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# *Analytic Functionals and Distributions*

Yoshimichi TSUNO

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## § 1. Introduction

Let $\Omega$ be a domain in the $n$-dimensional complex space $\boldsymbol{C}^n$. We shall denote by $\mathcal{A}(\Omega)$ the space of all holomorphic functions in $\Omega$, equipped with the topology of uniform convergence on compact subsets of $\Omega$. Clearly $\mathcal{A}(\Omega)$ is a Fréchet space. Elements of the dual space $\mathcal{A}'(\Omega)$ of $\mathcal{A}(\Omega)$ are called analytic functionals in $\Omega$. As usual, $\mathcal{E}'(\Omega)$ stands for the space of all distributions with compact supports in $\Omega$.

The purpose of this paper is to prove an isomorphism theorem (Theorem 3.2) which relates $\mathcal{A}'(\Omega)$ to $\mathcal{E}'(\Omega)$, and to use this isomorphism to give a simplified proof, without using $a$-priori-estimates, of Kiselman's theorem [5] concerning carriers of analytic functionals.

## § 2. Definition and comments

From the definition of the topology of $\mathcal{A}(\Omega)$ it follows that for any analytic functional $\mu \in \mathcal{A}'(\Omega)$ there exists a compact set K in $\Omega$ and a constant $C>0$ such that

$$|\mu(f)| \leqq C \sup_K |f|, \qquad \forall f \in \mathcal{A}(\Omega).$$

**Definition 2.1.** (Kiselman [5], Martineau [6]) A compact set K in $\Omega$ is called a *weak carrier* of $\mu \in \mathcal{A}'(\Omega)$ if for every neighborhood U of K which is in $\Omega$, there is a constant $C_U$ such that

$$|\mu(f)| \leqq C_U \sup_U |f|, \qquad \forall f \in \mathcal{A}(\Omega).$$

**Remark.** There is a concept that is stronger than that of a weak carrier (Martineau [6]). But in this paper, we shall use weak carriers only, so that in the following, we omit the term "weak", and "carrier" means always "weak carrier".

**Remark.** The following example shows that, different from a distribution, an analytic functional which is carried by one point is not always of finite order of differentiation.

**Example 2.2.** Let $\Omega$ be a complex plane $\boldsymbol{C}$, and for any entire function $f$, we set

$$\mu(f)=\sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)^2 f^{(n)}(0),$$

then, $\mu\in\mathcal{A}'(\boldsymbol{C})$ and the origin $\{0\}$ is a carrier of $\mu$.

PROOF. For any disc $U_r$ with center at the origin and radius $r$, Cauchy's integral formula yields

$$|f^{(n)}(0)|\leqq n!\frac{1}{r^n}\sup_{U_r}|f|.$$

Therefore

$$\begin{aligned}|\mu(f)|&\leqq\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{r^n}\sup_{U_r}|f|\\&=e^{1/r}\sup_{U_r}|f|.\end{aligned}$$

Q.E.D.

## § 3. Theorem of isomorphism and its direct application

If $\Omega$ is a domain in $\boldsymbol{C}^n$, $\mathcal{A}(\Omega)$ is a subspace of $\mathcal{E}^\infty(\Omega)$, and its topology is equivalent to the induced one from $\mathcal{E}^\infty(\Omega)$. From this point, we examine the relation between $\mathcal{A}'(\Omega)$ and $\mathcal{E}'(\Omega)$.

