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A Computation of Extremal Length in an Abstract Space

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In [2] and [3] we computed the extremal length of harmonic subflows in an n-dimensional $\mathscr{E}$-space. In this paper we shall compute the extremal length of a certain class of measures in an abstract space. The main results of [3] are special cases of the results in the present paper.

1. Let X be an abstract space and $\mathfrak{A}$ be a σ-field[1] of subsets of X. By a measure in this paper we shall mean a non-negative countably additive set-function. Let μ be a measure on $\mathfrak{A}$. With each $x \in X$, we associate an abstract space Y_x, a σ-field $\mathfrak{B}_x$ of subsets of Y_x and a measure ν_x defined on $\mathfrak{B}_x$. We shall denote by Z the set of all couples (x, y), $x \in X$, $y \in Y_x$. Suppose that there is a σ-field $\mathfrak{C}$ of sets in Z[2] which contains all sets of the form $\{(x, y);\ x \in A \in \mathfrak{A}, y \in Y_x\}$[3] and which, for every $E \in \mathfrak{C}$, satisfies

(1) $E_x = \{y \in Y_x;\ (x, y) \in E\}$ belongs to $\mathfrak{B}_x$ for every $x \in X$ not belonging to $A_E \in \mathfrak{A}$ with $\mu(A_E) = 0$,[4]

(2) $\nu_x(E_x)$ is an $\mathfrak{A}$-measurable function defined on $X - A_E$. We set

$$\alpha(E) = \int_X \nu_x(E_x)\, d\mu(x) \qquad \text{for } E \in \mathfrak{C}.$$

If $E^{(1)}, E^{(2)}, \cdots$ are mutually disjoint sets of $\mathfrak{C}$, then $\alpha(\bigcup_n E^{(n)}) = \sum_n \alpha(E^{(n)})$. Thus, α is a measure on $\mathfrak{C}$. If f is non-negative and $\mathfrak{C}$-measurable, it is inferred that $f(x, y)$ is a $\mathfrak{B}_x$-measurable function of y on Y_x for μ-a.e. $x \in X$, that $\int_{Y_x} f d\nu_x$ is an $\mathfrak{A}$-measurable function defined for μ-a.e. $x \in X$ and that

$$\int f d\alpha = \int_X \left(\int_{Y_x} f d\nu_x \right) d\mu(x).$$

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1) This means that $\mathfrak{A}$ is not empty, $A \in \mathfrak{A}$ implies $X - A$ and $A_1, A_2, \ldots \in \mathfrak{A}$ implies $\bigcup_n A_n \in \mathfrak{A}$. Sometimes, it is called a Borel field or σ-algebra.

2) The existence of $\mathfrak{C}$ will be discussed in Section 6.

3) Any set of this form satisfies conditions (1) and (2) imposed below, because $Z \in \mathfrak{C}$ satisfies condition (2).

4) This fact will be expressed as "for μ-a.e. $x \in X$".

Let π and κ be non-negative $\mathfrak{C}$-measurable functions in Z, and assume that $\kappa(x, y)$ restricted to Y_x is $\mathfrak{B}_x$-measurable for each $x \in X$. We shall call an $\mathfrak{C}$-measurable function $\rho \geqq 0$ in Z κ-admissible (in association with $\{\nu_x\}$) if $\int_{Y_x} \kappa\rho d\nu_x$[5] is well-defined and $\geqq 1$ for every $x \in X$. For $p, 0<p<\infty$, we define a *module* by

$$M_p(\{\nu_x\}; \pi, \kappa) = \inf_\rho \int \pi\rho^p d\alpha,$$

where inf is taken with respect to κ-admissible ρ. Although $M_p(\{\nu_x\}; \pi, \kappa)$ depends also on the choice of μ, $\mathfrak{A}$, $\mathfrak{C}$, etc. we shall not write them explicitly in it. If there is x such that $\int_{Y_x} \kappa d\nu_x = 0$, there exists no κ-admissible ρ. We set then $M_p(\{\nu_x\}; \pi, \kappa) = \infty$. If $\int_{Y_x} \kappa d\nu_x > 0$ for each $x \in X$, then $\rho \equiv \infty$ is κ-admissible. We shall call $1/M_p(\{\nu_x\}; \pi, \kappa)$ the *κ-extremal length of $\{\nu_x\}$ with weight π*. This is a special case of the extremal length defined in [1]. An admissible ρ which gives $\int \pi\rho^p d\alpha = M_p(\{\nu_x\}; \pi, \kappa)$ will be called *extremal*. In this paper we shall assume that $\int \kappa d\nu_x > 0$ for all x.

