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Modules over (qa)-rings

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Let R be a commutative ring with unit. When the total quotient ring Q of R is an Artinian ring we call R a (qa)-ring. In this paper we are mainly concerned with the theory of modules over such a ring. In §1, some preliminary results are summarized. In §2 we shall prove the following (Theorem 2. 10): Let R be a (qa)-ring with the self-injective total quotient ring, and let M be an h-divisible R-module such that $M/t(M)$ is an injective R-module. Then $t(M)$ is a direct summand. Some applications of the preceding result will be discussed in §3.

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1. Preliminaries

Let R be a commutative ring with 1 and let S be the set of all non zero-divisors in R. The total quotient ring R_S is denoted by Q, and K will denote the quotient module Q/R. Let M be a module (always assumed to be unitary) over the ring R. An element x in M is torsion if there is an element s in S such that $sx=0$, and torsion-free otherwise. M is called a torsion module if every element in M is torsion, and a torsion-free module if every element in M is trosion-free. Let M be an R-module. Then as is easily seen there is the unique maximal submodule which is torsion. This submodule will be denoted by $t(M)$ and will be called the torsion submodule of M. An R-module M is torsion-free if and only if $t(M)=0$.

Proposition 1.1. *Let M be an R-module. Then we have* $t(M)\cong \mathrm{Tor}_1^R(K, M)$.

Proof. From $0\to R\to Q\to K\to 0$, we have the following exact sequence: $0\to \mathrm{Tor}_1^R(Q, M)\to \mathrm{Tor}_1^R(K, M)\to M\to Q\otimes_R K$. But $\mathrm{Tor}_1^R(Q, M)=0$ since Q is a flat R-module, and by Proposition 1.4 $\mathrm{Tor}_1^R(K, M)$ is torsion. Thus $\mathrm{Tor}_1^R(K, M)\to t(M)$ is monomorphic. On the other hand, if N is a torsion-free module, then we have a canonical map: $N\to Q\otimes_R N$ is monomorphic. Therefore $\mathrm{Tor}_1^R(K, M)\to t(M)$ is an onto R-homomorphism. Thus $t(M)\cong \mathrm{Tor}_1^R(K, M)$.

Corollary 1. 2. *For any R-module M we have the following exact sequence:*

$$0\to M/t(M)\to Q\otimes_R M\to K\otimes_R M\to 0.$$

DEFINITON. *Let M be an R-module. Then M is called a divisible R-module in case $sM=M$ for any s in S,* i.e., *for any x in M and s is S there exists y in M such that $sy=x$.*

From the definition it follows immediately that for any R-module there is the unique maximal divisible submodule.

LEMMA 1.3. (1) *If M is a divisible R-module, then* $\mathrm{Hom}_R(M, N)$ *is a torsion-free R-module for any R-module N.*

(2) *If M is a torsion-free, divisible R-module, then* $\mathrm{Hom}_R(M, N)$ *is also a torsion-free, divisible R-module for any R-module N.*

PROOF. (1) Let $f \in \mathrm{Hom}_R(M, N)$ and assume that $sf=0$ for some s in S. For any x in M, there is y in M such that $sy=x$, and so $f(x)=f(sy)=sf(y)=0$. Therefore $f=0$.

(2) By (1), $\mathrm{Hom}_R(M, N)$ is torsion-free. In order to show $\mathrm{Hom}_R(M, N)$ is divisible, let us take $s \in S$ and $0 \neq f \in \mathrm{Hom}_R(M, N)$. Define $g: M \to N$ by $g(x)=f(x/s)$ for all x in M (x/s is well defined because M is torsion-free and divisible). It is easily seen that $g \in \mathrm{Hom}_R(M, N)$ and $sg=f$. Hence $\mathrm{Hom}_R(M, N)$ is divisible.

