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K- and KO-Rings of the Lens Space $L^n(p^2)$ for Odd Prime p

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§1. Introduction

In the previous note [4], the structures of the K- and KO-rings of the standard lens space $L^n(4)=S^{2n+1}/Z_4$ are investigated, by considering the canonical complex line bundle and the non-trivial real line bundle over $L^n(4)$.

In this note, we shall study the $(2n+1)$-dimensional standard lens space mod p^r:

$$L^n(p^r)(=L^n(p^r;1,\cdots,1))=S^{2n+1}/Z_{p^r},$$

for prime p, by the similar methods to those which were used to determine the K- and KO-rings of $L^n(p)$ due to T. Kambe [3].

Let η be the canonical complex line bundle over $L^n(p^r)$, and

$$\sigma=\eta-1 \in \tilde{K}(L^n(p^r)) \text{ and } \bar{\sigma}=r\sigma \in \widetilde{KO}(L^n(p^r))$$

be the stable class of η and the real restriction of σ. Then we have

THEOREM 1.1. (i) *Let p be a prime and $r\geqq 1$. Then, the order of the element σ^k of $\tilde{K}(L^n(p^r))$ is equal to p^{r+h}, $h=[(n-k)/(p-1)]$, for $1\leqq k\leqq n$; and $\sigma^{n+1}=0$.*

(ii) *Let p be an odd prime and $r\geqq 1$. Then, the order of the element $\bar{\sigma}^k$ of $\widetilde{KO}(L^n(p^r))$ is equal to $p^{r+h'}$, $h'=[(n-2k)/(p-1)]$, for $1\leqq k\leqq [n/2]$; and $\bar{\sigma}^{[n/2]+1}=0$.*

For the case $r=2$, the additive structures of $\tilde{K}(L^n(p^2))$ for prime p and $\widetilde{KO}(L^n(p^2))$ for odd prime p are determined as follows. Let

$$(1.2)\qquad n-p^i+1=a_i(p^{i+1}-p^i)+b_i \quad (0\leqq b_i<p^{i+1}-p^i) \quad \text{for } i=0, 1,$$

and consider the following elements of $\tilde{K}(L^n(p^2))$:

$$\sigma=\eta-1,\quad \sigma(1)=\eta^p-1=(1+\sigma)^p-1,$$

$$(1.3)\qquad \sigma(1,k)=\begin{cases}\sigma(1)\sigma^k+p^{[(n-k)/p]}\sigma^{p+k} \\ \qquad\qquad (\text{if } b_1\leqq k<b_1+p-1 \text{ or } k<b_1-(p-1)^2) \\ \sigma(1)\sigma^k \quad (\text{otherwise}),\end{cases}$$

for $0 \leqq k \leqq \min(p^2-p-1, n-p)$. Then we have the following

THEOREM 1.4. *Let p be a prime. Then*

$$\tilde{K}(L^n(p^2)) \cong \sum_{k=1}^{m} Z_{t_k},\ m=\min(p^2-1, n)\ (\textit{direct sum}),$$

where Z_t indicates a cyclic group of order t and

$$(1.5)\qquad t_k=\begin{cases} p^{2-i+a_i} & (\textit{if } p^i \leqq k < p^i+b_i\ (i=0, 1)) \\ p^{1-i+a_i} & (\textit{if } p^i+b_i \leqq k < p^{i+1}\ (i=0, 1)). \end{cases}$$

Also, the k-th direct summand Z_{t_k} is generated by the element

$$\sigma^k\ (\textit{if } 1 \leqq k < p),\qquad \sigma(1, k-p)\ (\textit{if } p \leqq k < p^2).$$[1]

Moreover, the ring structure of $\tilde{K}(L^n(p^2))$ is given by

$$\sigma^{p^2}=-\sum_{i=1}^{p^2-1}\binom{p^2}{i}\sigma^i,\qquad \sigma^{n+1}=0.$$

