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Approximate Computation of Errors in Numerical Integration of Ordinary Differential Equations by One-step Methods

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1. Introduction

Given a differential equation

$$y' = f(x, y), \tag{1.1}$$

where $f(x, y)$ is assumed to be a sufficiently smooth function.

In numerous papers [1-16][1], various methods are obtained for bounding or approximating the errors in numerical integration of (1.1) by one-step methods with the aids of the functions that bound or approximate the function $f_y(x, y)$, the truncation error and so on.

To avoid the use of such functions for practical purposes, in this paper, n steps of integration with a fixed step-size are considered as one step and a simple method is obtained for approximating the errors without computing explicitly any function other than $f(x, y)$. The method is illustrated by two numerical examples.

Since usually the step-size is not changed so often and the estimate of the error is not always necessary for each step of integration, it will not be a serious restriction to fix the step-size for the n steps of integration, and this method may be used as an integration method with a check on the accuracy of the numerical solution.

2. Preliminaries

In this paragraph and the next, we state five lemmas and one theorem without proof, but they are proved in paragraph 6.

In the sequel, for simplicity, we assume that $f(x, y)$ is defined, continuous and bounded in the strip

$$S : x_0 \leqq x \leqq x_0 + nh_0, \quad |y| < \infty,$$

and that the partial derivatives of $f(x, y)$ up to the necessary order, say s,

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1) Numbers in square brackets refer to the references listed at the end of this paper.

exist and are continuous and bounded in S, where n is a positive integer and h_0 is a positive number.[2)]

For any u and v such that $x_0 \leqq u \leqq x_0+(n-1)h_0$ and $|v|<\infty$, let $y(x; u, v)$ be the solution of (1.1) such that $y(u; u, v)=v$, and $\Phi(u, v; h)$ be the increment function of any one-step method of order p $(p \geqq 1)$ for approximating $y(u+h; u, v)$ $(0<h \leqq h_0)$ [6]. Then $y(u+h; u, v)$ can be written as follows:

$$y(u+h; u, v) = v + h\Phi(u, v; h) - T(u, v; h) \tag{2.1}$$

$$= v + h\Delta(u, v; h),$$

where $T(u, v; h)$ is a truncation error,

$$T(u, v; h) = O(h^{p+1})^{3)}, \tag{2.2}$$

and $\Delta(u, v; h)$ is the exact relative increment function [6].

We are concerned with the solution

$$y(x) = y(x; x_0, y_0) \tag{2.3}$$

and consider the case where the equation (1.1) is integrated numerically from x_0 to x_0+nh with step-size h $(0<h \leqq h_0)$. Hence we put

$$x_{j+1} = x_0 + (j+1)h, \tag{2.4}$$

$$p_{j+1} = \Phi(x_j, y_j; h), \tag{2.5}$$

$$y_{j+1} = y_j + hp_{j+1} \qquad (j = 0, 1, \cdots, n-1), \tag{2.6}$$

$$d_i = y_i - y(x_i), \tag{2.7}$$

and

$$f_i = f(x_i, y_i) \qquad (i = 0, 1, \cdots, n). \tag{2.8}$$

To take into consideration the propagation of error, along with $y(x)$ we consider a neighboring solution

$$y(x; e) = y(x; x_0, y_0 - e), \tag{2.9}$$

and put

$$c(x) = y(x) - y(x; e). \tag{2.10}$$

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2) The assumptions stated here are too severe for practical purposes but they are relaxed at the end of paragraph 3.

3) Here it is assumed that $s \geqq p$.

Then it is seen that, if e is an approximate value of the error of y_0, then $c(x_i)+d_i$ provides an approximation to the error of y_i and that d_i is the local error of y_i. For these reasons we try to obtain the approximate values of $c(x_i)+d_i$ and d_i $(0<i\leqq n)$.

We note that $c(x)$ is the solution of the differential equation

$$c' = F(x, y(x), c) \tag{2.11}$$

satisfying the initial condition

$$c(x_0) = e \tag{2.12}$$

and that it can be written as

$$c(x) = e\cdot O(1), \tag{2.13}$$

where

$$F(x, y, u) = f(x, y) - f(x, y-u). \tag{2.14}$$

Let r be a positive integer not greater than n, and let

$$m = 2n - r + 1, \tag{2.15}$$

$$T(x, y; h) = h^p[\sum_{i=1}^{m} h^i\varphi_i(x, y) + O(h^{m+1})]^{4)}, \tag{2.16}$$

$$h\Delta(x, y; h) = \sum_{i=1}^{m} h^i\Delta_i(x, y) + O(h^{m+1}), \tag{2.17}$$

and

$$d_j = h^p e_j \qquad (j=0, 1, \cdots, n). \tag{2.18}$$

Then we have the following

LEMMA 1. *e_j and $F(x_j, y_j, d_j)$ can be written as follows:*

$$e_j = \sum_{i=1}^{m} h^i P_i(j) + O(h^{m+1}), \tag{2.19}$$

and

$$F(x_j, y_j, d_j) = h^p[\sum_{i=1}^{m} h^i Q_i(j) + O(h^{m+1})] \qquad (j=0, 1, \cdots, n), \tag{2.20}$$

4) Here it is assumed that $s\geqq p+m$.

where $P_i(x)$ *and* $Q_i(x)$ *are polynomials in* x *of degree at most* i,

$$P_i(0)=0 \tag{2.21}$$

and

$$Q_i(0)=0 \qquad (i=1, 2, \cdots, m). \tag{2.22}$$

LEMMA 2. *There exist constants* a_{kj} *and* b_{kj} $(k, j=0, 1, \cdots, n)$ *that satisfy the conditions*

$$l\sum_{j=0}^{n} j^{l-1}a_{kj}+\sum_{j=0}^{n} j^{l}b_{kj}=k^{l} \qquad (l=1, 2, \cdots, m) \tag{2.23}$$

and

$$\sum_{j=0}^{n} j^{l}b_{kj}=0 \qquad (l=0, 1, \cdots, r). \tag{2.24}$$

LEMMA 3. *For* $y(x)$ *it is valid that*

$$y(x_k)-y(x_0)=h\sum_{j=0}^{n} a_{kj}y'(x_j)+\sum_{j=0}^{n} b_{kj}y(x_j)+O(h^{m+1}) \tag{2.25}$$

and

$$\sum_{j=0}^{n} b_{kj}y(x_j)=O(h^{r+1}) \qquad (k=0, 1, \cdots, n), \tag{2.26}$$

where a_{kj} *and* b_{kj} $(k, j=0, 1, 2, \cdots, n)$ *are constants satisfying* (2.23) *and* (2.24).