PROPOSITION 1.4. (1) *If M is a torsion-free, divisible R-module, then* $\mathrm{Ext}_R^i(M, N)$ *is also a torsion-free, divisible R-module for any R-module N and for all $i \geqq 0$.*

(2) *Let M and N be two R-modules. Then if M or N is torsion,* $\mathrm{Tor}_i^R(M, N)$ *is torsion, and if M or N is a torsion-free, divisible R-module,* $\mathrm{Tor}_i^R(M, N)$ *is torsion-free and divisible, for all $i \geqq 0$.*

PROOF. (1) Let $I: 0 \to N(=I_0) \to I_1 \to I_2 \to \cdots \to I_n \to \cdots$ be an injective resolution of N. Then $\mathrm{Ext}_R^i(M, N) \cong H^i(\mathrm{Hom}(M, I))$ for each value of i. As $\mathrm{Hom}_R(M, I_n)$ is torsion-free and divisible for all $n \geqq 0$ by Lemma 1.3, we have $H^i(\mathrm{Hom}(M, I))$ is torsion-free and divisible for all $i \geqq 0$. Thus $\mathrm{Ext}_R^i(M, N)$ is torsion-free and divisible for all $i \geqq 0$.

(2) Assume that M is torsion-free and divisible, and that $P: \cdots \to P_n \to \cdots \to P_2 \to P_1 \to N(=P_0) \to 0$ be a projective resolution on N. Then $\mathrm{Tor}_i^R(M, N) \cong H_i(M \otimes P)$ for each value of i. Since $M \otimes_R P_n$ is torsion-free and divisible, $H_i(M \otimes P)$ is also torsion-free and divisible for all $i \geqq 0$. Thus $\mathrm{Tor}_i^R(M, N)$ is torsion-free and divisible for all $i \geqq 0$. If M is a torsion R-module, it is easy to see that $\mathrm{Tor}_i^R(M, N)$ is torsion since $\mathrm{Tor}_i^R(M, N) \cong H_i(M \otimes P)$ for a projective resolution P of N and for all $i \geqq 0$, and $M \otimes_R P_n$ is torsion for $n=1, 2, \cdots$.

COROLLARY 1.5. $\mathrm{Ext}_R^n(Q, M)$ *is a torsion-free, divisible R-module and* $\mathrm{Tor}_n^R(K, M)$ *is a torsion R-module for any R-module M and for all $n \geqq 0$.*

DEFINITION. *We say that an R-module M has the property (D) in case there are a torsion-free, divisible R-module N and an R-homomorphism $f: N \to M$*

such that $f(N)=M$.

PROPOSITION 1.6. *For any R-module M there exists the unique maximal submodule which has the property* (D) (*this submodule is denoted by* $D(M)$ *and called the D-submodule of* M).

PROOF. If D_1 and D_2 are two submodules of M having the property (D), then D_1+D_2 also has the property (D). Thus the union of all submodules of M having the property (D) is the unique maximal submodule of M with the property (D).

PROPOSITION 1.7. *Let M be an R-module. Then we have the following exact sequence*: $0\to \mathrm{Hom}_R(K, M)\to \mathrm{Hom}_R(Q, M)\to D(M)\to 0$.

PROOF. From $0\to R\to Q\to K\to 0$, we have an exact sequence:

$$0\to \mathrm{Hom}_R(K, M)\to \mathrm{Hom}_R(Q, M)\overset{\beta}{\to} M.$$

Since $\mathrm{Hom}_R(Q, M)$ is torsion-free and divisible, $\beta(\mathrm{Hom}_R(Q, M))\subseteqq D(M)$.

Conversely, since $D(M)$ is a D-module there is a torsion-free, divisible R-module N such that $D(M)$ is a homomorphic image of N under an R-homorphism f. For any $0\neq x\in D(M)$, there is y in N such that $f(y)=x$. Define $g\colon R\to N$ by $g(1)=y$. Then there exists a unique $h\in \mathrm{Hom}_R(Q, N)$ such that the restriction of h to R is g because N is a torsion-free, divisible R-module, and so $fh(1)=f(y)=x$. Thus $x\in\beta(\mathrm{Hom}_R(Q, M))$.

COROLLARY 1.10. *For any R-module M we have the following exact sequence*: $0\to M/D(M)\to \mathrm{Ext}_R^1(K, M)\to \mathrm{Ext}_R^1(Q, M)\to 0$.

