few derivations as possible, that is, the Lie algebras L such that $D(L)=I(L)+C(L)$.

We first prove the following

THEOREM 4. *Let L be a Lie algebra which is the direct sum of the ideals $L_i (i=1, 2, \cdots, n)$. Then $D(L)=I(L)+C(L)$ if and only if $D(L_i)=I(L_i)+C(L_i)$ for each i.*

PROOF. By (1) in §1 we see that

$$D(L)=\sum_{i=1}^{n} D(L_i)+\sum_{i\neq j} D(L_i, L_j)$$

and that for $i\neq j$ $D(L_i, L_j)\subset C(L)$. Therefore if $D(L_i)=I(L_i)+C(L_i)$ for each i, it follows that $D(L)=I(L)+C(L)$.

Conversely suppose that $D(L)=I(L)+C(L)$. Any derivation D_i of L_i is trivially extended to a derivation D of L. Therefore we have

$$D=\mathrm{ad}_L x+\bar{D}$$

where $x=\sum_{j=1}^{n} x_j$ with x_j in L_j and $\bar{D}$ is in $C(L)$. Denote by $\bar{D}_i$ the restriction of $\bar{D}$ to L_i. Then

$$D_i=\mathrm{ad}_{L_i} x_i+\bar{D}_i.$$

It follows that

$$\bar{D}_i L_i=(D_i-\mathrm{ad}_{L_i} x_i)L_i\subset L_i\cap Z=Z_i,$$

where Z and Z_i are the centers of L and L_i respectively. Hence $\bar{D}_i$ is in $C(L_i)$ and therefore $D(L_i)=I(L_i)+C(L_i)$.

Thus the proof is complete.

Following Leger ([10], p. 643), a nilpotent Lie algebra L is called quasi-cyclic provided that L has a subspace U such that $L=U+[L, L]$ with $U\cap [L, L]=(0)$ and such that L is the direct sum of the spaces U^i where $U^1=U$ and $U^i=[U, U^{i-1}]$ for $i\geq 2$. We remark that any Lie algebra L such that $L^3=(0)$ is quasi-cyclic.

We shall prove the following

THEOREM 5. *Let L be a Lie algebra over a field of characteristic 0. If $D(L)=I(L)+C(L)$, then the radical is either non-quasi-cyclic or an abelian direct summand of L.*

To prove the theorem, we begin with recalling some known facts. Let

R be the radical of L and let $L=S+R$ be a Levi decomposition of L. Let $\mathfrak{A}(S)$ be the set of all derivations of L which map S into (0). Then $\mathfrak{A}(S)$ is a subalgebra of $D(L)$ and it is known ([7], p. 692) that

$$D(L)=I(L)+\mathfrak{A}(S).$$

Since R is stable under all derivations of L, for any derivation D of L its restriction to R is the derivation of R, which we denote by $\rho(D)$. Then ρ is a homomorphism of $D(L)$ into $D(R)$ and induces an isomorphism of $\mathfrak{A}(S)$ onto $\rho(\mathfrak{A}(S))$.

We here consider the set of all derivations of $\mathfrak{A}(S)$ which map R into the center of R. We denote the set by $\mathfrak{A}_0(S)$. Then $\mathfrak{A}_0(S)$ is a subalgebra of $\mathfrak{A}(S)$ such that

$$C(L)\subset\mathfrak{A}_0(S)\subset\mathfrak{A}(S).$$

We now show the following

LEMMA 4. (1) *$\rho(\mathfrak{A}(S))$ is the centralizer of $\mathrm{ad}_R S$ in $D(R)$.*

(2) $\rho(\mathfrak{A}(S)\cap I(L))=\rho(\mathfrak{A}(S))\cap I(R)$.

(3) $\rho(\mathfrak{A}(S)\cap I(L)^*)=\rho(\mathfrak{A}(S))\cap I(R)^*$.

(4) *$\rho(\mathfrak{A}_0(S))$ is the intersection of $\rho(\mathfrak{A}(S))$ and the centralizer of $I(R)$ in $D(R)$.*

PROOF. (1): If D is in $\mathfrak{A}(S)$, for any elements s of S and r of R we have

$$[\rho(D),\ \mathrm{ad}_R s]r=\rho(D)[s,r]-[s,\rho(D)r]$$
$$=[Ds,r]=0.$$

Hence $\rho(D)$ is contained in the centralizer of $\mathrm{ad}_R S$ in $D(R)$.