Corresponding to (2.25), we put

$$S_k=y_k-y_0-h\sum_{j=0}^{n} a_{kj}f_j-h\sum_{j=1}^{n} b_{kj}\sum_{q=1}^{j} p_q, \tag{2.27}$$

$$g_k=d_k-S_k \qquad (k=0, 1, \cdots, n), \tag{2.28}$$

and define δ_k by the formula

$$\delta_k=\begin{cases} 1, & \text{if } \sum_{j=0}^{n} j^{r+1}b_{kj}=0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.29}$$

Then we have the following

LEMMA 4. g_k *can be written as follows*:

$$(2.30) \qquad g_k = h^p\Big[\sum_{i=1}^{m} h^{i+1}R_{i+1}(k) + \sum_{i=r+1}^{m} h^i c_{ki}\Big] + O(h^{m+1}) \qquad (k=0, 1,\cdots, n),$$

where

$$(2.31) \qquad R_{i+1}(u) = \int_0^u Q_i(t)\,dt \qquad (i=1, 2,\cdots, m),$$

c_{ki}'s are constants,

$$(2.32) \qquad c_{0i} = 0,$$

and

$$(2.33) \qquad c_{k\ r+1} = 0 \qquad \textit{when} \qquad \delta_k = 1.$$

3. Approximate error formulas

From lemma 4 it is readily seen that, if

$$(3.1) \qquad m \geqq p+1,$$

then

$$(3.2) \qquad g_k = O(h^{p+2}).$$

Hence we have

THEOREM 1. *Under the condition* (3.1), *the local error of y_k satisfies*

$$(3.3) \qquad y_k - y(x_k) = S_k + O(h^{p+2}) \qquad (k=1, 2,\cdots, n).$$

To obtain better approximation to the error, in the sequel, we consider the case where the condition

$$(3.4) \qquad m \geqq p+r+1,$$

namely the condition

$$(3.5) \qquad 2(n-r) \geqq p \geqq 1$$

is satisfied.

Let

$$(3.6) \qquad x = x_u = x_0 + uh \qquad (0 \leqq u \leqq n)$$

and, corresponding to (2.19) and (2.30), we define the functions $e(x)$ and $g(x)$ by the formulas

$$e(x)=\sum_{i=1}^{m}h^{i}P_{i}(u)+h^{m+1}a(u) \tag{3.7}$$

and

$$g(x)=h^{p}[\sum_{i=1}^{m}h^{i+1}R_{i+1}(u)+\sum_{i=r+1}^{m}h^{i}c_{i}(u)]+h^{m+1}b(u), \tag{3.8}$$

where $a(u)$, $c_i(u)$ and $b(u)$ are interpolation polynomials of degree at most n such that

$$e(x_k)=e_k, \qquad c_i(k)=c_{ki}, \qquad g(x_k)=g_k \qquad (k=0,1,\cdots,n). \tag{3.9}$$

Making use of these functions and $y(x)$, we define the functions $d(x)$, $v(x)$ and $S(x)$ as follows:

$$d(x)=h^{p}e(x), \tag{3.10}$$

$$v(x)=y(x)+d(x), \tag{3.11}$$

and

$$S(x)=d(x)-g(x). \tag{3.12}$$

Then evidently

$$d(x_k)=d_k, \qquad v(x_k)=y_k, \qquad S(x_k)=S_k, \tag{3.13}$$

and we have the following

LEMMA 5. *Let $w(x;e)$ be the solution of the differential equation*

$$w'=F(x,v(x),S(x)+w) \tag{3.14}$$

satisfying the initial condition

$$w(x_0;e)=e. \tag{3.15}$$

Then, under the condition (3.5), *it is valid that*

$$y_k-y(x_k;e)=S_k+w(x_k;e)+O(h^{p+r+\delta_k+1}) \qquad (k=0,1,\cdots,n). \tag{3.16}$$

Since S_k $(k=0,1,\cdots,n)$ are computable, this lemma shows that, for the estimation of the error of y_k, we have only to obtain the approximate value of $w(x_k;e)$. For this purpose we use some one-step method. It is desired, however, to use such a one-step method as requires only the values of the derivative for equidistant abscissas, because the values of $S(x)$ and $v(x)$ are known only for $x=x_k$ $(k=0, 1,\cdots, n)$. Among such methods are Euler, improved Euler, modified Euler, Kutta, Heun, and Runge-Kutta methods and so on.

Let $\psi(t, w; h)$ be the increment function of such a one-step method of order q $(1\leqq q\leqq p+r+1)$ for approximating $w(x; e)$ that does not require the evaluations of the derivative for x other than $x=t+\frac{1}{k}jh$ $(j=0, 1, \cdots, k)$, where k is a positive integer not greater than n. Let l be an integral multiple of k not greater than n, t be the greatest integer not exceeding n/l and put $\nu=tl$.

We consider the case where the equation (3.14) is integrated numerically from x_0 to x_ν with step-size lh, and put

$$q_{j+l}(e)=\psi(x_j, w_j(e); lh), \tag{3.17}$$

$$w_{j+l}(e)=e+lhz_{j+l}(e) \qquad (j=0, l, 2l,\cdots, (t-1)l), \tag{3.18}$$

where

$$z_{j+l}(e)=z_j(e)+q_{j+l}(e), \tag{3.19}$$

$$w_0(e)=e, \tag{3.20}$$

and

$$z_0(e)=0. \tag{3.21}$$

Then we have the following

THEOREM 2. *Under the conditions* (3.5) *and*

$$p\geqq n-r-1, \qquad 1\leqq q\leqq p+r+1, \tag{3.22}$$

it is valid that

$$y_j-y(x_j; e)=T_j(e)+e\cdot O(h^{q+1})+O(h^{p+l_j+1}) \qquad (j=l, 2l,\cdots, tl), \tag{3.23}$$

where

$$T_j(e)=S_j+w_j(e), \tag{3.24}$$

and

$$l_j=\min(q, r+\delta_j). \tag{3.25}$$

$T_j(e)$ is the desired approximate error formula for y_j, where e is an approximate value of the error of y_0.