DEFINITION. *An R-module M is called an h-reduced R-module in case* $\mathrm{Hom}_R(Q, M)=0$, *M a cotorsion R-module in case* $\mathrm{Ext}_R^i(Q, M)=0$ *for* $i=1, 2$, *and M a strongly cotorsion R-module in case* $\mathrm{Ext}_R^i(Q, M)=0$ *for all* $i\geqq 0$. *A torsion R-module T is said to be of bounded order if* $sT=0$ *for some s in S.*

PROPOSITION 1.8. *If a torsion R-module T is of bounded order, then T is a strongly cotorsion R-module.*

PROOF. $\mathrm{Ext}_R^i(Q, T)$ is a torsion R-module of bounded order because T is of bounded order, for all $i\geqq 0$. On the other hand, by Proposition 1.4 $\mathrm{Ext}_R^i(Q, T)$ is torsion-free and divisible for all $i\geqq 0$. Thus $\mathrm{Ext}_R^i(Q, T)=0$ for all $i\geqq 0$.

DEFINITION. *An R-module M is called an h-divisible R-module if there are an injective R-module I and a surjective R-homomorphism $f\colon I\to M$, and a cotorsion-free R-module if M has no non-zero cotorsion factor modules.*

PROPOSITION 1.9. (1) *Every h-divisible R-module M has the property* (D).

(2) *Every R-module M with the property (D) is cotorsion-free.*
(3) *Every cotorsion-free R-module M is divisible.*

PROOF. (1) Since M is a homomorphic image of an injective R-module N, there is a torsion-free, injective R-module which has M as a homomorphic image. In fact, let F be a free R-module which has N as a homomorphic image under an R-homomorphism μ. Then since N is injective, μ can be extended to an R-homomorphism of $E(F)$ to N, that is onto homomorphism. On the other hand, any injective R-module is divisible. Thus M has the property (D).

(2) Let M have the property (D) and N a submodule of M. Then M/N also has the property (D), and so it is sufficient to show that M is not cotorsion. Since M has the property (D), there is a torsion-free, divisible R-module G such that G has M as a homomorphic image under a homomorphism μ. For $0\neq x\in M$, take $y\in G$ such that $\mu(y)=x$. Define $f\in \mathrm{Hom}_R(R, G)$ by $f(1)=y$. As G is a torsion-free, divisible R-module there is a unique g in $\mathrm{Hom}_R(Q, G)$ such that the restriction of g to R is f. Thus $0\neq \mu g\in \mathrm{Hom}_R(Q, M)$. Hence M is not cotorsion.

(3) Assume that $sM\neq M$. Then M/sM is a torsion R-module of bounded order, and so M is not cotorsion-free. This contradicts the hypothesis.

REMARKS. 1. For any R-module M the dual module M^* is defined as follows. Let us set $E=E(\sum\oplus R/P)$, $\sum$ runs through all the maximal ideals P of R. We shall set $M^*=\mathrm{Hom}_R(M, E)$. Then for M^* the above three conditions are equivalent though they are not known in general.

2. When R is an integral domain, E. Matlis proved the following fact. For an R-module M, $M^*$is cotorsion (cotorsion-free) if and only if M is torsion (torsion-free) ([6], Proposition 1.3). But this fact is also true even when R is not domain.

2. Some conditions for the torsion submodule to be a direct summand

Let M be a torsion-free, divisible R-module. Then M can be regarded as a Q-module, and so the following lemma is well defined.

LEMMA 2.1. *Let M be a torsion-free, divisible R-module. Then M is injective as an R-module if and only if M is injective as a Q-module.*

PROOF. Assume that M is injective as an R-module. Let $\mathfrak{A}'$ be any ideal of Q, $0\neq f\in \mathrm{Hom}_Q(\mathfrak{A}', M)$, and $\mathfrak{A}=\mathfrak{A}'\cap R$. Then since M is torsion-free and $\mathfrak{A}'/\mathfrak{A}$ is torsion as an R-module, the restriction of f to $\mathfrak{A}$ is not zero, and so there is a unique $g\in \mathrm{Hom}_R(R, M)$ such that the restriction of g to $\mathfrak{A}$ is f on $\mathfrak{A}$ because M is a torsion-free, injective R-module.

g can be extended to a unique Q-homomorphism $h\colon Q\to M$ because M is a torsion-free, divisible module. Furthermore it is easy to see that the restriction of h to $\mathfrak{A}'$ is f. Thus M is injective as a Q-module.