We shall state a property of $M_p(\{\nu_x\}; \pi, \kappa)$; see (1) for a proof.

(3) Let $A_1, A_2, \cdots \in \mathfrak{A}$ be all mutually disjoint, let $A_1', A_2', \cdots \in \mathfrak{A}$ and let $\bigcup_n (A_n \cup A_n') = X$. If $M_p(\{\nu_x; x \in A_n'\}; \pi, \kappa) = 0$ for every n, then

$$M_p(\{\nu_x\}; \pi, \kappa) = \sum_n M_p(\{\nu_x; x \in A_n\}; \pi, \kappa).$$

2. We begin with a preliminary remark that we may assume that $\pi > 0$ and $\kappa > 0$ everywhere on Z in computing $M_p(\{\nu_x\}; \pi, \kappa)$. Actually let ρ be κ-admissible, and denote by π^+, κ^+ and ν_x^+ the restrictions of π, κ and ν_x to $E_\kappa^+ = \{(x, y); \kappa(x, y) > 0\}$ and to $E_\kappa^+ \cap Y_x$ respectively. We observe easily that the restriction of ρ to E^+ is κ^+-admissible in association with $\{\nu_x^+\}$ and derive $M_p(\{\nu_x^+\}; \pi^+, \kappa^+) \leqq M_p(\{\nu_x\}; \pi, \kappa)$. The inverse inequality is evident, and the equality is established. Thus we may consider E_κ^+ instead of Z. For this reason, *we shall assume hereafter that $\kappa > 0$ everywhere on Z.*

Next we set $E_\pi^0 = \{(x, y); \pi(x, y) = 0\} \in \mathfrak{C}$ and

$$X_\pi^* = \{x; E_\pi^0 \cap Y_x \text{ is a set of } \mathfrak{B}_x \text{ with positive } \nu_x\text{-measure}\}.$$

This set belongs to $\mathfrak{A}$ and $M_p(\{\nu_x; x \in X_\pi^*\}; \pi, \kappa) = 0$, because ρ equal to ∞ on $\{(x, y) \in E_\pi^0; x \in X_\pi^*\}$ and to 0 elsewhere is κ-admissible in association with $\{\nu_x; x \in X_\pi^*\}$ and hence

5) In this paper we set $\infty \cdot 0 = 0 \cdot \infty = 0$.

$$M_p(\{\nu_x;\ x\in X_\pi^*\};\ \pi,\ \kappa)\leqq\int\pi\rho^p d\alpha=0.$$

On account of (3) it suffices to compute $M_p(\{\nu_x;\ x\in X-X_\pi^*\};\ \pi,\kappa)$. Furthermore we may assume that $\pi>0$ everywhere on $\{(x,y);\ x\in X-X_\pi^*\}$. For, if we change the values of π so that it is positive on $\{(x,y);\ x\in X-X_\pi^*\}$, the value of $M_p(\{\nu_x;\ x\in X-X_\pi^*\};\ \pi,\kappa)$ remains the same. Consequently, *we shall assume in the sequal that* $\pi>0$ *everywhere on* Z.

Before calculating the extremal length we prove

THEOREM 1. *Let* $p>1$. *We can find* $\{\nu_x'\}$, *each defined on* $\mathfrak{B}_x$, *such that*

$$M_p(\{\nu_x\};\ \pi,\ \kappa)=M_p(\{\nu_x'\};\ 1,\ \kappa), \tag{4}$$

where we allow $\nu_x'\equiv 0$ *for some* x.

PROOF. First we note that if $M_p(\{\nu_x\};\ \pi,\kappa)=\infty$ then (4) is true with $\{\nu_x'\equiv 0;\ x\in X\}$. We shall write q for $1/(p-1)$. We denote by E_π^∞ the set $\{(x,y);\ \pi(x,y)=\infty\}\in\mathfrak{C}$. First we consider the case that $E_\pi^\infty\cap Y_x\in\mathfrak{B}_x$ and $\nu_x(Y_x-E_\pi^\infty)=0$ for x belonging to $A\in\mathfrak{A}$ with $\mu(A)>0$. If ρ is κ-admissible, $\rho(x,y)>0$ on a set of $\mathfrak{B}_x$ of positive ν_x-measure for all x. For $x\in A$, $\int_{Y_x}\pi\rho^p d\nu_x=\infty$ and hence

$$\int\pi\rho^p d\alpha=\int_X\left(\int_{Y_x}\pi\rho^p d\nu_x\right)d\mu(x)=\infty.$$

It follows that $M_p(\{\nu_x\};\ \pi,\kappa)=\infty$. This case was already taken care of at the beginning of our proof.