REMARK 1. In the case where $n=\nu$, $y(x_n)$ can be written as follows:

$$y(x_0+nh)=y(x_0)+nh\Theta(x_0, y_0; nh)+O(h^{p+l_n+1}), \tag{3.26}$$

where

$$\Theta(x_0, y_0; h)=\frac{1}{n}\Big[\sum_{j=0}^{n} a_{nj}f_j+\sum_{j=1}^{n} b_{nj}\sum_{q=1}^{j} p_q - lz_n(0)\Big]. \tag{3.27}$$

Hence $\Theta(x, y; h)$ can be considered as the increment function of a one-step method of order $p+l_n$. In this way we can generate one-step methods of higher orders from those of lower orders.

REMARK 2. The assumptions stated at the beginning of paragraph 2 can be replaced by the assumption that there exists a positive number M that satisfies the following conditions:

1°. $f(x, y)$ is defined and continuous in the domain

$$G: x_0 \leqq x \leqq x_0+nh_0, \qquad |y| \leqq M$$

and the partial derivatives of $f(x, y)$ up to the order s exist and are continuous in G, where $s=\max(p+m, q+1)$;

2°. $y(x)$ and $y(x; e)$ exist over the interval $[x_0, x_n]$;

3°. $\Phi(x_i, y_i; h)$ $(i=0, 1, \cdots, n-1)$ are defined;

4°. $w(x)$ and $w(x; e)$ exist over $[x_0, x_n]$;

5°. $\psi(x_j, w_j(e); lh)$ $(j=0, l, 2l, \cdots, (t-1)l)$ are defined.

4. Round-off errors

In this paragraph, we take into consideration round-off errors and denote by $\hat{a}$ the computed value of a.

The computed values $\hat{y}_i$, $\hat{w}_j(e)$ and $\hat{T}_j(e)$ can be written as follows:

$$\hat{y}_i=\hat{y}_{i-1}+h\hat{p}_i-r'_i \qquad (i=1, 2, \cdots, n), \tag{4.1}$$

$$\hat{w}_j(e)=e+lh\hat{z}_j(e)-s_j \qquad (j=l, 2l, \cdots, tl), \tag{4.2}$$

and

$$\hat{T}_j(e)=\hat{S}_j+\hat{w}_j(e)-t_j, \tag{4.3}$$

where r'_i, s_j and t_j are round-off errors,

$$\hat{y}_0=y_0, \qquad \hat{w}_0=e. \tag{4.4}$$

From these we have

$$\hat{y}_i - y_i = \hat{y}_{i-1} - y_{i-1} + h(\hat{p}_i - p_i) - r_i' \tag{4.5}$$

and

$$\hat{w}_j(e) - w_j(e) = lh[\hat{z}_j(e) - z_j(e)] - s_j. \tag{4.6}$$

Let

$$\mu = \max\left(|\hat{p}_i - p_i|,\ |\hat{z}_j(e) - z_j(e)|,\ |\hat{f}_k - f_k|\right) \tag{4.7}$$

$$(i = 1, 2, \cdots, n;\ j = l, 2l, \cdots, tl;\ k = 0, 1, \cdots, n).$$

Then $\hat{S}_k$, $\hat{w}_j(e)$ and $\hat{T}_j(e)$ can be written as follows:

$$\hat{S}_k = \hat{y}_k - y_k + S_k + O(h\mu) - r_k \qquad (k = 1, 2, \cdots, n), \tag{4.8}$$

$$\hat{w}_j(e) = w_j(e) + O(h\mu) - s_j \qquad (j = l, 2l, \cdots, tl) \tag{4.9}$$

and

$$\hat{T}_j(e) = \hat{y}_j - y_j + T_j(e) + O(h\mu) - r_j - s_j - t_j, \tag{4.10}$$

where r_k's are round-off errors.

From these we obtain the following

THEOREM 3. *Under the condition* (3.1), *the local error of* $\hat{y}_k$ *satisfies*

$$\hat{y}_k - y(x_k) = \hat{S}_k + O(h^{p+2}) + O(h\mu) + r_k \qquad (k = 1, 2, \cdots, n) \tag{4.11}$$

and, under the conditions (3.5) *and* (3.22), *it is valid that*

$$\hat{y}_j - y(x;\ e) = \hat{T}_j(e) + e \cdot O(h^{q+1}) + O(h^{p+l_j+1}) + O(h\mu) + r_j + s_j + t_j \tag{4.12}$$

$$(j = l, 2l, \cdots, tl),$$

where μ *is defined by* (4.7) *and* r_k, s_j *and* t_j *are round-off errors in computing* S_k, $w_j(e)$ *and* $\hat{S}_j + \hat{w}_j(e)$ *respectively.*

From this result it is seen that round-off errors other than r_k, s_j and t_j appear in (4.11) and (4.12) in the form multiplied by h so that S_k, $w_j(e)$ and $\hat{S}_j + \hat{w}_j(e)$ should be computed minutely.

To compare S_k with round-off errors, we rewrite S_k as

$$R_k = y_k - y_0 - h\sum_{j=0}^{n} a_{kj} f_j - \sum_{j=0}^{n} b_{kj} y_j, \tag{4.13}$$

so that theoretically

$$v_k = R_k - S_k = 0. \tag{4.14}$$

Then $\hat{R}_k$ and $\hat{v}_k$ can be written as follows:

$$\hat{R}_k = \hat{y}_k - y_0 - h\sum_{j=0}^{n} a_{kj}\hat{f}_j - \sum_{j=1}^{n} b_{kj}\sum_{q=1}^{j}(h\hat{p}_q - r'_q) - t'_k, \tag{4.15}$$

and

$$\hat{v}_k = \sum_{j=1}^{n} b_{kj}\sum_{q=1}^{j} r'_q + r_k - t'_k - t''_k, \tag{4.16}$$

where t'_k and t''_k are round-off errors. Clearly $\hat{v}_k$ is an aggregate of round-off errors.

From these we may conclude that, if the local error of y_k dominates round-off errors, then $\hat{v}_k$ must be significantly smaller in magnitude than $\hat{S}_k$.