Conversely, let $\mathfrak{L}$ be any ideal of R, $f \in \mathrm{Hom}_R(\mathfrak{L}, M)$ and $\mathfrak{L}' = Q\mathfrak{L}$. Since M is a torsion-free, divisible R-module, f can be extended uniquely to $g: \mathfrak{L}' \to M$, and g can be regarded as a Q-homomorphism. Thus g can be extended to $h \in \mathrm{Hom}_Q(Q, M)$ since M is an injective Q-module. Moreover the restriction of h to $\mathfrak{L}$ is f and $Q \supseteqq R$. Hence M is injective as an R-module.

COROLLARY 2.2. *Let R be a ring with the Noetherian total quotient ring. Then a direct sum $M = \sum_{i \in I} \oplus M_i$ of torsion-free, divisible R-modules is injective if and only if each direct summand M_i is injective.*

PROOF. Since Q is Noetherian, M is injective as a Q-module if and only if M_i is injective as a Q-module for each i. Thus by Lemma 2.1 we have the result.

PROPOSITION 2.3.*) *If R is a ring with the Noetherian total quotient ring Q, then the following conditions are equivalent.*

(1) *Q is h-divisible.*

(2) *For any R-module M, M is h-divisible if and only if M has the property (D).*

PROOF. If M has the property (D), then there is a free Q-module F which has M as a homomorphic image because M is a torsion-free, divisible R-module. On the other hand, since Q is h-divisible, there is a torsion-free, injective R-module H having Q as a homomorphic image. As Q is Noetherian and H can be regarded as a Q-module, by Corollary 2.2 a direct sum of any number of H's is injective as an R-module, and so F is h-divisible. Hence M is h-divisible. The converse case is trivial.

THEOREM 2.4.**) *Let R be a (qa)-ring with the self-injective total quotient ring Q, and M a torsion-free, divisible R-module. Then M is injective as an R-module if and only if M is projective as a Q-module.*

PROOF. Assume that M is projective as a Q-module. Then M is a direct sum of a free Q-module F. Since Q is a self-injective and Noetherian ring, F is an injective Q-module. Thus M is injective as an R-module because M is a direct summand of the injective Q-module F, hence injective R-module by Lemma 2.1.

Conversely, assume that M is an injective R-module. Then M is an injective Q-module by Lemma 2.1.

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*) If Q is h-divisible, then Q is injective.

**) This result is contained in Theorem 18 of [7]. But since R is a commutative ring, we can give here a simple proof based on a different principle.

Since $Q=\sum_{i=1}^{n}\oplus Q_i$ as a ring, where Q_i is an Artinian local ring for $i=1, 2, \cdots, n$, we can write $M=\sum_{i=1}^{n}\oplus M_i$, where $M_i=Q_iM$ for $i=1, 2, \cdots, n$. Moreover M_i is injective as a Q_i-module because M_i is injective as a Q_i-module and $Q=\sum_{i=1}^{n}\oplus Q_i$ as a ring, for $i=1, 2, \cdots, n$.

By Theorem 2.5 and Theorem 3.1 of [4], we have $M_i=\sum_{\alpha\in\Gamma}\oplus E(Q_i/P_i)_\alpha$, where P_i is the maximal ideal of Q_i and $E(Q_i/P_i)_\alpha=E(Q_i/P_i)$ for each $\alpha\in\Gamma$, for $i=1, 2, \cdots, n$. On the other hand, since Q_i is a self-injective Noetherian local ring, Q_i is an indecomposable injective Q_i-module, and so by Theorem 3.1 of [4] $Q_i\cong E(Q_i/P_i)$ as a Q_i-module, for all i. Thus M_i is a free Q-module for $i=1, 2, \cdots, n$. From this M is a projective Q-module.

If R is a local ring any projective R-module is free. Then we have:

Corollary 2.5. *Let R be a self-injective Artinian local ring. Then any R-module M is injective if and only if M is free.*

The following Corollary was given by I. Levy in 1963 (see Theorem 16 in p. 172 of [9]). Now we can give here an easy proof, using Theorem 2.4.

Corollary 2.6. *The following statement are equivalent.*
(1) *Q is a semi-simple ring.*
(2) *Every torsion-free, divisible module is injective.*

Proof. (1)→(2). Since Q is semi-simple, every Q-module is Q-projective. Thus from Theorem 2.4 every torsion-free, divisible R-module is an injective R-module.