Conversely, let $\bar{D}$ be any element of the centralizer of $\mathrm{ad}_R S$ in $D(R)$. Define an endomorphism D of L in the following way:

$$Ds=0 \quad \text{for } s \text{ in } S \qquad \text{and} \qquad Dr=\bar{D}r \quad \text{for } r \text{ in } R.$$

Then D is a derivation of L, for

$$D[s,r]=D(\mathrm{ad}_R s)r=(\mathrm{ad}_R s)Dr$$
$$=[s,Dr]=[Ds,r]+[s,Dr].$$

Since $\bar{D}=\rho(D)$, $\bar{D}$ is contained in $\rho(\mathfrak{A}(S))$.

(2): This has been proved in [9] (p. 513). So we omit the proof.

(3): By considering a faithful representation of L, we may assume that L is a Lie subalgebra of $\mathfrak{gl}(V)$. Then by (4) in §1 R^* is the radical of L^* and $L^*=S+R^*$ is a Levi decomposition of L^*. By Lemma 1 we have

$$I(L)^* = \mathrm{ad}_L L^* \qquad \text{and} \qquad I(R)^* = \mathrm{ad}_R R^*.$$

Let $\mathrm{ad}_L x$ be any element of $\mathfrak{A}(S)\cap I(L)^*$. Since x is an element of L^*, x is expressed as the sum

$$x = s + r \qquad \text{with } s \text{ in } S \text{ and } r \text{ in } R^*.$$

Since $(\mathrm{ad}_L x)S = (0)$, if follows that

$$(\mathrm{ad}_L s)S = -(\mathrm{ad}_L r)S \subset S \cap R^* = (0).$$

Hence $[s, S]=(0)$ and $s=0$. Therefore x is in R^* and $\rho(\mathrm{ad}_L x)$ is contained in $I(R)^*$. Thus $\rho(\mathfrak{A}(S)\cap I(L)^*)\subset\rho(\mathfrak{A}(S))\cap I(R)^*$.

Conversely, let $\mathrm{ad}_R x$ be any element of $\rho(\mathfrak{A}(S))\cap I(R)^*$. Then $\mathrm{ad}_R x=\rho(D)$ with some D in $\mathfrak{A}(S)$. Since R^* and the center Z_1 of R^* are stable under $\mathrm{ad}_{R^*}S$ and since $\mathrm{ad}_{R^*}S$ is completely reducible, there exists a subspace U such that

$$R^* = U + Z_1, \qquad U \cap Z_1 = (0), \qquad (\mathrm{ad}_{R^*}S)U \subset U.$$

Since x is an element of R^*, x is expressed as the sum

$$x = u + z \qquad \text{with } u \text{ in } U \text{ and } z \text{ in } Z_1.$$

Since D is in $\mathfrak{A}(S)$, for any s in S and r in R we have

$$D[r, s] = [Dr, s]$$

and therefore $[u, [r, s]]=[[u, r], s]$. It follows that $[r, [s, u]]=0$. Therefore $[R, [s, u]]=(0)$, from which it follows that $[R^*, [s, u]]=(0)$. Thus $[s, u]$ is contained in Z_1 so that

$$[s, u] \in U \cap Z_1 = (0).$$

Thus $[s, u]=0$, which shows that $\mathrm{ad}_L u$ is in $\mathfrak{A}(S)$. Hence

$$\mathrm{ad}_R x = \mathrm{ad}_R u = \rho(\mathrm{ad}_L u) \in \rho(\mathfrak{A}(S) \cap I(L)^*).$$

Thus we see that $\rho(\mathfrak{A}(S)) \cap I(R)^* \subset \rho(\mathfrak{A}(S) \cap I(L)^*)$.

(4): Let D be any derivation in $\mathfrak{A}_0(S)$. Since $\mathfrak{A}_0(S) \subset \mathfrak{A}(S)$, $\rho(D)$ is in $\rho(\mathfrak{A}(S))$. Since $\rho(D)$ maps R into the center of R,

$$[\rho(D), \mathrm{ad}_R R] = \mathrm{ad}_R \rho(D) R = (0).$$

Therefore $\rho(D)$ is contained in the centralizer of $I(R)$ in $D(R)$.

Conversely, let $\bar{D}$ be any element of the intersection of $\rho(\mathfrak{A}(S))$ and the centralizer of $I(R)$ in $D(R)$. From the fact that $\bar{D}$ is in $\rho(\mathfrak{A}(S))$, it follows that there exists a derivation D in $\mathfrak{A}(S)$ such that $\rho(D) = \bar{D}$. By the fact that $\bar{D}$ is in the centralizer of $I(R)$ in $D(R)$, we see that D maps R into the center of R. Thus D is contained in $\mathfrak{A}_0(S)$ and therefore $\bar{D}$ is in $\rho(\mathfrak{A}_0(S))$.

Thus the lemma is proved.