PROPOSITION 3.1. *A natural projection map $i':\mathcal{E}'(\Omega)\to\mathcal{A}'(\Omega)$ is surjective. More precisely, for every $\mu\in\mathcal{A}'(\Omega)$, if* K *is a compact carrier of $\mu$, then for any neighborhood* U *of* K*, there exists a $T_\mu\in\mathcal{E}'(\Omega)$ such that*

$$\operatorname{supp} T_\mu\subset U$$

*and*

$$\mu(f)=<T_\mu, f>\qquad\forall f\in\mathcal{A}(\Omega).$$

PROOF. Let V be a relatively compact neighborhood of K such that $K\subset V\subset\bar{V}\subset U$. Since K is a carrier of $\mu$, there is a constant $C_V$ such that

$$|\mu(f)|\leqq C_V\sup_V|f|,\qquad\forall f\in\mathcal{A}(\Omega). \tag{1}$$

Noting that $p(g)=C_V\sup_V|g|$ for $g\in\mathcal{E}^\infty(\Omega)$ is a continuous semi-norm on $\mathcal{E}^\infty(\Omega)$, we extend the linear functional $\mu$ to that on $\mathcal{E}^\infty(\Omega)$ preserving the inequality (1) (The Hahn-Banach theorem). It follows that there exists a $T_\mu\in\mathcal{E}'(\Omega)$ such that

$$\mu(f)=<T_\mu, f>,\qquad\forall f\in\mathcal{A}(\Omega)$$

and

$$|<\mathrm{T}_\mu, g>| \leqq C_V \sup_V |g|, \qquad \forall g \in \mathcal{E}^\infty(\Omega).$$

The last inequality shows that the support of $\mathrm{T}_\mu$ is contained in $\bar{\mathrm{V}} \subset \mathrm{U}$. This completes the proof.

From this proposition, the following algebraic isomorphism holds:

$$\mathcal{A}'(\Omega) = \mathcal{E}'(\Omega)/(\text{Null space of } i'). \tag{2}$$

The distribution $\mathrm{T}_\mu$ in the above is called a representation of $\mu$ in $\mathcal{E}'(\Omega)$. This representation is, of course, not unique. For example, if $\mathrm{S}_j$ $(j=1, 2, \cdots, n)$ are any distributions in $\mathcal{E}'(\Omega)$, then $\mathrm{T}_\mu + \sum_{j=1}^n \bar{\partial}_j \mathrm{S}_j$ where $\bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \frac{1}{i}\frac{\partial}{\partial y_j}\right)$ is also a representation of $\mu$.

Conversely, from the next theorem, if $\Omega$ is a domain of holomorphy, then any two representations of an analytic functional differ by a distribution of the form $\sum_{j=1}^n \bar{\partial}_j \mathrm{S}_j$, $\mathrm{S}_j \in \mathcal{E}'(\Omega)$.

THEOREM 3.2. *(of isomorphism) If $\Omega$ is a domain of holomorphy in $\boldsymbol{C}^n$, then the following isomorphism holds*

$$\mathcal{A}'(\Omega) = \mathcal{E}'(\Omega) / \sum_{j=1}^n \bar{\partial}_j \mathcal{E}'(\Omega).$$

**Remark.** In this paper, we prove this isomorphism algebraically. But by the general theory of linear topological vector space, this isomorphism also holds topologically. (Grothendieck [3]).

PROOF. If $\Omega$ is a domain of holomorphy, we have the following exact sequence:

$$0 \to \mathcal{A}(\Omega) \xrightarrow{i} \mathcal{E}^\infty_{(0,0)}(\Omega) \xrightarrow{\bar{\partial}} \mathcal{E}^\infty_{(0,1)}(\Omega) \xrightarrow{\bar{\partial}} \cdots$$

where $\mathcal{E}^\infty_{(0,p)}(\Omega)$ is the space of all $\mathcal{E}^\infty(\Omega)$-differentiable forms of type $(0, p)$.

It is clear that the spaces $\mathcal{E}^\infty_{(0,p)}(\Omega)$ equipped with the product topologies of Fréchet spaces, are also Fréchet spaces. And exactness shows that each operator has a closed range. By the closed range theorem for Fréchet spaces (cf. [2], p 296, Théorème 3), the following equality holds for the operator $\bar{\partial}: \mathcal{E}^\infty_{(0,0)}(\Omega) \to \mathcal{E}^\infty_{(0,1)}(\Omega)$ and its adjoint $\bar{\partial}'$:

$$[\text{Null space of } \bar{\partial}]^0 = \text{Range of } \bar{\partial}' \text{ in } \mathcal{E}'(\Omega),$$

where $\mathrm{V}^0$ denotes the polar of V.