(2)→(1). As every torsion-free, divisible R-module is injective as an R-module, by Lemma 2.1 every Q-module is injective as a Q-module. Thus Q is semi-simple.

Corollary 2.7. *Let R be a (qa)-ring and let us set $Q=\sum_{i=1}^{n}\oplus Q_i$. Assume that the set of all ideals of Q_i is linearly ordered for all i, and that M is a torsion-free, divisible R-module. Then M is injective as an R-module if and only if M is projective as a Q-module.*

Proof. If the set of all ideals of Q_i is linearly ordered, then Q_i is a self-injective ring for $i=1, 2, \cdots, n$. Thus the result follows from Theorem 2.4.

Lemma 2.8.**) *If R is a Noetherian ring with the self-injective total quotient ring Q, then R is a (qa)-ring.*

Proof. By the assumption Q is Noetherian, and so (0) has a irredundant irreducible primary decomposition: $(0)=\bar{q}_1\cap\bar{q}_2\cap\cdots\cap\bar{q}_n$. By Theorem 2.3 of [4], the canonical imbedding of Q into $Q/\bar{q}_1\oplus Q/\bar{q}_2\oplus\cdots\oplus Q/\bar{q}_n$ can be extended to an isomorphism of Q onto $E(Q/\bar{q}_1)\oplus E(Q/\bar{q}_2)\oplus\cdots\oplus E(Q/\bar{q}_n)$. But $Q/\bar{q}_1\oplus Q/\bar{q}_2\oplus\cdots\oplus Q/\bar{q}_n$ is an essential extension of Q and $E(Q/\bar{q}_1)\oplus\cdots\oplus E(Q/\bar{q}_n)$ an essential extension of $Q/\bar{q}_1\oplus\cdots\oplus Q/\bar{q}_n$. Thus the canonical imbedding of Q into $Q/\bar{q}_1\oplus\cdots\oplus Q/\bar{q}_n$ is onto. Hence it is sufficient to show that $Q/\bar{q}_i$ is Artinian local. On the other hand, $Q/\bar{q}_i$ is an indecomposable injective R-module. Thus by Proposition 2.2 of [4] $Q/\bar{q}_i=E(R/q_i)$, where $q_i=R\cap\bar{q}_i$. Let $\bar{p}_i$ is a prime ideal of Q such that $\bar{q}_i$ is $\bar{p}_i$-primary, and set $p_i=R\cap\bar{p}_i$. Then $Q/\bar{q}_i\cong E(R/p_i)$ by Proposition 3.1 of [4]. In order to prove that $Q/\bar{q}_i$ is Artinian local, it is sufficient to show that for any $\bar{x}\in Q/\bar{q}-\bar{p}_i/\bar{q}_i$, $\bar{x}$ is unit in $Q/\bar{q}_i$. Let x be an element of Q such that $\bar{x}$ is a representation of x in $Q/\bar{q}_i$. Then $sx\in R-p_i$ for some s in S. By Lemma 3.2 of [4] the homomorphism: $Q/\bar{q}_i\rightarrow Q/\bar{q}_i$ defined by $y\rightarrow(sx)y$ is an automorphism of $Q/\bar{q}_i$. Thus $\overline{sx}$ is unit in $Q/\bar{q}_i$. Therefore $\bar{x}$ is unit.

THEOREM 2.9. *If R is a Noetherian ring, then the following conditions are equivalent.*

(1) *Q is a self-injective ring.*

(2) *For any torsion-free, divisible R-module M, M is injective as an R-module if and only if M is projective as a Q-module.*

PROOF. (1)→(2). By Lemma 2.8, R is a (qa)-ring. Hence the result follows from Theorem 2.4. The converse is immediate.

THEOREM 2.10. *Assume that R is a (qa)-ring such that Q is a self-injective ring. If an R-module M is h-divisible and $M/t(M)$ is an injective R-module, then $t(M)$ is a direct summand of M.*

PROOF. Since $M/t(M)$ is a torsion-free, injective R-module, by Theorem 2.4 $M/t(M)$ is a projective Q-module. Thus we may write $M/t(M)=M_1\oplus M_2\oplus\cdots\oplus M_n$, where $M_i=Q_iM/t(M)$ $(Q=\sum_{i=1}^{n}\oplus Q_i)$ for $i=1, 2, \cdots, n$, and so M_i is a projective Q-module for all i. Hence M_i is a free Q_i-module because $Q=\sum_{i=1}^{n}\oplus Q_i$ as a ring and Q_i is a local ring for $i=1, 2, \cdots, n$.