Next, we consider the case that $\nu_x(Y_x-E_\pi^\infty)>0$ for μ-a.e. x. We denote by X_0 the set of x, for which $\nu_x(Y_x-E_\pi^\infty)>0$ and $\pi(x,y)$ is $\mathfrak{B}_x$-measurable as a function of y. Then $\mu(X-X_0)=0$. For $x\in X_0$ we define ν_x' by $\int\pi^{-q}d\nu_x$, and for $x\in X-X_0$ we set $\nu_x'=\nu_x$. Evidently $M_p(\{\nu_x;\ x\in X-X_0\};\ \pi,\kappa)=M_p(\{\nu_x';\ x\in X-X_0\};\ 1,\kappa)=0$. On account of (3) it suffices to show $M_p(\{\nu_x;\ x\in X_0\};\ \pi,\kappa)=M_p(\{\nu_x';\ x\in X_0\};\ 1,\kappa)$. Consequently, we shall assume in the rest of the proof that $\pi(x,y)$ is a $\mathfrak{B}_x$-measurable function of y and $\nu_x(Y_x-E_\pi^\infty)>0$ for all x.

Let ρ be κ-admissible in association with $\{\nu_x\}$ such that $\int\pi\rho^p d\alpha<\infty$. It holds that $\rho=0$ α-a.e. on E_π^∞. If $\rho=0$ ν_x-a.e. on $E_\pi^\infty\cap Y_x$ we set $\rho'=\pi^q\rho$ on Y_x, and otherwise we set $\rho'=\infty$ constantly on Y_x. If $\rho'=\pi^q\rho$ on Y_x,

$$\int\kappa\rho' d\nu_x'=\int_{0<\pi<\infty}\kappa\rho' d\nu_x'=\int_{0<\pi<\infty}\kappa\pi^q\rho\pi^{-q}d\nu_x$$

$$=\int_{0<\pi<\infty}\kappa\rho d\nu_x=\int\kappa\rho d\nu_x\geqq 1.$$

If $\rho'=\infty$ on Y_x,

$$\int\kappa\rho' d\nu_x' \geqq \int_{0<\pi<\infty}\kappa\cdot\infty\, d\nu_x' = \infty.$$

Hence ρ' is κ-admissible in association with $\{\nu_x'\}$ and it follows that

$$M_p(\{\nu_x'\};\,1,\,\kappa)\leqq\iint\rho'^p d\nu_x' d\mu=\iint_{0<\pi<\infty}\pi^{pq}\rho^p d\nu_x' d\mu=\int\pi\rho^p d\alpha.$$

Hence

$$M_p(\{\nu_x'\};\,1,\,\kappa)\leqq M_p(\{\nu_x\};\,\pi,\,\kappa).$$

On the other hand, let ρ' be κ-admissible in association with $\{\nu_x'\}$. We define ρ by $\pi^{-q}\rho'$ everywhere. We observe

$$1\leqq\int\kappa\rho' d\nu_x'=\int_{0<\pi<\infty}\kappa\rho\pi^q\pi^{-q}\,d\nu_x\leqq\int\kappa\rho d\nu_x.$$

We derive

$$M_p(\{\nu_x\};\,\pi,\,\kappa)\leqq\int\pi(\rho'\pi^{-q})^p d\alpha=\iint_{0<\pi<\infty}\rho'^p\pi^{-q}d\nu_x d\mu=\iint\rho'^p d\nu_x' d\mu$$

and conclude

$$M_p(\{\nu_x\};\,\pi,\,\kappa)\leqq M_p(\{\nu_x'\};\,1,\,\kappa).$$

Now (4) follows.