5. Numerical examples

In the following examples, $n=l=4$ and the Runge-Kutta method is used for approximating $y(x)$ and $w(x; e)$, so that $p=q=4$. S_2, S_4 and R_4 are computed by the following formulas:

$$S_2 = y_2 - y_0 - hP + \frac{1}{2}h(p_4 - p_2 + p_3 - p_1), \tag{5.1}$$

$$S_4 = y_4 - y_0 - 2hP, \tag{5.2}$$

and

$$R_4 = \frac{1}{21}[5(y_4 - y_0) + 32(y_3 - y_1)] - 2h\left(2f_2 + \frac{4}{7}\Delta^2 f_1 + \frac{1}{35}\Delta^4 f_0\right), \tag{5.3}$$

where

$$P = 2f_2 + \frac{4}{7}\Delta^2 f_1 + \frac{1}{35}\Delta^4 f_0 + \frac{8}{21}(p_4 - p_3 + p_1 - p_2), \tag{5.4}$$

$$m = 8, \qquad r = 1, \qquad \delta_4 = 1, \tag{5.5}$$

and Δ is the forward difference operator.

Computation is carried out in the floating-point arithmetic with 39 bits mantissa and rounding is done by chopping. At the start of integration, e and h are set equal to 0 and 0.05 respectively. One step of integration in our extended sense is performed in accordance with the following program:

(1) set parameter $a=0$;

(2) compute y_i $(i=1, 2, 3, 4)$ and S_4;

(3) when the inequality

$$\varepsilon\,|\hat{y}_4| \geqq |\hat{S}_4| \qquad (\varepsilon = 5\times 10^{-7}) \tag{5.6}$$

is not satisfied, halve the step-size, set $a=1$ and go to (2);

(4) when (5.6) holds, compute v_4;

(5) when the inequality

$$\delta\,|\hat{S}_4| \geqq |\hat{v}_4| \qquad (\delta = 5\times 10^{-4}) \tag{5.7}$$

is not satisfied, if $a=0$, double the step-size and go to (2); if $a=1$, stop the machine (for then computation must be performed in multiple precision.);

(6) when (5.7) holds, compute S_2 and $T_4(e)$;

(7) replace y_0, x_0 and e with x_4, $\hat{y}_4$ and $\hat{T}_4(e)$ respectively.

The criterion (5.6) is set up to control the local error and (5.7) is made so that round-off errors may not dominate the local error. By our experience the criterion (5.7) improves the accuracy of the numerical solution in some cases.

EXAMPLE 1.

$$y' = 2xy, \qquad y(0)=1. \tag{5.8}$$

The solution is $y=\exp x^2$ and the results are shown in table 1. The actual errors of the numerical solution computed by the above program with process (5) omitted are $-1.487-06$, $-1.218-04$ and $-3.226-02$ for $x=1.0$, 2.0, and 3.0 respectively. Comparing these results with those in table 1, we can understand the significance of the criterion (5.7). The only difference in the computation procedure is that the step-size used in the first and the second step of integration in our sense is 0.1 and 0.05 for the program with process (5) but 0.05 and 0.1 for the program without process (5).

EXAMPLE 2.

$$y' = 12x^3 - \frac{8y}{x}, \qquad y(-1)=1. \tag{5.9}$$

The solution is $y=x^4$ and the results are given in table 2. For this equation the origin is a singular point. Since the general solution is $y=x^4+Cx^{-8}$, even a small error will cause a great trouble. In fact, integrating (5.9) by the

Runge-Kutta method with a uniform step-size $2^{-n}/10$ ($n=1, 2,\cdots, 9$), we could not have even one significant figure for $x=-0.1$. Without knowing the actual solution, however, we can readily see from table 2 that even one significant figure is not obtained for $x=-0.3$, -0.2 and -0.1.

Table 1

x \ error	computed	actual
1.0	−8.361−07	−8.720−07
2.0	−9.946−05	−9.941−05
3.0	−3.057−02	−3.039−02
4.0	−6.386+01	−6.343+01
5.0	−9.764+05	−9.687+05

Table 2

x \ error	computed	actual
−0.9	−2.374−07	−2.370−07
−0.8	−8.889−07	−8.877−07
−0.7	−2.925−06	−2.922−06
−0.6	−1.007−05	−1.006−05
−0.5	−4.331−05	−4.328−05
−0.4	−2.581−04	−2.580−04
−0.3	−2.575−03	−2.578−03
−0.2	−6.599−02	−6.706−02
−0.1	−1.688+01	−1.691+01

In both examples errors seem to be approximated fairly well and this method may be used as an integration method with a check on the accurracy of the numerical solution.

6. Proof

6.1 Proof of lemma 1

From (2.7), (2.6), (2.4) and (2.1), it follows that

$$d_{j+1}-d_j=h[\Delta(x_j, y(x_j)+d_j; h)-\Delta(x_j, y(x_j); h)]+ \\ +T(x_j, y(x_j)+d_j; h) \qquad (j=0, 1,\cdots, n-1). \tag{6.1}$$

Since $d_0=0$, from (2.2) it is readily seen that

$$d_i=O(h^{p+1}), \tag{6.2}$$

and so

$$e_i=O(h) \qquad (i=0, 1,\cdots, n). \tag{6.3}$$

By (2.16), (2.17) and (2.18), (6.1) can be written as follows:

$$e_{j+1}-e_j=\sum_{i=1}^{m} h^i\varphi_i(x_j, y(x_j))+ \\ +\sum_{\substack{i+l+(l-1)p\leqq m\\ i,l\geqq 1}}\frac{1}{l!}h^{(l-1)p+i}\frac{\partial^l}{\partial y^l}\Delta_i(x_j, y(x_j))e_j^l+ \tag{6.4}$$

$$+\sum_{\substack{i+l+lp\leqq m\\ i,l\geqq 1}}\frac{1}{l!}h^{lp+i}\frac{\partial^l}{\partial y^l}\varphi_i(x_j, y(x_j))e_j^l+O(h^{m+1}).$$

Making use of the fact that

$$y(x_j)=y_0+\sum_{q=1}^{m}\frac{1}{q!}(jh)^q y^{(q)}(x_0)+O(h^{m+1}), \tag{6.5}$$

and expanding $\varphi_i(x_j, y(x_j))$, $\frac{\partial^l}{\partial y^l}\Delta_i(x_j, y(x_j))$ and $\frac{\partial^l}{\partial y^l}\varphi_i(x_j, y(x_j))$ at x_0, we can write (6.4) as follows:

$$e_{j+1}-e_j=\sum_{i=1}^{m}h^i a_{i-1}(j+1)+\sum_{\substack{i+l\leqq m\\ i,l\geqq 1}}h^i b_{i-1}^{(l)}(j+1)e_j^l+O(h^{m+1}) \tag{6.6}$$

where $a_{i-1}(x)$ and $b_{i-1}^{(l)}(x)$ are polynomials in x of degree at most $i-1$.