By making use of Lemma 4 we can prove the following

LEMMA 5. *Let R be a solvable Lie algebra. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be semisimple subalgebras of $D(R)$ which are conjugate under an automorphism τ of $D(R)$ mapping $I(R)$ into itself. Let $L_1 = \mathfrak{S}_1 + R$ and $L_2 = \mathfrak{S}_2 + R$ be the semi-direct sums. Then $D(L_1) = I(L_1) + \mathfrak{A}_0(\mathfrak{S}_1)$ if and only if $D(L_2) = I(L_2) + \mathfrak{A}_0(\mathfrak{S}_2)$.*

PROOF. Let ρ_1 and ρ_2 be respectively the restriction homomorphisms of $D(L_1)$ and $D(L_2)$ into $D(R)$. By the definition of the semi-direct sum, $\mathrm{ad}_R \mathfrak{S}_1$ and $\mathrm{ad}_R \mathfrak{S}_2$ are identified with $\mathfrak{S}_1$ and $\mathfrak{S}_2$ as subalgebras of $D(R)$. From the facts that $D(L_i) = I(L_i) + \mathfrak{A}(\mathfrak{S}_i)$ and that $\mathfrak{A}_0(\mathfrak{S}_i) \subset \mathfrak{A}(\mathfrak{S}_i)$, it follows that $D(L_i) = I(L_i) + \mathfrak{A}_0(\mathfrak{S}_i)$ if and only if

$$\mathfrak{A}(\mathfrak{S}_i) = \mathfrak{A}(\mathfrak{S}_i) \cap I(L_i) + \mathfrak{A}_0(\mathfrak{S}_i) \qquad (i = 1, 2).$$

Therefore it is the case if and only if

$$\rho_i(\mathfrak{A}(\mathfrak{S}_i)) = \rho_i(\mathfrak{A}(\mathfrak{S}_i) \cap I(L_i)) + \rho_i(\mathfrak{A}_0(\mathfrak{S}_i)) \qquad (i = 1, 2).$$

Since τ maps $\mathrm{ad}_R \mathfrak{S}_1$ onto $\mathrm{ad}_R \mathfrak{S}_2$ and $I(R)$ onto itself, by using Lemma 4 we see that τ maps $\rho_1(\mathfrak{A}(\mathfrak{S}_1))$, $\rho_1(\mathfrak{A}(\mathfrak{S}_1) \cap I(L_1))$ and $\rho_1(\mathfrak{A}_0(\mathfrak{S}_1))$ onto $\rho_2(\mathfrak{A}(\mathfrak{S}_2))$, $\rho_2(\mathfrak{A}(\mathfrak{S}_2) \cap I(L_2))$ and $\rho_2(\mathfrak{A}_0(\mathfrak{S}_2))$ respectively. Thus we can conclude that $D(L_1) = I(L_1) + \mathfrak{A}_0(\mathfrak{S}_1)$ if and only if $D(L_2) = I(L_2) + \mathfrak{A}_0(\mathfrak{S}_2)$, completing the proof.

In virtue of Lemmas 4 and 5, we can now prove the following

LEMMA 6. *Let R be the radical of a Lie algebra L and let $L = S + R$ be a*

Levi decomposition of L. If R is quasi-cyclic and non-abelian, then $D(L)\neq I(L)+\mathfrak{A}_0(S)$.

PROOF. If L has a semisimple ideal, we denote by L_1 the largest semisimple ideal of L. Then L is the direct sum of L_1 and an ideal L_2 whose radical is R. From the fact that maximal semisimple subalgebras of L are conjugate to each other by Malcev's theorem, it follows that L_1 is contained in S. Therefore there exists a semisimple subalgebra S_2 such that

$$S=L_1+S_2 \qquad \text{and} \qquad L_2=S_2+R.$$

Since the center of L_1 is (0) and $L_1=[L_1, L_1]$, by using (1) in §1 we see that

$$D(L_1, L_2)=D(L_2, L_1)=(0)$$

and therefore that $D(L)=D(L_1)+D(L_2)$. Since $D(L_1)=I(L_1)$, it is sufficient to prove the assertion of the lemma for L_2. Thus we assume that L has no semisimple ideals.

Suppose that R is quasi-cyclic and non-abelian. Then there exists a subspace U such that

$$R=\sum_i U^i \qquad \text{with} \quad U^i\cap U^j=(0) \quad \text{for} \quad i\neq j.$$