3. Let $p>1$. We write $f(x,y)$ for $\kappa^{p/(p-1)}\pi^{-1/(p-1)}$, and $h(x)$ for $\int_{Y_x} f(x,y)\,d\nu_x(y)$. This is an $\mathfrak{A}$-measurable function defined for μ-a.e. $x\in X$. We shall prove

THEOREM 2. *Let $p>1$. Then the equality*

$$M_p(\{\nu_x\};\,\pi,\,\kappa)=\int_X\frac{d\mu(x)}{h^{p-1}(x)} \tag{5}$$

holds if and only if $M_p(\{\nu_x;\,x\in X_h^\infty\};\,\pi,\,\kappa)=0$ for $X_h^\infty=\{x;\,h(x)=\infty\}$. If, in particular, $0<h(x)<\infty$ for μ-a.e. $x\in X$, then an extremal function is given by

$$\rho_0(x,y)=\begin{cases}\dfrac{\kappa^{\frac{1}{p-1}}\pi^{-\frac{1}{p-1}}}{h} & \textit{if } 0<h(x)<\infty,\\ \infty & \textit{otherwise.}\end{cases} \tag{6}$$

PROOF. First we consider the case that $E_{\pi}^{\infty}\cap Y_x$ belongs to $\mathfrak{B}_x$ and $h(x)$ vanishes for x belonging to $A\in\mathfrak{A}$ with $\mu(A)>0$. Since $\kappa>0$, $\pi=\infty$ ν_x-a.e. on Y_x for $x\in A$. Let ρ be any κ-admissible function. It follows that $\int\pi\rho^p d\nu_x=\infty$ for every $x\in A$ and hence that $\iint\pi\rho^p d\nu_x d\mu=\infty$. Thus $M_p(\{\nu_x\};\pi,\kappa)=\infty$. Both sides of (5) are now equal to ∞.

Next we assume that $h(x)>0$ for μ-a.e. $x\in X$. We begin with the special case that $\pi\equiv 1$. Since $\int\kappa d\nu_x>0$ for each x, $h(x)=\int\kappa^{p/(p-1)}d\nu_x>0$ for each x. Suppose $h(x)<\infty$ for μ-a.e. x. Let ρ be κ-admissible. If $h(x)<\infty$ for x, we apply Hölder's inequality and obtain

$$1\leqq\left(\int_{Y_x}\rho^p d\nu_x\right)^{1/p}\left(\int_{Y_x}\kappa^{p/(p-1)}d\nu_x\right)^{(p-1)/p}=\left(\int_{Y_x}\rho^p d\nu_x\right)^{1/p}h^{(p-1)/p}(x)$$

or

$$\frac{1}{h^{p-1}(x)}\leqq\int_{Y_x}\rho^p d\nu_x.$$

This is true for all x and it holds that

$$\int_X\frac{d\mu(x)}{h^{p-1}(x)}\leqq\int\rho^p d\alpha.$$

Hence

$$\int_X\frac{d\mu(x)}{h^{p-1}(x)}\leqq M_p(\{\nu_x\};1,\kappa).$$

On the other hand, we observe that ρ_0 is $\mathfrak{C}$-measurable and check that ρ_0 is κ-admissible. It follows that

$$M_p(\{\nu_x\};1,\kappa)\leqq\iint\rho_0^p d\nu_x d\mu=\int_X\frac{d\mu(x)}{h^{p-1}(x)}$$

and the equality is concluded. It is seen that ρ_0 is extremal.

In the case when $\mu(X_h^{\infty})>0$, we have by (3)

$$M_p(\{\nu_x\};1,\kappa)=M_p(\{\nu_x;x\in X_h^{\infty}\};1,\kappa)+M_p(\{\nu_x;x\in X-X_h^{\infty}\};1,\kappa),$$

and infer that

$$M_p(\{\nu_x;x\in X-X_h^{\infty}\};1,\kappa)=\int_{X-X_h^{\infty}}\frac{d\mu(x)}{h^{p-1}(x)}=\int_X\frac{d\mu(x)}{h^{p-1}(x)}.$$

Hence (5) holds if and only if $M_p(\{\nu_x;\ x\in X_h^\infty\};\ 1,\ \kappa)=0$.

Finally we are concerned with a general π. The inequality $\nu_x(Y_x-E_\pi^\infty)>0$ is equivalent to $h(x)>0$. Let us assume this for each x. Let $\nu_x'=\int\pi^{-1/(p-1)}d\nu_x$. We observe that $h'(x)$ defined by $\int\kappa^{p/(p-1)}d\nu_x'$ is equal to $h(x)$. Hence the set $X_{h'}^\infty=\{x;\ h'(x)=\infty\}$ is identical to X_h^∞. By Theorem 1 we have $M_p(\{\nu_x\};\ \pi,\ \kappa)=M_p(\{\nu_x'\};\ 1,\ \kappa)$ and $M_p(\{\nu_x;\ x\in X_h^\infty\};\ \pi,\ \kappa)=M_p(\{\nu_x';\ x\in X_{h'}^\infty\};\ 1,\ \kappa)$. The relation