Therefore we have $M/t(M)=\sum_{(i,\alpha)}\oplus Q_{i\alpha}$, where $Q_{i\alpha}\cong Q_i$ as a Q_i-module for each α, for $i=1, 2, \cdots, n$.

By the hypothesis that M is an h-divisible R-module, there exists an injective R-module D such that M is a homomorphic image of D. Moreover we may assume that D is a torsion-free, injective R-module. In fact, for the injective module D there is a free R-module F such that D is a homomorphic image of F under a homomorphism μ. Since D is injective, μ can be extended to an R-homomorphism g of $E(F)$ to D. Thus M is a homomorphic image of a torsion-free, injective R-module $E(F)$.

As D is a torsion-free, injective R-module, by Theorem 2.4 we can write

$D=\sum_{i=1}\oplus D_i$, where D_i is a free Q_i-module for $i=1, 2, \cdots, n$.

Let h be the given surjection of D to M and g the given surjection of M to $\sum_{(i,\alpha)}\oplus Q_{i\alpha}$.

By Lemma 2 of [3] it is sufficient to show that $t(M)$ is a direct summand of each $B_{i\alpha}=g^{-1}(Q_{i\alpha})$.

Since D and $\sum_{(i,\alpha)}\oplus Q_{i\alpha}$ are torsion-free, divisible R-modules, gh can be regarded as a Q-homomorphism.

Let $e_{i\alpha}$ be a generator of $Q_{i\alpha}$ as a Q_i-module. Then there is y in D such that $gh(y)=e_{i\alpha}$. Moreover, if $y=\sum_{i=1}^{n} y_i$ ($y_i \in D_i$ for $i=1, 2, \cdots, n$), then $gh(y_i)=1_i gh(y)=e_{i\alpha}$, where 1_i is the identity of Q_i. Thus we have $gh(y_i)=e_{i\alpha}$.

Consider $Q_i y_i$ in D_i. Then gh is an isomorphism on $Q_i y_i$ as Q_i-homomorphism and $h(ry_i)\neq 0$ for any ry_i in $Q_i y_i$ because of $g(h(ry_i))=gh(ry_i)=rgh(y_i)=re_{i\alpha}\neq 0$. Thus $Q_i y_i \cong h(Q_i y_i)\cong Q_i$ as an R-module.

As $gh(Q_i y_i)=Q_{i\alpha}$, $h(Q_i y_i)\subseteq B_{i\alpha}$. From the facts that $g\colon h(Q_i y_i)\to Q_{i\alpha}$ is surjective and $gh\colon Q_i y_i \to Q_{i\alpha}$ is isomorphic, we have $h(Q_i y_i)+t(M)=B_{i\alpha}$ and $h(Q_i y_i)\cap t(M)=0$. Thus $B_{i\alpha}=h(Q_i y_i)\oplus t(M)$ for each (i, α). Therefore $t(M)$ is a direct summand of M.

COROLLARY 2.11. *Let R be a Noetherian ring such that Q is a self-injective ring and let M be an h-divisible R-module such that $M/t(M)$ is an injective R-module. Then $t(M)$ is a direct summand of M.*

COROLLARY 2.12. *If R is a ring with a semi-simple total quotient ring Q, then the torsion submodule of every h-divisible R-module is a direct summand.*

PROOF. Since Q is semi-simple, every torsion-free, divisible R-module is injective by Corollary 2.6. Thus we have the result by Theorem 2.10.

LEMMA 2.13. *Let C be a cotorsion R-module. Then* $\mathrm{Ext}_R^i(M, C)=0$ *for* $i=1, 2,$ *for any torsion-free, divisible R-module M.*

PROOF. Since M is a torsion-free, divisible R-module, M can be regarded as a Q-module. Consider the following exact sequence of Q-modules:

$$0\to N\to F\to M\to 0,$$

where F is a free Q-module.