We assume that $U^n\neq(0)$ and $U^{n+1}=(0)$. Then $n>1$ since R is not abelian. The identity mapping of U extends to the derivation of R, which we denote by $\bar{D}$. Then it is easy to see that U is the only subspace of R such that

$$R=U+R^2, \qquad U\cap R^2=(0) \qquad \text{and} \qquad \bar{D}U\subset U.$$

Take a maximal abelian subalgebra $\mathfrak{A}_1$ consisting of semisimple elements of $D(R)$ which contains $\bar{D}$. Then, by (6) in §1, there exists a maximal semisimple subalgebra $\mathfrak{S}$ of $D(R)$ such that

$$\mathfrak{A}_1=(\mathfrak{S}\cap\mathfrak{A}_1)+\mathfrak{A}$$

where $\mathfrak{A}$ is a maximal abelian subalgebra of semisimple elements of the radical of $D(R)$, and such that $[\mathfrak{S}, \mathfrak{A}]=(0)$. Hence $\mathfrak{S}+\mathfrak{A}$ is completely reducible and therefore there exists a subspace U' of R such that

$$R=U'+R^2, \qquad U'\cap R^2=(0) \qquad \text{and} \qquad (\mathfrak{S}+\mathfrak{A})U'\subset U'.$$

Since $\bar{D}$ is contained in $\mathfrak{S}+\mathfrak{A}$, it follows that $\bar{D}U'\subset U'$ and therefore that

$U'=U$. Thus we see that $\bar{D}$ is a scalar mapping on each U^i and U^i is stable under $\mathfrak{S}+\mathfrak{A}$, from which it follows that $[\bar{D}, \mathfrak{S}]=(0)$. Therefore $\bar{D}$ is contained in the centralizer of any subalgebra $\mathfrak{S}_1$ of $\mathfrak{S}$ in $D(R)$.

We now assert that, for the semi-direct sum $L_1=\mathfrak{S}_1+R$,

$$D(L_1)\neq I(L_1)+\mathfrak{A}_0(\mathfrak{S}_1).$$

In fact, suppose that $D(L_1)=I(L_1)+\mathfrak{A}_0(\mathfrak{S}_1)$. Then

$$\mathfrak{A}(\mathfrak{S}_1)=\mathfrak{A}(\mathfrak{S}_1)\cap I(L_1)+\mathfrak{A}_0(\mathfrak{S}_1)$$

and therefore

$$\rho_1(\mathfrak{A}(\mathfrak{S}_1))=\rho_1(\mathfrak{A}(\mathfrak{S}_1)\cap I(L_1))+\rho_1(\mathfrak{A}_0(\mathfrak{S}_1)),$$

where ρ_1 is the restriction homomorphism of $D(L_1)$ into $D(R)$. By Lemma 4 (1), $\rho_1(\mathfrak{A}(\mathfrak{S}_1))$ is the centralizer of $\mathfrak{S}_1$ in $D(R)$, where $\mathfrak{S}_1$ is identified with $\mathrm{ad}_R\mathfrak{S}_1$. Hence $\bar{D}$ is contained in $\rho_1(\mathfrak{A}(\mathfrak{S}_1))$. By Lemma 4 (2) $\bar{D}$ is expressed in the form

$$\bar{D}=\mathrm{ad}_R r+\tilde{D}$$

where r is in R and $\tilde{D}$ is a derivation of R mapping R into the center $\tilde{Z}$ of R. For any element u of U,

$$u=\bar{D}u=[r, u]+\tilde{D}u\in R^2+\tilde{Z},$$

whence

$$[U, U^{n-1}]\subset[R^2, U^{n-1}]\subset U^{n+1}=(0).$$

Therefore we see that $U^n=(0)$, which is a contradiction. Thus we have $D(L)\neq I(L_1)+\mathfrak{A}_0(\mathfrak{S}_1)$, as was asserted.

For a given Levi decomposition $L=S+R$, we see that ad_RS is a semisimple subalgebra of $D(R)$. Let $\mathfrak{S}'$ be a maximal semisimple subalgebra of $D(R)$ containing ad_RS. Then by Malcev-Harish-Chandra's theorem, there exists an automorphism τ of $D(R)$ which maps $\mathfrak{S}'$ onto $\mathfrak{S}$ and $I(R)$ onto itself. Let $\mathfrak{S}_1$ be the image of ad_RS under τ and let L_1 be the semi-direct sum $\mathfrak{S}_1+R$. Then, as shown above,

$$D(L_1)\neq I(L_1)+\mathfrak{A}_0(\mathfrak{S}_1).$$

Since L has no semisimple ideals, L can be considered as the semi-direct sum

$\mathrm{ad}_R S+R$. Hence we can use Lemma 5 to see that $D(L)\neq I(L)+\mathfrak{A}_0(S)$.
Thus the proof of the lemma is complete.

PROOF OF THEOREM 5. Suppose that the radical R of L is quasi-cyclic and not abelian. Take a Levi decomposition $L=S+R$. Then by Lemma 6 we see that

$$D(L)\neq I(L)+\mathfrak{A}_0(S).$$

Since $C(L)\subset\mathfrak{A}_0(S)$, it follows that $D(L)\neq I(L)+C(L)$.