From the definition of the adjoint operator, the range of $\bar{\partial}'$ is equal to the space $\sum_{j=1}^n \bar{\partial}_j \mathcal{E}'(\Omega)$.

Now, $\mathcal{A}(\Omega)$ is equal to the null space of $\bar{\partial}$, therefore

$$\begin{aligned} \text{Null space of } i' &= [\mathcal{A}(\Omega)]^0 \quad \text{in} \quad \mathcal{E}'(\Omega) \\ &= [\text{Null space of } \bar{\partial}]^0 \\ &= \text{Range of } \bar{\partial}' \\ &= \sum_{j=1}^{n} \bar{\partial}_j \mathcal{E}'(\Omega). \end{aligned} \tag{3}$$

From (2) and (3), we get

$$\mathcal{A}'(\Omega) = \mathcal{E}'(\Omega) / \sum_{j=1}^{n} \bar{\partial}_j \mathcal{E}'(\Omega). \qquad \text{Q.E.D.}$$

In the one dimensional case, $\sum_{j=1}^{n} \bar{\partial}_j \mathcal{E}'(\Omega)$ reduces to $\bar{\partial}\mathcal{E}'(\Omega)$, and every domain is a domain of holomorphy. For this reason, we can prove the following proposition.

Proposition 3.3. *Let* $\mathrm{P(D)}$ *be any differential operator with constant coefficients of order* $\geqq 1$, *where* $\mathrm{D} = \frac{d}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$, *and let* $\Omega$ *be any non-simply connected domain in* $\boldsymbol{C}$.
*Then* $\mathrm{P(D)}\mathcal{A}(\Omega)$ *is not dense in* $\mathcal{A}(\Omega)$.

Proof. It is sufficient to prove that the adjoint operator $\mathrm{P}(-\mathrm{D}): \mathcal{A}'(\Omega) \to \mathcal{A}'(\Omega)$ is not injective. Now, since the degree of the polynomial $\mathrm{P(X)}$ is greater than or equal to one, there exists a complex number $\alpha$ such that

$$\mathrm{P}(-\alpha) = 0.$$

And there exists a compact component K of $\mathrm{C}\Omega$, because $\Omega$ is not simply connected.

Then we construct a distribuition $\mathrm{S}_1 \in \mathcal{E}'(\boldsymbol{R}^2)$ such that

$$\mathrm{S}_1 = e^{\alpha z} \quad \text{in some neighborhood of K},$$

and

$$\operatorname{supp} \mathrm{S}_1 \subset \mathrm{K} \cup \Omega.$$

Moreover, we set

$$\mathrm{S}_2 = \bar{\partial}\mathrm{S}_1, \qquad \text{and} \qquad \mathrm{S}_3 = \mathrm{P}(-\mathrm{D})\mathrm{S}_1.$$

Then an easy calculation shows that

$$\operatorname{supp} \mathrm{S}_2 \subset \Omega \qquad \text{and} \qquad \operatorname{supp} \mathrm{S}_3 \subset \Omega$$

therefore

$$S_2 \in \mathcal{E}'(\Omega) \qquad \text{and} \qquad S_3 \in \mathcal{E}'(\Omega).$$

Now, let $\mu \in \mathcal{A}'(\Omega)$ be the restriction of $S_2$ to $\mathcal{A}(\Omega)$, i.e.

$$\mu(f) = \langle S_2, f \rangle, \qquad \forall f \in \mathcal{A}(\Omega).$$

Then $\mu \neq 0$ as an element of $\mathcal{A}'(\Omega)$. In fact, if $\mu = 0$, then by the isomorphism theorem there exists a $T \in \mathcal{E}'(\Omega)$ such that

$$S_2 = \bar{\partial} T.$$

Since the operator $\bar{\partial}$ is one to one, it follows that $T = S_1$. But this is impossible because $S_1$ is not an element of $\mathcal{E}'(\Omega)$.