From this we have the following exact sequence:

$$0\to \mathrm{Hom}_R(M, C)\to \mathrm{Hom}_R(F, C)\to \mathrm{Hom}_R(N, C)\to \mathrm{Ext}_R^1(M, C)$$
$$\to \mathrm{Ext}_R^1(F, C)\to \mathrm{Ext}_R^1(N, C).$$

Since C is cotorsion, $\mathrm{Hom}_R(F, C)=0$ and $\mathrm{Ext}_R^1(F, C)=0$. Thus we have $\mathrm{Hom}_R(M, C)=0$ and $\mathrm{Hom}_R(N, C)\cong \mathrm{Ext}_R^1(M, C)$. In the same way, we have

$\mathrm{Hom}_R(N, C)=0$. Therefore $\mathrm{Ext}_R^1(M, C)=0$.

THEOREM 2.14. *If M is an R-module such that $t(M)$ is cotorsion and $M/t(M)$ is divisible, then $t(M)$ is a direct summand of M.*

PROOF. It is sufficient to show that $\mathrm{Ext}_R^1(M/t(M), t(M))=0$ because $t(M)$ is a direct summand of M if $\mathrm{Ext}_R^1(M/t(M), t(M))=0$. Since $M/t(M)$ is a torsion-free, divisible R-module and $t(M)$ is cotorsion by Lemma 2.13 $\mathrm{Ext}_R^1(M/t(M), t(M))=0$.

COROLLARY 2.15. *Let R be a ring and M be a divisible R-module such that $t(M)$ is cotorsion. Then $t(M)$ is a direct summand of M.*

COROLLARY 2.16. *Let R be a (qa)-ring such that $Q=\sum_{i=1}^{n} Q_i$ is a self-injective ring and let N be an R-module such that $M/t(M)$ is injective and $Q_i(M/t(M))\neq 0$ for all i. Then the following conditions are equivalent.*

i) *$t(M)$ is cotorsion.*

ii) *$t(M)$ is a direct summand of M and h-reduced.*

PROOF. By the assumption, we have $M/t(M)=\sum_{(i,\alpha)}\oplus Q_{i\alpha}$, where $Q_{i\alpha}\cong Q_i$ as a Q_i-module for each α, for $i=1, 2, \cdots, n$ (see in the proof of Theorem 2.4.) Thus $\mathrm{Ext}_R^1(M/t(M), t(M))=0$ if and only if $\prod_{(i,\alpha)}\mathrm{Ext}_R^1(Q_{i\alpha}, t(M))=0$ if and only if $t(M)$ is cotorsion since $t(M)$ is h-reduced and $Q_iM/t(M)=0$ for all i.

COROLLARY 2.17 *If R is a ring with the semi-simple total quotient ring $Q=\sum_{i=1}^{n}\oplus Q_i$, then the following conditions are equivalent for any divisible R-module M such that $Q_i(M/t(M))\neq 0$ for all i.*

i) *$t(M)$ is cotorsion.*

ii) *$t(M)$ is a direct summand of M and h-reduced.*

PROOF. Since Q is semi-simple, $M/t(M)$ is injective by Corollary 2.6. Hence this corollary follows from the preceding one.

3. (qa)-rings with the self-injective total quotient rings

PROPOSITION 3.1. *If R is a Noetherian (qa)-ring and the total quotient ring Q is a self-injective ring, then an R-module M is torsion if and only if* $\mathrm{Hom}_R(M, Q)=0$.

PROOF. If M is a torsion module, then since a homomorphic image of a torsion module is torsion and Q is a torsion-free module, $\mathrm{Hom}_R(M, Q)=0$.

Conversely, assume that $\mathrm{Hom}_R(M, Q)=0$. If $0\neq x\in M$ is not torsion, then there is $f\in\mathrm{Hom}_R(M, Q)$ such that $f(x)\neq 0$. In fact, since R is a Noetherian (qa)-ring and Q is a self-injective ring, we may write $Q=\sum_{i=1}^{n}\oplus Q_i$, where Q_i is a self-injective Artinian local ring associated to minimal prime ideal P_i for $i=1, 2, \cdots, n$, and so we have that Q_i is an indecomposable injective R-module for all i. Thus $Q_i \cong E(R/P_i)$ by Proposition 2.2 of [4], for $i=1, 2, \cdots, n$.