$$M_p(\{\nu_x'\};\ 1,\ \kappa)=\int\frac{d\mu(x)}{(h'(x))^{p-1}}=\int\frac{d\mu(x)}{h^{p-1}(x)}$$

holds if and only if $M_p(\{\nu_x;\ x\in X_h^\infty\};\ \pi,\ \kappa)=0$. Furthermore, in case $h(x)<\infty$ for μ-a.e. x, we define ρ_0' by $\kappa^{1/(p-1)}h^{-1}$ if $h(x)<\infty$ and by ∞ otherwise. It is extremal in association with $\{\nu_x'\}$ as seen above. We observe that ρ_0 is κ-admissible in association with $\{\nu_x\}$ and

$$\begin{aligned}\int\pi\rho_0^p d\alpha&=\iint_{0<\pi<\infty}\pi\rho_0^p d\nu_x d\mu=\iint_{0<\pi<\infty}\rho_0'^p d\nu_x' d\mu=M_p(\{\nu_x'\};\ 1,\ \kappa)\\&=M_p(\{\nu_x\};\ \pi,\ \kappa).\end{aligned}$$

Thus ρ_0 is extremal.

It remains to treat the case when $h(x)=0$ for some x. We set $X_h^0=\{x;\ h(x)=0\}$. Since $\mu(X_h^0)=0$,

$$M_p(\{\nu_x;\ x\in X_h^0\};\ \pi,\ \kappa)=0=\int_{X_h^0}\frac{d\mu(x)}{h^{p-1}(x)}.$$

We know that

$$M_p(\{\nu_x;\ x\in X-X_h^0\};\ \pi,\ \kappa)=\int_{X-X_h^0}\frac{d\mu(x)}{h^{p-1}(x)}$$

if and only if $M_p(\{\nu_x;\ x\in X_h^\infty\};\ \pi,\ \kappa)=0$. It is concluded that (5) holds if and only if $M_p(\{\nu_x;\ x\in X_h^\infty\};\ \pi,\ \kappa)=0$. It is easy to check that (6) is extremal if $0<h(x)<\infty$ for μ-a.e. x. Our proof is completed.

4. A condition for $M_p(\{\nu_x;\ x\in X_h^\infty\};\ \pi,\ \kappa)=0$ is found in

THEOREM 3. *Let $p>1$. Suppose that $h(x)=\infty$ for all $x\in X$. Then $M_p(\{\nu_x\};\ \pi,\ \kappa)=0$ if and only if μ is σ-finite and there exists $Z_0\in\mathfrak{C}$ with the following property: The restriction of α to Z_0 is σ-finite and*

$$\int_{Y_x\cap Z_0}f d\nu_x=\infty \qquad \textit{for } \mu\textit{-a.e. } x, \tag{7}$$

where $f=\kappa^{p/(p-1)}\pi^{-1/(p-1)}$ *as before.*

PROOF. We may assume that $\pi(x, y)$ is $\mathfrak{B}_x$-measurable on Y_x for each x. It does not happen for any x that $\pi(x, y)=\infty$ ν_x-a.e. on Y_x, because it implies $h(x)=0$. Hence by taking $\int \pi^{-1/(p-1)}d\nu_x$ for ν'_x, $M_p(\{\nu'_x\}; 1, \kappa)=M_p(\{\nu_x\}; \pi, \kappa)$ by Theorem 1, and for any set $B\in\mathfrak{B}_x$, it holds that $\int_B fd\nu_x=\int_B \kappa^{p/(p-1)}d\nu'_x$. Consequently, it suffices to consider the case $\pi\equiv 1$.

Assume the condition in the theorem. Let X^* be the set of x for which (7) holds. If we denote by ν_x^* the restriction of ν_x to $Y_x\cap Z_0$ for $x\in X^*$, then $M_p(\{\nu_x^*; x\in X^*\}; 1, \kappa)\geqq M_p(\{\nu_x\}; 1, \kappa)$. Hence we may assume that $Z=Z_0$ and (7) holds for every x. Let $\{E_n\}$ be an increasing sequence in $\mathfrak{E}$ such that $\bigcup_n E_n=Z$ and $\alpha(E_n)<\infty$ for every n. If we set $\min(\kappa, n)=\kappa_n$, we have

$$\int_X\left(\int_{Y_x\cap E_n}\kappa_n^{p/(p-1)}d\nu_x\right)d\mu(x)\leqq n^{p/(p-1)}\int_X \nu_x(Y_x\cap E_n)\,d\mu(x)$$
$$=n^{p/(p-1)}\alpha(E_n)<\infty$$