Let $p=2q+1$ be an odd prime, and consider the following elements of $\widetilde{KO}(L^n(p^2))$:

$$\bar{\sigma}=r\sigma,\ \bar{\sigma}(1)=\sum_{i=1}^{q+1}\frac{p}{2i-1}\binom{q+i-1}{2i-2}\sigma^i,$$

$$(1.6)\quad \bar{\sigma}(1, k)=\begin{cases} \bar{\sigma}(1)\bar{\sigma}^k+p^{[(n-2k-1)/p]}\bar{\sigma}^{q+k+1} \\ \qquad (\text{if } [b_1/2] \leqq k < [b_1/2]+q \text{ or } k < [b_1/2]-2q^2) \\ \bar{\sigma}(1)\bar{\sigma}^k \quad (\text{otherwise}), \end{cases}$$

for $0 \leqq k \leqq \min(pq-1, [n/2]-q-1)$.

THEOREM 1.7. *Let $p=2q+1$ be an odd prime. Then*

$$\widetilde{KO}(L^n(p^2)) \cong \begin{cases} \sum_{k=1}^{m'} Z_{s_k} & (\textit{if } n \not\equiv 0 \bmod 4) \\ \sum_{k=1}^{m'} Z_{s_k} \oplus Z_2 & (\textit{if } n \equiv 0 \bmod 4), \end{cases}$$

where $m'=\min(q(p+1), [n/2])$ and $s_k=t_{2k}$ (the number given by (1.5)). Also, the k-th summand Z_{s_k} is generated by the element

$$\bar{\sigma}^k\ (\textit{if } 1 \leqq k \leqq q),\quad \bar{\sigma}(1, k-q-1)\ (\textit{if } q < k \leqq q(p+1)).$$

[1] We notice that these generators are slightly different from those in [4, Th. A] for $p=2$.

Moreover, the ring structure of $\widetilde{KO}(L^n(p^2))$ *is given by*

$$\bar{\sigma}^{q(p+1)+1}=\sum_{i=1}^{q(p+1)}-\frac{p^2}{2i-1}\binom{q(p+1)+i-1}{2i-2}\bar{\sigma}^i, \quad \bar{\sigma}^{[n/2]+1}=0.$$

In §2, we prepare some known results of $\tilde{K}(L^n(m))$ for any m and $\widetilde{KO}(L^n(m))$ for odd m. Th. 1.1 is proved in §3 by studying some relations on $\sigma^k (1\leqq k\leqq n)$ by means of the two relations:

$$(1+\sigma)^{p^r}=1 \text{ and } \sigma^{n+1}=0.$$

Also we have a non-immersion (-embedding) theorem for $L^n(p^r)$ as a corollary (Cor. 3.6), by the methods of M. F. Atiyah [2].

In §4, we study some relations on $\sigma(1)^l\sigma^k$ and prove Th. 1.4. The proofs are based only on the above two relations and the known facts that $\tilde{K}(L^n(m))$ contains exactly m^n elements.[2] Th. 1.7 is proved in §5, by making use of the $2n$-skeleton $L_0^n(m)$ of the standard cell complex $L^n(m)$, and the complexification

$$c: \widetilde{KO}(L_0^n(m))\to\tilde{K}(L_0^n(m))\cong\tilde{K}(L^n(m))$$

which is a monomorphism for odd m.

§2. Some results on $\tilde{K}(L^n(m))$ and $\widetilde{KO}(L^n(m))$

The standard lens space mod m is defined to be the orbit space:

$$L^n(m)=S^{2n+1}/Z_m, \quad n>1,$$

where the operation on S^{2n+1} of Z_m generated by γ is given by

$$\gamma(z_0, z_1,\dots, z_n)=(e^{2\pi i/m}z_0, e^{2\pi i/m}z_1,\dots, e^{2\pi i/m}z_n).$$

As is well known, $L^n(m)$ has a cell structure given by

$$L^n(m)=e^0\cup e^1\cup\cdots\cup e^{2n}\cup e^{2n+1},$$

and let $L_0^n(m)$ be the $2n$-skeleton of this CW-complex:

$$L_0^n(m)=e^0\cup e^1\cup\cdots\cup e^{2n},$$

then

2) According to N. Mahammed [5], it is announced that $K(L^n(m))=Z[\eta]/<(\eta-1)^{n+1}, \eta^m-1>$ for any m.