Since $e_0=0$, summing (2.28) from $j=0$ to $k-1$, we have

$$e_k=\sum_{i=1}^{m}h^i\sum_{j=1}^{k}a_{i-1}(j)+\sum_{\substack{i+l\leqq m\\ i,l\geqq 1}}h^i\sum_{j=1}^{k}b_{i-1}^{(l)}(j)e_{j-1}^l+O(h^{m+1}) \tag{6.7}$$

$$(k=1, 2,\dots, n).$$

Substitution of (6.3) into (6.7) yields

$$e_k=hka_0+O(h^2), \tag{6.8}$$

so that (2.19) and (2.21) are valid for $m=1$ and $i=1$ respectively, where $a_0=a_0(0)$.

Let $B_n(x)$ be the Bernoulli polynomial of degree n. Then it is well known that

$$\sum_{j=1}^{k}j^l=\frac{1}{l+1}[B_{l+1}(k+1)-B_{l+1}(1)] \qquad (l\geqq 0). \tag{6.9}$$

The right hand side of (6.9) is a polynomial in k of degree $l+1$ without the constant term. From this fact follow readily (2.19) and (2.21) by induction.

Although (2.19) is proved only for $j\geqq 1$, since $e_0=0$ and (2.21) holds, (2.19) is valid also for $j=0$.

By (2.14), (2.17) and (2.18) we have

$$\begin{aligned}F(x_j, y_j, d_j)&=f(x_j, y(x_j)+d_j)-f(x_j, y(x_j))\\ &=h^p\Big[\sum_{\substack{l+(l-1)p\leqq m\\ l\geqq 1}}\frac{1}{l!}h^{(l-1)p}\frac{\partial^l}{\partial y^l}f(x_j, y(x_j))e_j^l+O(h^{m+1})\Big].\end{aligned} \tag{6.10}$$

Substituting (2.19) into (6.10), making use of (6.5) and expanding $\frac{\partial^l}{\partial y^l}f(x_j, y(x_j))$ at x_0, we obtain (2.20) and (2.22). This completes the proof of lemma 1.

6.2 Proof of lemma 2 and lemma 3

Let E, Δ and D be the shifting, forward difference and differential operators respectively [7]. Then by the well-known formulas

(6.11) $$E = 1 + \Delta$$

and

(6.12) $$hD = \log(1 + \Delta),$$

Δ and $E^k - 1$ can be rewritten as follows:

(6.13) $$\Delta = (a_0 + a_1\Delta + a_2\Delta^2 + \cdots)hD$$

and

(6.14) $$E^k - 1 = (1 + \Delta)^k - 1 = \sum_{i=1}^{k}\binom{k}{i}\Delta^i$$
$$= (\gamma_{k0} + \gamma_{k1}\Delta + \gamma_{k2}\Delta^2 + \cdots)hD \qquad (k = 1, 2, \cdots),$$

where

(6.15) $$\gamma_{ki} = \sum_{j=1}^{i}(-1)^{j+1}\frac{1}{j+1}\gamma_{k\,i-j}\binom{k}{i+1} \qquad (i = 1, 2, \cdots),$$

(6.16) $$\gamma_{k0} = k,$$

(6.17) $$a_i = \gamma_{1i} \qquad (i = 0, 1, \cdots)$$

and

(6.18) $$\binom{k}{l} = 0 \quad \text{for} \quad l > k.$$

Then by (6.13) and (6.14), for any function $z(x)$ which is $m+1$ times continuously differentiable on $[x_0, x_0 + mh]$, it is valid that

(6.19) $$z(x_k) - z(x_0) = h\sum_{i=0}^{m-1}\gamma_{ki}\Delta^i z'(x_0) + O(h^{m+1}),$$

and

$$\Delta^l z(x_0) = h\sum_{i=0}^{m-1} a_i\Delta^{i+l-1}z'(x_0) + O(h^{m+1}) \qquad (l=1, 2,\cdots, m). \tag{6.20}$$

Suppose that $n \geqq r+1$ and let A be the matrix

$$A = \begin{pmatrix} a_2, & a_3, & \cdots\cdots\cdots, & a_{l+1} \\ a_3, & a_4, & \cdots\cdots\cdots, & a_{l+2} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{l+1}, & a_{l+2}, & \cdots\cdots, & a_{2l} \end{pmatrix} \tag{6.21}$$

where $l=n-r$. Then, from (6.20), we have a system of linear equations

$$A\begin{pmatrix} h\Delta^{n+1}z'(x_0) \\ \cdot \\ \vdots \\ h\Delta^{n+l}z'(x_0) \end{pmatrix} = \begin{pmatrix} \Delta^n z(x_0) - h\sum_{i=0}^{1} a_i\Delta^{n-1+i}z'(x_0) + O(h^{m+1}) \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \Delta^{r+1}z(x_0) - h\sum_{i=0}^{l} a_i\Delta^{r+i}z'(x_0) + O(h^{m+1}) \end{pmatrix}. \tag{6.22}$$

To show that A is non-singular, making use of (6.15) and (6.16), we rewrite $|A|$ as follows:

$$|A| = \begin{vmatrix} a_0, a_1, \cdots, a_l & \\ 0, & \\ \cdot & A \\ \cdot & \\ 0, & \end{vmatrix} = (-1)^l \begin{vmatrix} \frac{1}{1+0}, & \frac{1}{1+1}, & \cdots\cdots\cdots\cdots, & \frac{1}{1+l} \\ \frac{1}{2+0}, & \frac{1}{2+1}, & \cdots\cdots\cdots\cdots, & \frac{1}{2+l} \\ \frac{1}{l+1+0}, & \frac{1}{l+1+1}, & \cdots\cdots, & \frac{1}{l+1+l} \end{vmatrix}. \tag{6.23}$$

Then, from the well-known identity due to Cauchy, it follows that

$$|A| = (-1)^l \frac{[1!\,2!\cdots l!]^3}{(l+1)!\,(l+2)!\cdots(2l+1)!} \neq 0. \tag{6.24}$$