On the other hand,

$$P(-D)S_2 = \bar{\partial} P(-D) S_1 = \bar{\partial} S_3,$$

so $P(-D)\mu = 0$ as an element of $\mathcal{A}'(\Omega)$. This proves that $P(-D)$ is not injective. Q.E.D.

Conversely, if $\Omega$ is simply connected, then it is well known that for any $P(D)$, $P(D)\mathcal{A}(\Omega) = \mathcal{A}(\Omega)$. (cf. [1]) Therefore the surjection theorem for one dimensional case becomes complete.

## § 4. Carriers of an analytic functional. Another proof of Kiselman's theorem.

We defined the carrier of an analytic functional in §2. But in general, it is not determined uniquely for an analytic functional; it even happens that an analytic functional has many compact carriers which are mutually disjoint.

**Example.** $\Omega$ is the complex plane $\boldsymbol{C}$. $\mu \in \mathcal{A}'(\Omega)$ is defined as follows: $\mu(f) = f(0)$. Then, $\{0\}$ and any circle about the origin are compact carriers of $\mu$ which are mutually disjoint.

The purpose of this section is to prove Theorem 4.1 concerning the intersection of two carriers of an analytic functional. This theorem was proved by C. O. Kiselman [5] using $\bar{\partial}$-cohomology and its *a*-priori-estimates. In this paper, we shall give a completely differrent proof of this theorem.

THEOREM 4.1. (Kiselman [5]) *Let $\Omega$ be a domain of holomorphy in $\boldsymbol{C}^n$, and $K_1$, $K_2$ be carriers of an analytic functional $\mu$ in $\Omega$, $\mu \neq 0$. If $K_1 \cup K_2$ is an $\mathcal{A}(\Omega)$-convex compact set, then $K_1 \cap K_2 \neq \phi$ and it carries $\mu$.*

For the proof of this theorem, we recall the following well known lemma.

LEMMA 4.2. *If $\Omega$ is a domain of holomorphy in $\boldsymbol{C}^n$, $K$ is an $\mathcal{A}(\Omega)$-convex* *compact set in* $\Omega$ *and* $\omega$ *is any neighborhood of* K, *then there exists an open set* U (*not necessarily connected*) *such that*

(i) $K \subset U \subset \bar{U} \subset \omega$,

(ii) $\mathcal{A}(\Omega)|_U$ *is dense in* $\mathcal{A}(U)$,

(iii) U *is a domain of holomorphy.*

If in particular U is an analytic polyhedron, then these conditions are satisfied. For the detail, we refer to [4]. (Theorems 2.5.13, 4.3.3 and Lemma 5.3.7).

Proof of Theorem 4.1. First, we shall show that the intersection of $K_1$ and $K_2$ is not empty.

If $K_1 \cap K_2 = \phi$, then by Lemma 4.2, there are neighborhoods $V_1$, $V_2$ of $K_1$, $K_2$, respectively, such that

$$V_1 \cap V_2 = \phi,$$

and any $f \in \mathcal{A}(V_1 \cup V_2)$ is a uniform limit of elements of $\mathcal{A}(\Omega)$ on every compact set of $V_1 \cup V_2$.

Since $\mu$ is carried by $K_1$ and $K_2$, there are compact sets $L_1$, $L_2$ which are contained in $V_1$, $V_2$, respectively, and constants $C_1$, $C_2$ such that

$$|\mu(f)| \leqq C_j \sup_{L^j} |f|, \qquad \forall f \in \mathcal{A}(\Omega), \qquad j=1, 2. \tag{4}$$

We, then, fix $g \in \mathcal{A}(\Omega)$ such that $\mu(g) \neq 0$, define $\tilde{g} \in \mathcal{A}(V_1 \cup V_2)$ by

$$\tilde{g} = \begin{cases} g & \text{in} \quad V_1 \\ 0 & \text{in} \quad V_2, \end{cases}$$

and approximate $\tilde{g}$ uniformely in $\mathcal{A}(V_1 \cup V_2)$ by a sequence $f_n$ in $\mathcal{A}(\Omega)$.