(2.1) $$L_0^n(m)/L_0^{n-1}(m) = S^{2n-1} \bigcup_m e^{2n}$$

where the attaching map $m: S^{2n-1} \to S^{2n-1}$ means the map of degree m.

The following lemmas are proved by the same way as [3, §§2-3].

LEMMA 2.2. (i) $$\tilde{K}(S^{2n-1} \bigcup_m e^{2n}) \cong Z_m,$$
and $\tilde{K}^{\pm 1}(S^{2n-1} \bigcup_m e^{2n}) = 0$. *Also, the induced homomorphism*

$$\pi^! : \tilde{K}(S^{2n}) \to \tilde{K}(S^{2n-1} \bigcup_m e^{2n})$$

is an epimorphism, where $\pi: S^{2n-1} \bigcup_m e^{2n} \to S^{2n}$ *is the projection collapsing* S^{2n-1} *to a point.*

(ii) *If* m *is an odd number, then*

$$\widetilde{KO}(S^{2n-1} \bigcup_m e^{2n}) \cong Z_m \text{ (for even } n), \quad = 0 \text{ (for odd } n);$$

and the other results of (i) *hold for* $\widetilde{KO}$ *instead of* $\tilde{K}$.

PROOF. (i) In the Puppe exact sequence

$$\cdots \to \tilde{K}^{-1}(S^{2n-1}) \xrightarrow{\delta} \tilde{K}(S^{2n}) \xrightarrow{\pi^!} \tilde{K}(S^{2n-1} \bigcup_m e^{2n}) \to \tilde{K}(S^{2n-1}) \to \cdots,$$

the boundary homomorphism $\delta: \tilde{K}^i(S^{2n-1}) \to \tilde{K}^{i+1}(S^{2n}) \cong \tilde{K}^i(S^{2n-1})$ is equal to $m^!$, and $m^!(x) = mx$. Therefore, we have (i) since $\tilde{K}(S^i) \cong Z$ (for even i) and $=0$ (for odd i). Similarly we have (ii) using the exact sequence for $\widetilde{KO}$, since $\widetilde{KO}(S^i) \cong Z$ (for $i \equiv 0, 4 \bmod 8$), $\cong Z_2$ (for $i \equiv 1, 2 \bmod 8$) and $=0$ (otherwise).
q.e.d.

LEMMA 2.3. (i) *The following sequence is exact:*

$$0 \to \tilde{K}(S^{2n-1} \bigcup_m e^{2n}) \to \tilde{K}(L_0^n(m)) \to \tilde{K}(L_0^{n-1}(m)) \to 0,$$

and $\tilde{K}(L_0^n(m))$ *contains exactly* m^n *elements. Also* $\tilde{K}^{\pm 1}(L_0^n(m)) = 0$.

(ii) *If* m *is odd, then* $\widetilde{KO}(L_0^n(m))$ *contains exactly* $m^{[n/2]}$ *elements, and* $\widetilde{KO}^{\pm 1}(L_0^n(m)) = 0$.

PROOF. Considering the Puppe exact sequence of (2.1), we can prove inductively the desired results by the above lemma. q.e.d.

LEMMA 2.4. *Let* $i: L_0^n(m) \to L^n(m)$ *be the inclusion. Then*

(i) $$i^! : \tilde{K}(L^n(m)) \cong \tilde{K}(L_0^n(m)).$$

(ii) *If* m *is odd, then we have the following split exact sequence:*

$$0 \longrightarrow \widetilde{KO}(S^{2n+1}) \longrightarrow \widetilde{KO}(L^n(m)) \xrightarrow{i^!} \widetilde{KO}(L_0^n(m)) \longrightarrow 0.$$

PROOF. This lemma follows immediately from the above lemma and the Puppe exact sequence of $L^n(m)/L_0^n(m) = S^{2n+1}$. q.e.d.