Hence (6.22) can be solved and the solution can be written in the following form:

$$h\Delta^{n+j}z'(x_0) = \sum_{k=1}^{n-r} \alpha_{jk}\Delta^{r+k}z(x_0) + h\sum_{k=r}^{n} \beta_{jk}\Delta^k z'(x_0) + O(h^{m+1}) \tag{6.25}$$

$$(j=1, 2,\cdots, n-r).$$

Substituting this into (6.19), we can rewrite (6.19) as follows:

(6.26) $$z(x_k)-z(x_0)=h\sum_{j=0}^{n}\mu_{kj}\Delta^j z'(x_0)+\sum_{j=r+1}^{n}\nu_{kj}\Delta^j z(x_0)+O(h^{m+1}).$$

Since

(6.27) $$\Delta^j=(E-1)^j=\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}E^l,$$

if we put

(6.28) $$\sum_{l=0}^{n}(-1)^{l-j}\mu_{kj}\binom{l}{j}=a_{kj}$$

and

(6.29) $$\sum_{l=r+1}^{n}(-1)^{l-j}\nu_{kj}\binom{l}{j}=b_{kj}\qquad (j=0,1,\cdots,n),$$

then it follows that

(6.30) $$z(x_k)-z(x_0)=h\sum_{j=0}^{n}a_{kj}z'(x_j)+\sum_{j=0}^{n}b_{kj}z(x_j)+O(h^{m+1}),$$

(6.31) $$\sum_{j=r+1}^{n}\nu_{kj}\Delta^j z(x_0)=\sum_{j=0}^{n}b_{kj}z(x_j),$$

and

(6.32) $$a_{0j}=b_{0j}=0\qquad (j=0,1,\cdots,n),$$

because $\gamma_{0i}=0$ $(i=0,1,2,\cdots,m-1)$.

It is readily seen from (6.19) that (6.30), (6.31) and (6.32) are valid also for the case $n=r$ with $\nu_{kj}=b_{kj}=0$ because $m-1=n$.

Since

(6.33) $$z(x_j)=\sum_{l=0}^{m}j^l\frac{1}{l!}h^l z^{(l)}(x_0)+O(h^{m+1}),$$

(6.34) $$z'(x_j)=\sum_{l=1}^{m}lj^{l-1}\frac{1}{l!}h^{l-1}z^{(l)}(x_0)+O(h^{m+1}),$$

(6.35) $$\Delta^j z(x_0)=O(h^j),$$

and $z(x)$ is any function smooth enough, from (6.30) and (6.31) follow (2.33) and (2.24). This proves lemma 2.

Lemma 3 follows readily from (6.30), (6.31) and (6.35).

6.3 Proof of lemma 4

From (2.7), (2.6), (2.18) and (2.19) it follows that

$$y(x_k) - y(x_0) = y_k - y_0 - d_k, \tag{6.36}$$

$$\sum_{j=0}^{n} b_{kj} y(x_j) = \sum_{j=0}^{n} b_{kj} y_j - \sum_{j=0}^{n} b_{kj} d_j, \tag{6.37}$$

$$\sum_{j=0}^{n} b_{kj} y_j = y_0 \sum_{j=0}^{n} b_{kj} + h \sum_{j=1}^{n} b_{kj} \sum_{q=1}^{j} p_q, \tag{6.38}$$

and

$$\sum_{j=0}^{n} b_{kj} d_j = h^p [\sum_{i=1}^{m} h^i \sum_{j=0}^{n} b_{kj} P_i(j) + O(h^{m+1})]. \tag{6.39}$$

By (2.24), further (6.38) and (6.39) can be written as follows:

$$\sum_{j=0}^{n} b_{kj} y_j = h \sum_{j=1}^{n} b_{kj} \sum_{q=1}^{j} p_q, \tag{6.40}$$

and

$$\sum_{j=0}^{n} b_{kj} d_j = h^p \sum_{i=r+1}^{m} h^i \sum_{j=0}^{n} b_{kj} P_i(j) + O(h^{p+m+1}). \tag{6.41}$$

From (6.10) and (2.20) it follows that

$$y'(x_j) = f(x_j, y(x_j)) = f_j - h^p \sum_{i=1}^{m} h^i Q_i(j) + O(h^{p+m+1}). \tag{6.42}$$

Substituting (6.36), (6.42), (6.37), (6.40) and (6.41) into (2.25), we have

$$\begin{aligned} y_k - y_0 - d_k = h \sum_{j=0}^{n} a_{kj} f_j + h \sum_{j=1}^{n} b_{kj} \sum_{q=1}^{j} p_q - \\ - h^p [\sum_{i=1}^{m} h^{i+1} \sum_{j=0}^{n} a_{kj} Q_i(j) + \sum_{i=r+1}^{m} b_{kj} P_i(j)] + O(h^{m+1}). \end{aligned} \tag{6.43}$$

Since by (2.23)

$$\sum_{j=0}^{n} j^{l-1} a_{kj} = \frac{1}{l} k^l - \frac{1}{l} \sum_{j=0}^{n} b_{kj} j^l \qquad (l = 1, 2, \cdots, m), \tag{6.44}$$

from (2.31) it follows that

$$\sum_{j=0}^{n} a_{kj} Q_i(j) = R_{i+1}(k) - \sum_{j=0}^{n} b_{kj} R_{i+1}(j). \tag{6.45}$$

Further by (2.24) it holds that

$$\sum_{j=0}^{n} b_{kj} R_{i+1}(j) = 0 \qquad (i = 0, 1, \cdots, r-1). \tag{6.46}$$

Substituting (6.45) and (6.46) into (6.43) and making use of (2.27) and (2.28), we have (2.30), where

$$c_{ki} = \sum_{j=0}^{n} b_{kj} [P_i(j) - R_i(j)]. \tag{6.47}$$

Clearly (2.33) holds and (2.32) follows from (6.32). Thus lemma 4 has been proved.