We next suppose that R is abelian and not a direct summand of L. Then the center Z of L is a proper subalgebra of R. Let R_1 be the complementary subspace of Z in R. Then $R_1\neq(0)$. The identity mapping of R can be trivially extended to a derivation of L, which we denote by D. We assert that D is not contained in $I(L)+C(L)$. In fact, let $L=S+R$ be a Levi decomposition of L. If D is contained in $I(L)+C(L)$, then

$$D=\mathrm{ad}_L(s+r)+\tilde{D} \qquad \text{with } s \text{ in } S, r \text{ in } R_1 \text{ and } \tilde{D} \text{ in } C(L).$$

Since $DS=(0)$, it follows that $[s, S]=(0)$ and therefore that $s=0$. Then for any element r' of R_1 we have

$$r'=Dr'=\tilde{D}r'\in R_1\cap Z=(0),$$

which is a contradiction. Therefore D is not contained in $I(L)+C(L)$, as was asserted. Hence we have $D(L)\neq I(L)+C(L)$.

Thus we conclude that if $D(L)=I(L)+C(L)$ then either R is not quasi-cyclic or R is abelian and a direct summand of L. The proof of Theorem 5 is complete.

As an immediate consequence of Theorem 5 we have the following corollary, which was proved by Leger ([10], p. 643).

COROLLARY 1. *If the center of L is not* (0) *and if the nilpotent radical is quasi-cyclic, then* $D(L)\neq I(L)$.

PROOF. Suppose that the center $Z\neq(0)$ and $D(L)=I(L)$. It is evident that L is not the direct sum of a semisimple ideal and a central ideal. Hence by Theorem 5 the radical R is not quasi-cyclic. But by Corollary to Theorem 2 R is nilpotent. Thus the nilpotent radical is not quasi-cyclic, completing the proof.

Before giving another consequence of Theorems 4 and 5, we recall a notion of Lie algebras introduced in [17]. A Lie algebra L is called characteristically solvable provided that $D(L)$ is solvable and the center of L is contained in $[L, L]$.

Corollary 2. *Let L be a solvable Lie algebra such that* $D(L)=I(L)+C(L)$. *Then L is the direct sum of a central ideal and a characteristically solvable ideal which is not quasi-cyclic.*

Proof. Let Z be the center of L. Put $Z_1=Z\cap[L, L]$ and choose a complementary subspace Z_2 of Z_1 in Z. Take a subspace L_1 of L containing $[L, L]$ such that

$$L=L_1+Z_2, \qquad L_1\cap Z_2=(0).$$

Then L_1 is an ideal of L, the center of L_1 is Z_1 and $Z_1\subset[L_1, L_1]$.

By Theorem 4 we see that $D(L_1)=I(L_1)+C(L_1)$. For any derivations D, D' in $C(L_1)$ and for any element x of L_1, we have

$$\begin{aligned}[D, D']x=(DD'-D'D)x \in DZ_1-D'Z_1 \\ \subset D[L_1, L_1]-D'[L_1, L_1]=(0)\end{aligned}$$

and

$$[D, \mathrm{ad}_{L_1}x]=\mathrm{ad}_{L_1}Dx \in \mathrm{ad}_{L_1}Z_1=(0).$$

Hence $C(L_1)$ is a central ideal of $D(L_1)$. It follows that $D(L_1)$ is solvable. Thus L_1 is characteristically solvable. By Theorem 5 we also see that L_1 is not quasi-cyclic. Thus L is the direct sum of a characteristically solvable and non-quasi-cyclic ideal L_1 and of a central ideal Z_2. The proof is complete.

Remark 3. Let L be a Lie algebra whose radical R is not an abelian direct summand. Then Theorem 5 states:

(a) If $D(L)=I(L)+C(L)$, then R is not quasi-cyclic.

Corollary 1 to Theorem 5, Leger's result, is an easy consequence of Corollary to Theorem 2 and the following statement:

(b) If $D(L)=I(L)$, then R is not quasi-cyclic.

(b) is a special case of (a). We shall here note that there exists a nilpotent Lie algebra which is not quasi-cyclic, which satisfies the assumption of (a) and which does not satisfy the assumption of (b). Dixmier-Lister [5] gave the following nilpotent Lie algebra as an example of a characteristically nilpotent Lie algebra. Let H be the 8 dimensional Lie algebra over a field of characteristic 0 described in terms of a basis $x_1, x_2, \cdots, x_8$ by the following table:

$$[x_1, x_2] = x_5, \quad [x_1, x_3] = x_6, \quad [x_1, x_4] = x_7,$$

$$[x_1, x_5] = -x_8, \quad [x_2, x_3] = x_8, \quad [x_2, x_4] = x_6,$$

$$[x_2, x_6] = -x_7, \quad [x_3, x_4] = -x_5, \quad [x_3, x_5] = -x_7,$$

$$[x_4, x_6] = -x_8.$$

In addition $[x_i, x_j] = -[x_j, x_i]$ and for $i<j$ $[x_i, x_j]=0$ if it is not in the table above. It has been shown that $D(H)=I(H)+C(H)$, $C(H)$ is 8 dimensional and intersects $I(H)$ in a 2 dimensional space. Hence $D(H)\neq I(H)$. Since H has no semisimple derivations, H is not quasi-cyclic.