We shall identify the rings of (i) of the above by $i^!$, and denote the element of $\tilde{K}(L^n(m))$ and its $i^!$-image by the same letter.

Let $CP^n = S^{2n+1}/S^1$ be the n-dimensional complex projective space, and

$$\pi : L^n(m) \rightarrow CP^n \text{ and } \pi : L_0^n(m) \rightarrow CP^n$$

be the natural projection and its restriction. Then, it is clear that the map $\pi : (L_0^n(m), L_0^{n-1}(m)) \rightarrow (CP^n, CP^{n-1})$ induces the projection

$$\pi : S^{2n-1} \bigcup_m e^{2n} = L_0^n(m)/L_0^{n-1}(m) \rightarrow CP^n/CP^{n-1} = S^{2n}$$

of Lemma 2.2.

LEMMA 2.5. *We have the following commutative diagram of the Puppe exact sequences*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{K}(S^{2n}) & \longrightarrow & \tilde{K}(CP^n) & \longrightarrow & \tilde{K}(CP^{n-1}) & \longrightarrow & 0 \\ & & \downarrow \pi^! & & \downarrow \pi^! & & \downarrow \pi^! & & \\ 0 & \longrightarrow & \tilde{K}(S^{2n-1} \bigcup_m e^{2n}) & \longrightarrow & \tilde{K}(L_0^n(m)) & \longrightarrow & \tilde{K}(L_0^{n-1}(m)) & \longrightarrow & 0 \end{array}$$

where all of $\pi^!$ are epimorphic.

PROOF. The upper sequence is the Puppe sequence of $CP^n/CP^{n-1} = S^{2n}$ (cf. [1, Th. 7.2]). Since $\pi^!$ in the left is epimorphic by Lemma 2.2 (i), we see inductively that $\pi^!$ in the middle is also epimorphic. q.e.d.

Let η be the canonical complex line bundle over CP^n, and denote also by η the canonical complex line bundle $\pi^! \eta$ over $L^n(m)$ or $L_0^n(m)$, and by

$$\sigma = \eta - 1 \in \tilde{K}(L^n(m)) = \tilde{K}(L_0^n(m))$$

the stable class of η. Then

PROPOSITION 2.6. *The ring $\tilde{K}(L^n(m))$ is generated by σ and contains exactly m^n elements. Furthermore*

(2.7) $$(1+\sigma)^m = 1.$$

(2.8) $$\sigma^{n+1} = 0.$$

(2.9) *The order of the element σ^n is equal to m.*

PROOF. (2.7) follows from the fact that the first Chern class $c_1(\eta^m)$ is equal to $mc_1(\eta) = 0$ in $H^2(L^n(m)) \cong Z_m$.

The ring $\tilde{K}(CP^n)$ is generated by $\eta-1$ and $(\eta-1)^{n+1}=0$, and also the element $(\eta-1)^n$ generates the subgroup of $\tilde{K}(CP^n)$ which is the image of $\tilde{K}(S^{2n})\cong Z$ in the diagram of Lemma 2.5, (cf. [1, Th. 7.2]). Therefore we have the desired results by Lemmas 2.5 and 2.4 (i). q.e.d.

Consider the complexification $c\colon \widetilde{KO}(X)\to\tilde{K}(X)$ and the real restriction $r\colon \tilde{K}(X)\to\widetilde{KO}(X)$ (cf. [1]), and the element

$$\bar{\sigma}=r\sigma\in\widetilde{KO}(L^n(m)) \text{ or } \widetilde{KO}(L_0^n(m)). \tag{2.10}$$

PROPOSITION 2.11. *Let m be an odd number. Then*

(i) $$c\colon \widetilde{KO}(L_0^n(m))\to\tilde{K}(L_0^n(m))=\tilde{K}(L^n(m))$$

is a monomorphism. Also, the ring $\widetilde{KO}(L_0^n(m))$ is generated by $\bar{\sigma}$ and contains exactly $m^{[n/2]}$ elements, and it holds $\bar{\sigma}^{[n/2]+1}=0$.