On the other hand, $0(x)$ (order ideal of x in R) $\subsetneqq P_j$ for some j because x is not torsion. Define $f\colon Rx \to Q_j$ by $f(x)=y(\neq 0) \in A_1$, where $A_1=\{t \in Q_j/P_j t =0\} \neq 0$ by Theorem 3.4 of [4]. Since Q is injective, there is $g \in \mathrm{Hom}_R(M, Q)$ such that the restriction of g to Rx is f. This is the contradiction to $\mathrm{Hom}_R(M, Q)=0$.

Proposition 3.2. *Assume that R is a ring with the semi-simple total quotient ring Q. Then $E(M)$ is a torsion module for any torsion module M.*

Proof. Since Q is semi-simple, by Corollary 2.12 $E(M)=t(E(M))\oplus N$, where N is torsion-free and divisible. From the facts that N is torsion-free and $E(M)$ is an essential extension of M, $N=0$.

Theorem 3.3. *Let R be a (qa)-ring with the self-injective total quotient ring Q. Then the following conditions are equivalent.*

(1) *Q is a semi-simple ring.*

(2) *$E(M)/M$ is a torsion module for any R-module M.*

Proof. (1)→(2). For any R-module M, by Corollary 2.12 $E(M)=t(E(M)) \oplus N$, where N is a torsion-free, divisible R-module. It is easily seen that the class of element in $t(E(M))$ is torsion in $E(M)/M$.

Since N is torsion-free and divisible, we can write $N=N_1\oplus N_2\oplus\cdots\oplus N_n$, where $N_i=Q_iN(Q=\sum_{i=1}^{n}\oplus Q_i)$ is a vector space over Q_i for $i=1, 2, \cdots, n$. For $0\neq x \in N$ $x=\sum_{i=1}^{n} x_i(x_i \in N_i)$, and since $E(M)$ is an essential extension of M, $Rx_i \cap M\neq 0$. Thus there is an ideal $\mathfrak{A}_i \supsetneqq P_i$, where P_i is the minimal prime divisor of (0) in R, associated to Q_i, such that $\mathfrak{A}_i x_i \subseteqq M$ for all i.

Since $\mathfrak{A}_i$ properly contains P_i, there exists an element $s_i \in \mathfrak{A}_i - P_i$ such that $s_i x_i \in M$ and $s_i \in P_j$ if $i \neq j$, for $i=1, 2, \cdots, n$. Put $s=\sum_{i=1}^{n} s_i$. Then $sx=\sum_{i=1}^{n} s_i x_i \in M$. Thus x is torsion modulo M because $s \in S$.

Conversely, assume that Q is not semi-simple. By the assumption $Q=\sum_{i=1}^{n}\oplus Q_i$, where Q_i is a self-injective Artinian local ring with the maximal ideal $\mathfrak{M}_i$ for $i=1, 2, \cdots, n$. Thus there is at least one j such that Q_j is not a field because Q is not semi-simple, and so $Q_j/\mathfrak{M}_j$ is a torsion free R-module. But since Q_j is a self-injective local ring, Q_j is an indecomposable, injective R-module. Hence by Proposition 2.2 of [4] $E(\mathfrak{M}_j)=Q_j$, and so $E(\mathfrak{M}_j)/\mathfrak{M}_j=$

$Q_j/\mathfrak{M}_j$ is torsion-free. This contradicts the hypothesis.

Using Theorem 3.3 we can give the generalizations of Theorem 1.2 and Theorem 1.3 of [5].

COROLLARY 3.4. *Let R be a ring with the semi-simple total quotient ring Q. Suppose that for any divisible R-module M, $t(M)$ is a direct summand. Then $hd_RQ=1$ if $R\neq Q$.*

PROOF. Let N be any R-module. Then $E(N)/N$ is a torsion module by Theorem 3.3. Using this fact, we can prove the result by the similar method used in the proof of Theorem 1.2 of [5].

COROLLARY 3.5. *Let R be a ring with the semi-simple total quotient ring Q. Suppose that Q is countably generated as an R-module. Then every divisible R-module is h-divisible, and so $hd_RQ=1$ if $R\neq Q$.*

PROOF. Since Q is injective and Noetherian, a direct sum of any number of Q's is injective as an R-module by Corollary 2.2. From this we can prove the result, modifying the proof of Theorem 1.3 of [5].

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