6.4 Proof of lemma 5

Since by (2.14) and (3.11)

$$\begin{aligned} F(x, v(x), S(x) + g(x)) &= F(x, v(x), d(x)) \\ &= f(x, y(x) + d(x)) - f(x, y(x)), \end{aligned} \tag{6.48}$$

by the same reasoning as for (6.10), it is easily seen that

$$F(x, v(x), S(x) + g(x)) = h^p [\sum_{i=1}^{m} h^i Q_i(u) + O(h^{m+1})]. \tag{6.49}$$

Differentiating (3.8), we have

$$g'(x) = h^p [\sum_{i=1}^{m} h^i Q_i(u) + \sum_{i=r+1}^{m} h^{i-1} c_i'(u)] + h^m b'(u). \tag{6.50}$$

Since $m \geqq p+r$ by (3.4), by our assumptions there exists a constant K such that

$$|g'(x) - F(x, v(x), S(x) + g(x))| \leqq K h^{p+r} \qquad (0 \leqq u \leqq n). \tag{6.51}$$

Let L be a constant such that the inequality

$$|f_y(x, y)| \leqq L \tag{6.52}$$

is valid for any point (x, y) belonging to S, then by (2.14) the inequality

$$|F(x, v(x), S(x) + w_1) - F(x, v(x), S(x) + w_2)| \leqq L |w_1 - w_2| \tag{6.53}$$

holds for any numbers w_1 and w_2.

Since $g(x_0)=0$, if we put

(6.54) $$w(x)=w(x;\,0),$$

from the well-known theorem in the theory of ordinary differential equations, it follows that

(6.55) $$|g(x)-w(x)|\leq h^{p+r}\frac{K}{L}(e^{Luh}-1) \qquad (0\leq u\leq n),$$

so that

(6.56) $$g(x)=w(x)+O(h^{p+r+1}).$$

Therefore we put

(6.57) $$w(x)=h^p\sum_{i=1}^{m}h^{i+1}R_{i+1}(u)+h^{p+r+1}z(u)+h^{p+r+2}z(u,\,h),$$

where

(6.58) $$z(0)=z(0,\,h)=0.$$

Then we have

(6.59) $$w'(x)=h^p\sum_{i=1}^{m}h^iQ_i(u)+h^{p+r}z'(u)+h^{p+r+1}z_u(u,\,h).$$

On the other hand, since by (3.12), (3.8) and (6.57)

(6.60) $$S(x)+w(x)=d(x)+h^{p+r+1}[z(u)-c_{r+1}(u)]+O(h^{p+r+2}),$$

it follows that

(6.61) $$\begin{aligned}F\big(x,\,v(x),\,S(x)+w(x)\big)&=f\big(x,\,y(x)+d(x)\big)-f\big(x,\,y(x)\big)+\\&\quad+f\big(x,\,y(x)\big)-f\big(x,\,y(x)+d(x)-S(x)-w(x)\big)\\&=h^p\sum_{i=1}^{m}h^iQ_i(u)+h^{p+r+1}f_y(x_0,\,y_0)\,[z(u)-c_{r+1}(u)]+O(h^{p+r+2}).\end{aligned}$$

Comparing (6.61) with (6.59), we find that

(662) $$z(u)=0,$$

so that

$$w(x)=h^{p}\sum_{i=1}^{m}h^{i+1}R_{i+1}(u)+O(h^{p+r+2}), \tag{6.63}$$

and

$$d(x)=S(x)+w(x)+h^{p+r+1}c_{r+1}(u)+O(h^{p+r+2}). \tag{6.64}$$

Now, from (2.11) and (3.14), it follows that

$$\begin{aligned} w'(x)+c'(x) &= F(x, v(x), S(x)+w(x))+F(x, y(x), c(x)) \\ &= F(x, v(x), S(x)+w(x)+c(x))+F(x, y(x), S(x)+w(x)-d(x)) \\ &\quad -F(x, y(x)-c(x), S(x)+w(x)-d(x)) \\ &= F(x, v(x), S(x)+w(x)+c(x))+O(h^{p+r+1}). \end{aligned} \tag{6.65}$$

Since by (2.12), (6.54) and (3.15)

$$w(x_0)+c(x_0)=e, \tag{6.66}$$

by the same reasoning as for $g(x)$ and $w(x)$, it is seen that

$$w(x)+c(x)=w(x;\,e)+O(h^{p+r+2}). \tag{6.67}$$

From (6.67), (6.64), (2.10) and (3.11), it follows that

$$v(x)-y(x;\,e)=S(x)+w(x;\,e)+h^{p+r+1}c_{r+1}(u)+O(h^{p+r+2}). \tag{6.68}$$

Then, by (3.13), (3.16) holds. This completes the proof of lemma 5.

6.5 Proof of theorem 2

From (3.7) and (3.8) we have

$$h^{k}\frac{d^{k}}{dx^{k}}e(x)=\sum_{i=1}^{m}h^{i}P_{i}^{(k)}(u)+h^{m+1}a^{(k)}(u) \tag{6.69}$$

and

$$h^{k}\frac{d^{k}}{dx^{k}}g(x)=h^{p}\Big[\sum_{i=1}^{m}h^{i+1}R_{i+1}^{(k)}(u)+\sum_{i=r+1}^{m}h^{i}c_{i}^{(k)}(u)\Big]+h^{m+1}b^{(k)}(u) \tag{6.70}$$

$$(k=1,\,2,\cdots).$$

Since $a(u)$, $c_i(u)$ and $b(u)$ are polynomials of degree at most n, $P_i(u)$ and $R_i(u)$ are polynomials of degree at most i and $m\geqq n+1$ by (3.5) and (2.15), from

(6.69) and (6.70) it follows that

$$h^k \frac{d^k}{dx^k} e(x) = O(h^k) \tag{6.71}$$

and

$$h^k \frac{d^k}{dx^k} g(x) = \begin{cases} O(h^{p+k}), & 0 \leqq k \leqq r+1 \quad \text{or} \quad k \geqq n+1, \\ O(h^{p+r+1}), & r+1 < k \leqq n, \end{cases} \tag{6.72}$$

so that by (3.10) and (3.12) we have

$$h^k \frac{d^k}{dx^k} d(x) = O(h^{p+k}) \tag{6.73}$$

and

$$h^k \frac{d^k}{dx^k} S(x) = O(h^{p+\alpha_k}), \tag{6.74}$$

where

$$\alpha_k = \min(k, r+1). \tag{6.75}$$

Further by (3.11) we have

$$\frac{d^k}{dx^k} v(x) = O(1) \tag{6.76}$$

and

$$\frac{d^k}{dx^k} S(x) = O(1) \qquad (k=0, 1, \cdots), \tag{6.77}$$

because $p \geqq 1$ by (3.5) and $p+r+1-n \geqq 0$ by (3.22). Also by (2.13), (6.60) and (6.67) it is seen that

$$S(x) + w(x; e) = e \cdot O(1) + O(h^{p+1}). \tag{6.78}$$

Now, to determine the order of the truncation errors of $w_i(e)$ $(i=l, 2l, \cdots, tl)$, we shall show by induction that

$$h^{k+1} \frac{d^{k+1}}{dx^{k+1}} w(x; e) = e \cdot O(h^{k+1}) + O(h^{p+\alpha_k+1})^{5)} \qquad (k=0, 1, \cdots, p+r+1). \tag{6.79}$$

By definition $w(x; e)$ satisfies the equation

5) Here it is assumed that $s \geqq k+1$.