(ii) $$\widetilde{KO}(L^n(m))\cong\begin{cases}\widetilde{KO}(L_0^n(m)) & (\textit{for } m \not\equiv 0 \bmod 4)\\ \widetilde{KO}(L_0^n(m))\oplus Z_2 & (\textit{for } m\equiv 0 \bmod 4),\end{cases}$$

and the subring of $\widetilde{KO}(L^n(m))$ generated by $\bar{\sigma}$ is isomorphic to $\widetilde{KO}(L_0^n(m))$.

(iii) *The following equality holds:*

$$c\bar{\sigma}=\sigma^2/(1+\sigma)=\sigma^2-\sigma^3+\sigma^4-\cdots. \tag{2.12}$$

PROOF. (i) It is well-known that $rc=2$, and so this is isomorphic for $\widetilde{KO}(L_0^n(m))$ by Lemma 2.3 (ii). Therefore c is monomorphic and r is epimorphic. We see $\sigma^{[n/2]+1}=0$ by (iii) and (2.8). In the commutative diagram

$$\begin{array}{ccc}\tilde{K}(CP^n) & \xrightarrow{r} & \widetilde{KO}(CP^n)\\ \downarrow{\scriptstyle\pi^!} & & \downarrow{\scriptstyle\pi^!}\\ \tilde{K}(L_0^n(m)) & \xrightarrow{r} & \widetilde{KO}(L_0^n(m)),\end{array}$$

$\pi^!$ on the left side is epimorphic by Lemma 2.5, and hence $\pi^!$ on the right is also so. Therefore we see the desired results because the ring $\widetilde{KO}(CP^n)$ is generated by $r(\eta-1)$ [6, Th. (3.9)].

(ii) σ is of odd order by the above proposition, and so is $\sigma\in\widetilde{KO}(L^n(m))$. Therefore (ii) follows from (i) and Lemma 2.4 (ii).

(iii) This equality is well known since $\sigma+1=\eta$ is a complex line bundle (cf. [3, Lemma (3.5), ii)]). q.e.d.

§3. Proof of Theorem 1.1

Henceforth, we consider the case $m=p^r$ where p is a prime and $r\geqq 1$.

Let $B \in K(L^n(p^r))$ be the element such that

$$B=\sum_{i=1}^{p-1}\frac{1}{p^r}\binom{p^r}{i}\sigma^{i-1}=\sum_{i=1}^{p-1}\frac{1}{i}\binom{p^r-1}{i-1}\sigma^{i-1},$$

then we have

PROPOSITION 3.1. *In* $\tilde{K}(L^n(p^r))$,

$$p^{r-2+h}(pB\sigma^k+\sigma^{k+p-1})=0 \quad \textit{for } 1\leqq k\leqq n-p+1, \tag{3.2}$$

$$p^{r+h}\sigma^k=0 \quad \textit{for } 1\leqq k\leqq n, \tag{3.3}$$

where $h=[(n-k)/(p-1)]$. *Furthermore,*

$$p^{r-2+h}\sigma^{n-(h-1)(p-1)}=-p^{r-1+h}\sigma^{n-h(p-1)}, \tag{3.4}$$

for $n-h(p-1)\geqq 1,\ h>0$.

PROOF. Multiplying σ^{k-1} to $(1+\sigma)^{p^r}-1=0$ of (2.7), we have

$$p^rB\sigma^k+\binom{p^r}{p}\sigma^{k+p-1}+\sum_{i=p+1}^{p^r}\binom{p^r}{i}\sigma^{i+k-1}=0. \tag{*}$$

Since the constant term of B is 1, this equality and $\sigma^{n+1}=0$ of (2.8) imply (3.3) for $k=n,\ n-1,\cdots,\ n-p+2$, i.e., for $h=0$.