Remark 4. If L is a Lie algebra over K whose radical is an abelian direct summand, L is the direct sum of a semisimple ideal S and a central ideal. Therefore $D(L)\neq I(L)$. Since the center of S is (0) and $S=[S, S]$, by using (1) in §1 we see that $D(L)=I(L)+C(L)$.

Remark 5. We shall give an example of a Lie algebra L such that $D(L)\neq I(L)+C(L)$. Let L be the Lie algebra over a field of characteristic 0 described in terms of a basis $x_1, x_2, \cdots, x_5$ by the following table:

$$[x_1, x_2] = 2x_2, \quad [x_1, x_3] = -2x_3, \quad [x_2, x_3] = x_1,$$

$$[x_1, x_4] = -x_4, \quad [x_1, x_5] = x_5, \quad [x_2, x_4] = -x_5,$$

$$[x_3, x_5] = -x_4, \quad [x_2, x_5] = [x_3, x_4] = [x_4, x_5] = 0.$$

In addition $[x_i, x_j] = -[x_j, x_i]$. $S=(x_1, x_2, x_3)$ is a semisimple subalgebra and $R=(x_4, x_5)$ is the radical of L. After calculation we see that $C(L)=(0)$ and $D(L)$ contains a 1 dimensional space of outer derivations. Therefore $D(L)\neq I(L)+C(L)$. For an example of a Lie algebra such that $D(L)\neq I(L)+C(L)$ and $C(L)\neq(0)$, it suffices to take the direct sum of the Lie algebra above and of another abelian Lie algebra.

6. Lie algebras with radicals whose central derivations are inner and with radicals which have as few derivations as possible

In this section we are concerned with the Lie algebras L whose radicals R satisfy the conditions considered in the preceding sections. We can not always expect that L satisfies the corresponding conditions.

We shall first study the Lie algebras whose radicals R satisfy each of the conditions $C(R)\subset I(R)$, $C(R)\subset I(R)^*$, $C(R)\cap I(R)=(0)$ and $C(R)\cap I(R)^*=(0)$. Namely, we shall prove the following

THEOREM 6. *Let L be a Lie algebra over a field of characteristic* 0 *and let R be the radical of L. Then:*

(1) *If* $C(R)\subset I(R)$, *then* $C(L)\subset I(L)$.

(2) *If* $C(R)\subset I(R)^*$, *then* $C(L)\subset I(L)^*$.

(3) *If* $C(R)\cap I(R)=(0)$, *then* $C(L)\cap I(L)=(0)$.

(4) *If* $C(R)\cap I(R)^*=(0)$, *then* $C(L)\cap I(L)^*=(0)$.

PROOF. Let ρ be the restriction homomorphism of $D(L)$ into $D(R)$ and let $L=S+R$ be a Levi decomposition of L.

(1): Since $C(L)\subset\mathfrak{A}(S)$, we have $\rho(C(L))\subset\rho(\mathfrak{A}(S))$. From the fact that the center of L is contained in that of R, it follows that $\rho(C(L))\subset C(R)$. Therefore by the assumption we have

$$\rho(C(L))\subset\rho(\mathfrak{A}(S))\cap I(R).$$

Using Lemma 4 (2) we see that

$$\rho(C(L))\subset\rho(\mathfrak{A}(S)\cap I(L)).$$

Since the restriction of ρ to $\mathfrak{A}(S)$ is an isomorphism, we obtain that $C(L)\subset\mathfrak{A}(S)\cap I(L)$ and therefore $C(L)\subset I(L)$.

(3): By using Lemma 4 (2), we have

$$\rho(C(L)\cap I(L))\subset\rho(\mathfrak{A}(S)\cap I(L))=\rho(\mathfrak{A}(S))\cap I(R).$$

From the fact that $\rho(C(L))\subset C(R)$, it follows that

$$\rho(C(L)\cap I(L))\subset C(R)\cap I(R)=(0).$$

Since ρ is an isomorphism on $\mathfrak{A}(S)$, we have $C(L)\cap I(L)=(0)$.

The proofs of (2) and (4) can be given in a similar way as in the proofs of (1) and (3), by using Lemma 4 (3) instead of Lemma 4 (2). Therefore we omit the proofs.