Then (4) with $j=1$ tells us

$$\lim_{n \to \infty} \mu(f_n) = \mu(g) \neq 0$$

and (4) with $j=2$,

$$\lim_{n \to \infty} \mu(f_n) = 0,$$

which is a contradiction. This prove that $K_1 \cap K_2 \neq \phi$.

Now we shall prove the remainder of the theorem.

We fix any neighborhood W in $\Omega$ of $K_1 \cap K_2$. It is sufficient to show that there exists a representation of $\mu$ in $\mathcal{E}'(\Omega)$, the support of which is in W, because the desired estimate of the modulus of $\mu(f)$ then follows easily from Cauchy's integral formula.

We choose neighborhoods $U_1'$, $U_2'$ of $K_1$, $K_2$, respectively, such that

$$\bar{U}_1' \cap \bar{U}_2' \subset W.$$

$U_1'\cup U_2'$ is a neighborhood of the $\mathcal{A}(\Omega)$-convex compact set $K_1\cup K_2$, therefore by Lemma 4.2, there exists a neighborhood U of $K_1\cup K_2$, which satisfies the conditions (i) (ii) (iii) in Lemma 4.2 with $\omega$ equal to $U_1'\cup U_2'$. We set

$$U_1=U\cap U_1', \qquad U_2=U\cap U_2'.$$

Hence, there exist neighborhoods $U_1$ and $U_2$ of $K_1$ and $K_2$, respectively, such that

(5) $U_1\cup U_2$ satisfies the conditions (i) (ii) (iii) in Lemma 4.2,

(6) $\bar{U}_1\cap\bar{U}_2$ is contained in W.

Then, $\bar{U}_1\cap CW$ and $\bar{U}_2\cap CW$ are mutually disjoint compact sets in $\Omega$, so that there exists a smooth function $\varphi$ such that

$$\varphi=\begin{cases} 1 \text{ on a neighborhood of } \bar{U}_1\cap CW \\ 0 \text{ on a neighborhood of } \bar{U}_2\cap CW. \end{cases}$$

Let $S_1$ and $S_2$ be representations of $\mu$ in $\mathcal{E}'(\Omega)$ the supports of which are in $U_1$ and $U_2$, respectively. And $\mu$ can be extended to an analytic functional $\tilde{\mu}$ on $\mathcal{A}(U)$ and condition (5) shows that this extension is unique. Therefore $S_1$ and $S_2$ in the above are also representations of $\tilde{\mu}$ in $\mathcal{E}'(U)$, that is,

$$<S_1, f>=<S_2, f>=\tilde{\mu}(f), \qquad \forall f\in\mathcal{A}(U).$$

By condition (iii) of Lemma 4.2 and Theorem 3.2, there exist distributions $T_j\in\mathcal{E}'(U)$, $j=1, 2, \cdots, n$, such that

$$S_1-S_2=\sum_{j=1}^{n}\bar{\partial}_jT_j.$$

Then we set

$$S=S_1-\sum_{j=1}^{n}\bar{\partial}_j(\varphi T_j).$$

By the definition, S also represents $\mu$ and it remains to examine the support of S.

In a neighborhood of $\bar{U}_1\cap CW$ where $\varphi$ is equal to one,

$$S=S_1-\sum_{j=1}^{n}\bar{\partial}_jT_j=S_2=0,$$

because the support of $S_2$ is contained in $U_2$. Similarly, in a neighborhood of $\bar{U}_2\cap CW$ where $\varphi$ vanishes,

$$S=S_1=0.$$

Consequently, the support of S is contained in W.
This completes the proof.

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*Department of Mathematics*
*Faculty of Science*
*Hiroshima University*