Assume (3.3) inductively for $h<h_0$, and consider the case $h=h_0$. In the equality $(*)\times p^{h-1}$,

$$p^{h-1}\binom{p^r}{p}\sigma^{k+p-1}=p^{r-2+h}\binom{p^r-1}{p-1}\sigma^{k+p-1}=p^{r-2+h}\sigma^{k+p-1},$$

because $\binom{p^r-1}{p-1}\equiv 1 \bmod p$ and $p^{r-1+h}\sigma^{k+p-1}=0$ by the inductive assumptions. Also, if $i=jp^s>p$ and $(j,p)=1$, then

$$(p-1)(h-s)-(n-k-i+1)\geqq jp^s-(s+1)(p-1)>0,$$

and so $p^{h-1}\binom{p^r}{i}\sigma^{k+i-1}=0$ for $i>p$ by the inductive assumptions. Therefore, we have (3.2) for $h=h_0$, and so (3.3) for $h=h_0$ multiplying p to (3.2). Thus we have (3.2–3).

Consider (3.2) for $k=n-h\,(p-1)$, then we have

$$p^{r-2+h}\sigma^{n-(h-1)(p-1)}=-p^{r-1+h}B\sigma^{n-h(p-1)},$$

and so (3.4), since the constant term of B is equal to 1 and $p^{r-1+h}\sigma^{n-h(p-1)+i}=0$ for $i>0$ by (3.3). q.e.d.

REMARK. By the above proofs, we see that the above proposition follows from (2.7) for $m=p^r$ and

(3.5) $$p^{r-k}\sigma^{n+k}=0 \qquad \text{for } 0<k\leqq r,$$

instead of (2.8).

Now, we are ready to prove Th. 1.1.

PROOF OF THEOREM 1.1. (i) The order of σ^k is a power of p by Prop. 2.6 for $m=p^r$, and $p^{r+h}\sigma^k=0$ by (3.3). Assume $p^{r-1+h}\sigma^k=0$ for some $k\leqq n$. Then $p^{r-1+h}\sigma^{n-h(p-1)}=0$ since $n-h(p-1)\geqq k$, and hence $p^{r-1}\sigma^n=0$ by (3.4). This contradicts to (2.9) for $m=p^r$.

(ii) Since the complexification c is a ring homomorphism, we have

$$c(\bar{\sigma}^k)=\sigma^{2k}/(1+\sigma)^k$$

by (2.12), and so the desired results by (i) and Prop. 2.11. q.e.d.

COROLLARY 3.6. *For an odd prime p and $r\geqq 1$, the lens space $L^n(p^r)$ cannot be immersed in the Euclidean space $R^{2n+2L(n,p^r)}$, and cannot be embedded in $R^{2n+2L(n,p^r)+1}$, where*

$$L(n, p^r)=\max\left\{i \,\middle|\, i\leqq[n/2],\ \binom{n+i}{i}\not\equiv 0 \bmod p^{r+[(n-2i)/(p-1)]}\right\}.$$

PROOF. By the methods of M. F. Atiyah [2] using the γ-operation of the stable tangent bundle, we have the desired results by taking

$$L(n, p^r)=\max\left\{i \,\middle|\, \binom{n+i}{i}\bar{\sigma}^i\neq 0\right\},$$

(cf. [4, Prop. 7.6]). This number is equal to the above one by (ii) of Th. 1.1. q.e.d.

§4. Proof of Theorem 1.4

Now, consider the following elements of $\tilde{K}(L^n(p^r))$ if $n\geqq p$ and $r\geqq 2$:

(4.1) $$\sigma(1)=\eta^p-1=(1+\sigma)^p-1=pA\sigma+\sigma^p,$$

where

$$A=\sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}\sigma^{i-1}=\sum_{i=1}^{p-1}\frac{1}{i}\binom{p-1}{i-1}\sigma^{i-1}.$$

Then we have the following lemmas in $\tilde{K}(L^n(p^r))$.