so that $\int_{Y_x\cap E_n}\kappa_n^{p/(p-1)}d\nu_x<\infty$ for μ-a.e. x. Let us set $X_n=\left\{x;\ 1\leqq \int_{Y_x\cap E_n}\kappa_n^{p/(p-1)}d\nu_x\right\}$. We shall denote by $\nu_x^{(n)}$ the restriction of ν_x to E_n. By Theorem 2 we have, for $m\geqq n$,

$$M_p(\{\nu_x; x\in X_n\}; 1, \kappa)\leqq M_p(\{\nu_x^{(m)}; x\in X_n\}; 1, \kappa_m)=\int_{X_n}\frac{d\mu(x)}{\left(\int\kappa_m^{p/(p-1)}d\nu_x^{(m)}\right)^{p-1}}\leqq\mu(X).$$

If $\mu(X)<\infty$, the integral decreases to zero as $m\to\infty$. Since $X=\bigcup_n X_n$, $M_p(\{\nu_x\}; 1, \kappa)=0$ by (3). We obtain the same conclusion on account of (3) if μ is σ-finite.

Conversely, suppose $M_p(\{\nu_x\}; 1, \kappa)=0$. There exists a κ-admissible ρ_n satisfying $\int\rho_n^p d\alpha<1/n$. Since $\int\kappa\rho_n d\nu_x\geqq 1$ for every x, $\int\rho_n^p d\nu_x>0$ for every x. We set $X_m=\{x;\ 1/m\leqq\int_{Y_x}\rho_n^p d\nu_x\}$. Evidently, $X=\bigcup_m X_m$. We have

$$\frac{\mu(X_m)}{m}=\frac{1}{m}\int_{X_m}d\mu\leqq\int_{X_m}\int_{Y_x}\rho_n^p d\nu_x d\mu\leqq\int\rho_n^p d\alpha<1/n.$$

This shows that μ is σ-finite. We set next $Z_n=\{(x, y);\ \rho_n(x, y)>0\}$. We shall show that $\bigcup_n Z_n$ may be taken for Z_0. Let $Z_n^{(m)}=\{(x, y);\ \rho_n(x, y)>1/m\}$. Evidently $Z_n=\bigcup_m Z_n^{(m)}$. Since the fact $\int_{Z_n^{(m)}}\rho_n^p d\alpha<1/n<\infty$ implies $\alpha(Z_n^{(m)})<\infty$, the restriction of α to Z_n and hence to $\bigcup_n Z_n$ is σ-finite. Suppose that there were $M<\infty$ and $A\in\mathfrak{A}$ with $\mu(A)>0$ such that $\int_{Y_x\cap(\bigcup_n Z_n)} f d\nu_x < M$ for every $x\in A$. It would hold

$$1\leqq\left(\int_{Y_x\cap Z_n}\kappa\rho_n d\nu_x\right)^p\leqq\left(\int_{Y_x\cap Z_n}\rho_n^p d\nu_x\right)\left(\int_{Y_x\cap Z_n}\kappa^{p/(p-1)}d\nu_x\right)^{p-1}$$

$$=\left(\int_{Y_x\cap Z_n}\rho_n^p d\nu_x\right)\left(\int_{Y_x\cap Z_n} f d\nu_x\right)^{p-1}\leqq M^{p-1}\int\rho_n^p d\nu_x.$$

Hence

$$0<\frac{\mu(A)}{M^{p-1}}=\frac{1}{M^{p-1}}\int_A d\mu\leqq\iint\rho_n^p d\nu_x d\mu=\int\rho_n^p d\alpha<\frac{1}{n}.$$

This is impossible if n is large. Consequently $\int_{Y_x\cap(\bigcup_n Z_n)} f d\nu_x=\infty$ for μ-a.e. x. Thus all conditions on Z_0 are satisfied.

5. We shall apply Theorem 2. Using the notations of [3], we take $\tau\cap[\Gamma]$ for X, the flux φ restricted to $\tau\cap[\Gamma]$ for μ, c_Q for Y_x and the length s on c_Q for ν_x. As $\mathfrak{C}$ we take the class of all Lebesgue measurable sets in $[\Gamma]$, and given π' in $\mathscr{E}$, we take $\pi'/|\text{grad } H|$ for π. Then, for any $E\in\mathfrak{C}$,

$$\alpha(E)=\int\nu_x(E_x)d\mu(x)=\int_{Q\in\tau\cap[\Gamma]}\left(\int_{E\cap c_Q}ds\right)d\varphi(Q)$$

and

$$\int\pi\rho^p d\alpha=\int\left(\int_{c_Q}\frac{\pi'\rho^p}{|\text{grad } H|}ds\right)d\varphi(Q)=\int_{[\Gamma]}\pi'\rho^p dv.$$

It is easy to derive Theorem 1 and its generalizations in [3] from our present results.