REMARK 6. The converse of the statement (1) in Theorem 6 is not generally true. For example, let L be the Lie algebra given in Remark 5.

Since the center of L is (0) and since the radical R is abelian, we have $C(L)=(0)$ and $I(R)=(0)$. Hence $C(L)\subset I(L)$, but $C(R)\not\subset I(R)$.

Remark 7. We shall show, by example, that the converse of the statement (3) in Theorem 6 does not hold generally. Let L be the Lie algebra over a field of characteristic 0 described in terms of a basis $x_1, x_2, \cdots, x_6$ by the following table:

$$[x_1, x_2]=2x_2, \qquad [x_1, x_3]=-2x_3, \qquad [x_2, x_3]=x_1,$$

$$[x_1, x_4]=-x_4, \qquad [x_1, x_5]=x_5, \qquad [x_2, x_4]=-x_5,$$

$$[x_3, x_5]=-x_4, \qquad [x_4, x_5]=x_6.$$

In addition $[x_i, x_j]=-[x_j, x_i]$ and for $i<j$ $[x_i, x_j]=0$ if it is not in the table above. $R=(x_4, x_5, x_6)$ is the radical of L. Since $L=[L, L]$, we have $C(L)=(0)$ and therefore $C(L)\cap I(L)=(0)$. However $C(R)=I(R)$ and it is a 2 dimensional space, whence $C(R)\cap I(R)\neq(0)$.

Remark 8. We remark that if H is a Cartan subalgebra of L and if $C(H)\subset I(H)$, then L is solvable. In fact, there exists a Levi decomposition $L=S+R$ of L such that H is the sum of a Cartan subalgebra H_1 of S and a subalgebra $H\cap R$ and H_1 is a central ideal of H ([4], p. 18). If $H_1\neq(0)$, we have $I(H)H_1=(0)$. However there exists a central derivation of H which is the idendity mapping on H_1. Hence $C(H)\not\subset I(H)$, contradicting the assumption. Therefore we have $H_1=(0)$, whence $S=(0)$, that is, L is solvable.

$D(R)=I(R)$ if and only if $D(L)=I(L)$ and L is the direct sum of a semisimple ideal and the radical ([16], p. 74). If $I(R)\subset C(R)$ (resp. $D(R)=C(R)$), we can not assert that $I(L)\subset C(L)$ (resp. $D(L)=C(L)$), since $\mathrm{ad}_L S$ with S a semisimple subalgebra of L is not contained in $C(L)$.

In the rest of this section, we shall study the Lie algebras whose radicals have as few derivations as possible. Such Lie algebras do not necessarily have the same property. Let L be the Lie algebra given in Remark 5. Then $D(R)=C(R)$ for the radical R, but $D(L)\neq I(L)+C(L)$. L has further the properties that the center is (0) and that $L=[L, L]$.

However, for a Lie algebra L with radical R we can generally show that if $D(R)=I(R)+C(R)$, then L is the direct sum of an ideal L_1 with $D(L_1)=I(L)+C(L_1)$ and of an ideal L_2 with $D(L_2)\neq I(L_2)+C(L_2)$, and L_2 is a Lie algebra of such a type as noted above. It is the aim of this section to prove this fact.

We begin with the following

Lemma 7. *Let L be a Lie algebra and let R be the radical of L. If $D(R)=I(R)+C(R)$, then L is the direct sum of a characteristically solvable ideal L_1*

with $D(L_1)=I(L_1)+C(L_1)$ *and of an ideal* L_2 *whose radical is abelian.*

Proof. Let $L=S+R$ be a Levi decomposition of L and let Z be the center of R. Put $Z_1=Z\cap[R, R]$. Then Z and Z_1 are stable under all derivations of R. Since $\mathrm{ad}_R S$ is completely reducible, there exists a subspace Z_2 of Z such that

$$Z = Z_1 + Z_2, \qquad Z_1 \cap Z_2 = (0), \qquad (\mathrm{ad}_R S)Z_2 \subset Z_2.$$

Since R and $[R, R]$ are stable under $\mathrm{ad}_R S$, there exists a subspace R_1 of R containing $[R, R]$ such that

$$R = R_1 + Z_2, \qquad R_1 \cap Z_2 = (0), \qquad (\mathrm{ad}_R S)R_1 \subset R_1.$$

It follows that R_1 is an ideal of L.

We assert that R_1 is characteristically solvable. In fact, the center of R_1 contains Z_1 and is contained in Z. Hence it is equal to Z_1 and therefore contained in $[R_1, R_1]$. Therefore R_1 has no abelian direct summands. Since R is the direct sum of the ideals R_1 and Z_2, from our assumption and Theorem 4 it follows that $D(R_1)=I(R_1)+C(R_1)$. Therefore we can use Corollary 2 to Theorem 5 to see that R_1 is characteristically solvable, as was asserted.