(6.80) $$w'(x; e) = F(x, v(x), S(x) + w(x; e))$$
$$= f(x, v(x)) - f(x, v(x) - S(x) - w(x; e)).$$

From this we have by (6.78)

(6.81) $$h \frac{d}{dx} w(x; e) = [S(x) + w(x; e)] \cdot O(1)$$
$$= e \cdot O(h) + O(h^{p+2}),$$

because $f_y(x, y)$ is bounded in S. Thus (6.79) is valid for $k=0$. Hence suppose that (6.79) holds for $k=0, 1, \cdots, j-1$ $(j \leqq p+r+1)$. Then we have

(6.82) $$\frac{d^{k+1}}{dx^{k+1}} w(x; e) = e \cdot O(1) + O(h^{p+\alpha_k - k}) = O(1) \qquad (k=0, 1, \cdots, j-1),$$

and so

(6.83) $$S^{(i)}(x) + w^{(i)}(x; e) = O(1) \qquad (i=0, 1, \cdots, j),$$

because $p+\alpha_k - k \geqq 0$ provided $k \leqq p+r+1$, where $f^{(i)}(x)$ denotes $\frac{d^i}{dx^i} f(x)$. Further, from the assumption, it follows that

(6.84) $$h^{j+1}[S^{(i)}(x) + w^{(i)}(x; e)]$$
$$= h^{j+1-i}[h^i S^{(i)}(x) + h^i w^{(i)}(x; e)]$$
$$= h^{j+1-i}[O(h^{p+\alpha_i}) + e \cdot O(h^i) + O(h^{p+\alpha_{i-1}+1})]$$
$$= e \cdot O(h^{j+1}) + O(h^{p+\alpha_j+1}) \qquad (i=0, 1, \cdots, j),$$

because (6.84) is evidently valid by (6.78) for $i=0$, and

(6.85) $$j+1-i+p+\alpha_i = \begin{cases} p+1+r+1+(j-i) & (i \geqq r+1) \\ p+1+j & (i < r+1) \end{cases}$$

and

(6.86) $$j+1-i+p+\alpha_{i-1}+1 = \begin{cases} p+1+r+1+(j+1-i) & (i-1 \geqq r+1) \\ p+1+j & (i-1 < r+1) \end{cases}$$

for $i \geqq 1$.

Differentiating (6.80) j times and multiplying it by h^{j+1}, we see that $h^{j+1} w^{(j+1)}(x; e)$ can be expressed as a linear combination of the terms of the form as follows:

$$(6.87)\quad U(\mu_1, \mu_2,\cdots, \mu_j; \nu_1, \nu_2,\cdots, \nu_j; \alpha, \beta) = h^{j+1}\Big[\prod_{i=1}^{j}(v^{(\mu_i)}(x))^{\nu_i}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}f(x, v(x)) - $$

$$-\prod_{i=1}^{j}(v^{(\mu_i)}(x) - S^{(\mu_i)}(x) - w^{(\mu_i)}(x; e))^{\nu_i}\frac{\partial^{\alpha+\beta}}{\alpha x^\alpha \partial y^\beta}f(x, v(x) - S(x) - w(x; e))\Big],$$

where μ_i, ν_i $(i=1, 2,\cdots, j)$, α, and β are non-negative integers not greater than j such that

$$(6.88)\qquad \alpha+\beta \leqq j, \qquad \sum_{i=1}^{j}\mu_i\cdot\nu_i \leqq j.$$

Making use of (6.76), 6.83) and (6.84), we can rewrite U as follows:

$$(6.89)\quad U(\mu_1, \mu_2,\cdots, \mu_j; \nu_1, \nu_2,\cdots, \nu_j; \alpha, \beta) = h^{j+1}\prod_{i=1}^{j}(v^{(\mu_i)}(x))^{\nu_i}\Big[\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}f(x, v(x)) -$$

$$-\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}f(x, v(x) - S(x) - w(x; e))\Big] + h^{j+1}\Big[\prod_{i=1}^{j}(v^{(\mu_i)}(x))^{\nu_i} -$$

$$-\prod_{i=1}^{j}(v^{(\mu_i)}(x) - S^{(\mu_i)}(x) - w^{(\mu_i)}(x; e))^{\nu_i}\Big]\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}f(x, v(x) - S(x) - w(x; e))$$

$$= e\cdot O(h^{j+1}) + O(h^{p+\alpha_j+1})$$

because the partial derivatives of $f(x, y)$ are bounded by the the assumption. Hence (6.79) is valid also for $k=j$.

From (6.79) it follows that

$$(6.90)\qquad w(x_i; e) = w_i(e) + e\cdot O(h^{q+1}) + O(h^{p+\alpha_q+1}) \qquad (i = l, 2l,\cdots, tl),$$

because a one-step method of order q $(q \leqq p+r+1)$ is applied for approximating $w(x; e)$. Substitution of (6.90) into (3.16) yields (3.23). Thus theorem 2 has been proved.

Remark 3. When $m=p+r$, replacing $c_{r+1}(u)$ by $c_{r+1}(u)+b(u)$ in (6.60), (6.61), (6.64) and (6.68), and modifying (3.16) as

$$(3.16)'\qquad y_k - y(x_k; e) = S_k + w(x_k; e) + O(h^{p+r+1}) \qquad (k=0, 1,\cdots, n),$$

we have the following

Theorem 2′. *Under the conditions* (3.22) *and*

$$(3.5)'\qquad 2(n-r)+1 = p \geqq 1,$$

it is valid that

$$y_j - y(x_j; e) = T_j(e) + e \cdot O(h^{q+1}) + O(h^{p+\alpha+1}) \qquad (j = l, 2l, \cdots, tl), \tag{3.23$'$}$$

where $T_j(e)$ *is defined by* (3.24) *and*

$$\alpha = \min(q, r). \tag{3.25$'$}$$

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