Another choices are, with the notations of § 6 in [3], T for X, t for μ, S_t for Y_x and σ for ν_x. Then Theorem 5 of [3] is obtained immediately.

6. Next we are interested in the existence of $\mathfrak{C}$. Suppose that Z, $\mathfrak{A}$, μ, $\mathfrak{B}_x$ and ν_x are given. At the beginning we assumed the existence of $\mathfrak{C}$ satisfying the required conditions. One might wonder if $\mathfrak{C}$ really exists. Obviously it is a necessary condition that $\nu_x(Y_x)=\nu_x\{y\in Y_x;\,(x,y)\in Z\}$ is an $\mathfrak{A}$-measurable function. Conversely, assume that $\nu_x(Y_x)$ is an $\mathfrak{A}$-measurable function. Then the class of all sets of the form $\{(x,y);\,x\in A\in\mathfrak{A},\,y\in Y_x\}$ may be taken as $\mathfrak{C}$ and is, in fact, the smallest one. Furthermore, if Z can be written as $\bigcup_n Z_n$ such that each $\nu_x(Z_n \cap Y_x)$ is a finite-valued $\mathfrak{A}$-measurable function defined for μ-a.e. x, then the class of all sets $E \subset Z$ satisfying (1) and (2) may be taken as $\mathfrak{E}$ and is the largest one.

As an example, we take the interval $0<x<1$ on the x-axis as X and the linear measure as μ. Let X_1 be a non-measurable subset of X with the inner measure $\underline{m}X_1=0$ and the outer measure $\overline{m}X_1=1$. For $X_2=X-X_1$, $\underline{m}X_2=0$ and $\overline{m}X_2=1$. At each point of X_1 we take the interval $0<y<1$ for Y_x, and at each point of X_2 we take $-1/2<y<1/2$ for Y_x. The linear measure is taken for ν_x on each Y_x. Since $\nu_x(Y_x)\equiv 1$ is a measurable function on X, there exists $\mathfrak{E}$ satisfying the required conditions. By our Theorem 2 we have $M_p(\{\nu_x\};\, 1,\, 1)=1$ for any choice of $\mathfrak{E}$.

7. So far we have assumed that each $\mathfrak{B}_x$ consists of subsets of Y_x and ν_x is defined on $\mathfrak{B}_x$. Suppose now that Y_x is contained in a larger space Y'_x for each x. We take any σ-field $\mathfrak{B}'_x$ in Y'_x whose restriction to Y_x coincides with $\mathfrak{B}_x$, and define ν'_x on $\mathfrak{B}'_x$ so that $\nu_x=\nu'_x$ on $\mathfrak{B}_x$ and $\nu'_x(Y'_x-Y_x)=0$. Let $\mathfrak{E}$ and $\mathfrak{E}'$ be σ-fields of sets in $Z=\{(x,y);\, x\in X,\, y\in Y_x\}$ and $Z'=\{(x,y);\, x\in X,\, y\in Y'_x\}$ respectively which satisfy the required conditions, and suppose $\mathfrak{E}\subset\mathfrak{E}'$. We define $\alpha(E)$ by $\int_X \nu_x(E_x)d\mu(x)$ for $E\in\mathfrak{E}$, and $\alpha'(E')$ by $\int_X \nu'_x(E'_x)\,d\mu(x)$ for $E'\in\mathfrak{E}'$. If $E\in\mathfrak{E}$,

$$\alpha'(E)=\int_X \nu'_x(E_x)d\mu(x)=\int_X \nu_x(E_x)d\mu(x)=\alpha(E).$$

Hence α' is an extension of α. Furthermore, if $f(x,y)$ is non-negegative and $\mathfrak{E}'$-measurable,

$$\int f d\alpha'=\int_X\left(\int_{Y'_x} f d\nu'_x\right)d\mu(x)=\int_X\left(\int_{Y_x} f d\nu_x\right)d\mu(x)=\int f d\alpha.$$

We see easily that $M_p(\{\nu_x\}:\,\pi,\,\kappa)=M_p(\{\nu'_x\};\,\pi',\,\kappa')$ if π' and κ' are non-negative $\mathfrak{E}'$-measurable extensions of π and κ in Z' such that the restriction of κ' to Y'_x is $\mathfrak{B}'_x$-measurable for each x. Roughly speaking, the extremal length does not change for any extension of Y_x if ν_x is extended by the value 0.