Since $\mathrm{ad}_R S$ maps R_1 into R_1, $\mathrm{ad}_{R_1} S$ is a semisimple subalgebra of $D(R_1)$. By the fact that R_1 is characteristically solvable, we see that $\mathrm{ad}_{R_1} S=(0)$. That is, $[S, R_1]=(0)$.

Now we put $L_2=S+Z_2$. Then taking account of the fact that $[S, Z_2]\subset Z_2$, we see that L_2 is an ideal of L. Thus L is the direct sum of the ideals R_1 and L_2 satisfying the conditions in the statement of the lemma.

In virtue of Lemma 7 we can now prove the following

Theorem 7. *Let L be a Lie algebra over a field of characteristic* 0 *and let R be the radical of L. Then* $D(R)=I(R)+C(R)$ *if and only if L is the direct sum of the ideals* L_1 *and* L_2 *satisfying the following conditions*:

(1) L_1 *is the direct sum of a semisimple ideal, a central ideal and a characteristically solvable ideal* R_1 *with* $D(R_1)=I(R_1)+C(R_1)$.

(2) *The radical of* L_2 *is abelian, the center of* L_2 *is* (0) *and* $L_2=[L_2, L_2]$.

And then L_1 *and* L_2 *are characteristic ideals of L and*

$$D(L_1) = I(L_1) + C(L_1), \qquad D(L_2) \neq I(L_2) + C(L_2).$$

Proof. Suppose that $D(R) = I(R) + C(R)$. Let S_1 be the largest semisimple ideal of L. Then it is a direct summand of L. Hence L is the direct sum of S_1 and an ideal H whose radical is R. Using Lemma 7, we see that H is the direct sum of a characteristically solvable ideal R_1 with $D(R_1)=I(R_1)+C(R_1)$ and of an ideal H_1 whose radical A is abelian. Let $H_1=S+A$ be a Levi decomposition of H_1. We denote by Z the center of H_1. Since Z is stable under completely reducible mappings $\mathrm{ad}_A S$, there exists a subspace R_2 of A such that

$$A = Z + R_2, \qquad Z \cap R_2 = (0), \qquad (\mathrm{ad}_A S)R_2 \subset R_2.$$

We see that $S+R_2$ is an ideal of H_1 and therefore that H_1 is the direct sum of the ideals Z and $S+R_2$. Now we put

$$L_1 = S_1 + Z + R_1 \qquad \text{and} \qquad L_2 = S + R_2.$$

Then L_1 is an ideal of L and is the direct sum of ideals S_1, Z and R_1. L_2 is an ideal of L whose radical is abelian and whose center is (0).

We assert that $L_2=[L_2, L_2]$. In fact, if $L_2\neq[L_2, L_2]$, then from the fact that $[L_2, L_2]=S+[L_2, R_2]$ it follows that $R_2\neq[L_2, R_2]$. Both R_2 and $[L_2, R_2]$ are stable under the derivations of L_2. Hence there exists a subspace $U\neq(0)$ of R_2 such that

$$R_2 = U + [L_2, R_2], \qquad U\cap[L_2, R_2] = (0), \qquad (\mathrm{ad}_{R_2}S)U\subset U.$$

It follows that

$$[S, U]\subset U\cap[L_2, R_2] = (0).$$

Therefore U is contained in the center of L_2. This is a contradiction since the center of L_2 is (0). Hence we see that $L_2=[L_2, L_2]$, as was asserted.

Thus L is the direct sum of the ideals L_1 and L_2 satisfying the conditions (1) and (2).

Conversely, if L is such a direct sum of ideals, then R is the direct sum of a solvable ideal R_1 with $D(R_1)=I(R_1)+C(R_1)$ and of a central ideal. By using Theorem 3 (3) and Theorem 4 we see that $D(R)=I(R)+C(R)$.

Since the center of L_2 is (0) and since $L_2=[L_2, L_2]$, by using (1) in §1 we see that

$$D(L_1, L_2) = D(L_2, L_1) = (0)$$

and therefore that $D(L)=D(L_1)+D(L_2)$. It follows that L_1 and L_2 are stable under all derivations of L.

Since $D(S_1)=I(S_1)$ and since $D(Z)=C(Z)$, by Theorem 4 we obtain that $D(L_1)=I(L_1)+C(L_1)$. As for L_2, the radical R_2 is abelian and not a direct summand. Therefore we can use Theorem 5 to see that $D(L_2) \neq I(L_2)+C(L_2)$.

The proof of the theorem is complete.

It is to be noted that the Lie algebra referred to in Remark 3 gives an example of characteristically solvable Lie algebras L such that $D(L)=I(L)+C(L)$.

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