Next we consider the special case that Y_x is common. Namely, Z is the product space $\{(x,y);\, x\in X,\, y\in Y\}$. Furthermore, $\mathfrak{B}_x$ and ν_x may or may not be common. To illustrate it, let us be concerned with the example discussed at the end of Section 6. We take as Y the y-axis or any interval containing the interval $-1/2<y<1$, and take as $\mathfrak{B}_x$ the common class $\mathfrak{B}$ of linearly measurable subsets of Y. We take as ν_x the linear measure on $\{(x,y);\, 0<y<1\}$ (on $\{(x,y);\, -1/2<y<1/2\}$ resp.) for $x\in X_1$ (X_2 resp.), extended by 0 elsewhere. Then again $M_p(\{\nu_x\};\, 1,\, 1)=1$ for any choice of $\mathfrak{E}$. However, we cannot take the class of Lebesgue measurable sets in Z as $\mathfrak{E}$, because $E=\{0<x<1,\, 0<y<1/2\}$, for instance, does not satisfy condition (2). We add as a remark that $\inf_{\rho} \iint \rho^2 dxdy = 4/3$ as computed in § 2 of [2], where ρ is a non-negative Lebesgue measurable function satisfying $\int_0^1 \rho dy \geqq 1$ if $x \in X_1$ and $\int_{-1/2}^{1/2} \rho dy \geqq 1$ if $x \in X_2$.

As another example in which Y and $\mathfrak{B}$ are common but ν_x are different, we consider again a harmonic subflow Γ treated in Section 5. As before we take $\tau \frown [\Gamma]$ for X and take the flux φ restricted to $\tau \frown [\Gamma]$ for μ. Here, we take the t-axis for the common Y and the class of linearly measurable sets for the common $\mathfrak{B}$. As ν_x we take the measure which is equal to the linear measure on the image of c_Q by $t = H(P)$ and which vanishes outside the image. As $\mathfrak{C}$ we take the class of Lebesgue measurable sets in the product space Z. Given π' in $\mathscr{E}$, we take $\pi'/|\text{grad}\, H|^2$ for π, and obtain the same value of the module as in Section 5. If we want to keep the same value of the module while taking the linear measure on the t-axis as the common ν, we take κ' which is the extension of κ by 0. The same remark holds for the preceding example.

Finally, we remark that we may limit ourselves to the case when Y and $\mathfrak{B}$ are common if we want. Let X, $\mathfrak{A}$, μ, Y_x, ν_x and $\mathfrak{C}$ be given as in Section 1. We consider the sum-space $Y = \sum_x Y_x$. In order to avoid a possible confusion between the points in the product space $X \times Y$ and the points of Y_x, we shall write $Y = \sum_u Y_u$. Let $\mathfrak{B}$ be the σ-field in Y whose restriction to Y_u is equal to $\mathfrak{B}_u$ for each u. If $\mathfrak{B}$ is regarded as a σ-field on $\{(x, y); y \in Y\}$, then we shall use the notation $\mathfrak{B}^{(x)}$. Let ν'_x be the measure on $\mathfrak{B}^{(x)}$ such that $\nu'_x = \nu_x$ on $\mathfrak{B}_x$ and $\nu'_x(Y - Y_x) = 0$, let $\mathfrak{C}'$ be a σ-field in $X \times Y$ containing $\mathfrak{C}$ and let π' and κ' be respectively non-negative $\mathfrak{C}'$-measurable extensions of π and κ in $X \times Y$ such that the restriction of κ' to $\{(x, y); y \in Y\}$ is $\mathfrak{B}^{(x)}$-measurable for each x. Then we have

$$M_p(\{\nu_x\}; \pi, \kappa) = M_p(\{\nu'_x\}; \pi', \kappa').$$

If we want to take the common ν_x, we define ν on $\mathfrak{B}$ by the equality $\nu = \nu_u$ on $\mathfrak{B}_u$ for each u and set $\kappa' = 0$ in $X \times Y$ outside $\{(x, y); x \in X, y \in Y_x\}$. As a measure on $\{(x, y); y \in Y\}$, ν being denoted by $\nu^{(x)}$, it holds that $M_p(\{\nu_x\}; \pi, \kappa) = M_p(\{\nu^{(x)}\}; \pi', \kappa')$.

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