LEMMA 4.2. *Let $h=[(n-k)/(p-1)]$, then*

(4.3) $$p^{r-2+h}(pA\sigma^k+\sigma^{k+p-1})=p^{r-2+h}\sigma(1)\sigma^{k-1}=0,$$

(4.4) $$p^{r-2+h}\sigma^{k+p-1}=-p^{r-1+h}A\sigma^k,$$

for $1\leqq k\leqq n-p+1$; *and*

(4.5) $$p^{r-1+h}\sigma^k=-p^{r-1+h+p}\sigma^{k-p(p-1)} \qquad \textit{for } p(p-1)<k\leqq n.$$

PROOF. Since $\frac{1}{i}\binom{p^r-1}{p-1}\equiv\frac{1}{i}\binom{p-1}{i-1} \bmod p$ for $1\leqq i<p$, we have (4.3–4) by Prop. 3.1 and the definitions of B and A. By (4.4),

$$p^{r-1+h}\sigma^k=(-1)^p p^{r-1+h+p}A^p\sigma^{k-p(p-1)}.$$

It is easy to see that the constant term of the integral polynomial A^p of σ is equal to 1 and the coefficient of σ^i is a multiple of p for $1\leqq i<p-1$. Therefore, we have (4.5) using (3.3). q.e.d.

LEMMA 4.6. *Let*

$$l_k=[(n+p-1-k)/p], \quad \textit{i.e.,} \quad n\leqq pl_k+k\leqq n+p-1,$$

then we have

$$p^{r-1-l}\sigma(1)^{l_k+l}\sigma^k=0 \qquad \textit{for } 0<l\leqq r-1.$$

PROOF. In fact, the left hand side is equal to

$$p^{r-1-l}(pA\sigma+\sigma^p)^{l'}\sigma^k=\sum_{i=0}^{l'}\binom{l'}{i}A^ip^{r-l-1+i}\sigma^{pl'+k-i(p-1)}, \ l'=l_k+l,$$

and each term of this summation is 0 by (3.3), since $[(n-(pl_k+pl+k)+i(p-1))/(p-1)]\leqq -l-1+i$ by the definition of l_k. q.e.d.

By this lemma and the equality

(4.7) $$((1+\sigma(1))^{p^{r-1}}-1)\sigma^k=0,$$

which follows from (2.7) for $m=p^r$ and (4.1), we have the following

PROPOSITION 4.8. *Let* $r\geqq 2$ *and* $n\geqq p$. *In* $\tilde{K}(L^n(p^r))$,

(4.9) $$p^{r-1+j}\sigma(1)^l\sigma^k=0, \ j=[(n+p-1-pl-k)/p(p-1)],$$

for $l\geqq 1$, $k\geqq 0$ *and* $j<r$.

(4.10) $$p^{r-3+j}\sigma(1)^{l_k-(j-1)(p-1)}\sigma^k=-p^{r-2+j}\sigma(1)^{l_k-j(p-1)}\sigma^k$$

for $l_k-j(p-1)\geqq 1$ *and* $j>0$, *where* l_k *is the number defined in the above lemma.*

PROOF. We notice that j of (4.9) is equal to $[(l_k-l)/(p-1)]$. Then, we can prove (4.9–10) using the above lemma and (4.7), by the same methods to prove (3.3–4) using (3.5) and (2.7) for $m=p^r$, (cf. Remark after Prop. 3.1). q.e.d.

LEMMA 4.11. *If* $n<pl+k\leqq n+p-1$, *then*

$$p^{r-2+j}\sigma(1)^{l-j(p-1)}\sigma^k=-p^{r-2+jp}\sigma^{pl+k-jp(p-1)},$$

for $l-j(p-1)\geqq 1$, $j>0$.

PROOF. By the definition of l_k in Lemma 4.6 and the assumption, we have $l=l_k$. Therefore,

$$\begin{aligned} p^{r-2+j}\sigma(1)^{l-j(p-1)}\sigma^k &=(-1)^j p^{r-2}\sigma(1)^l\sigma^k && \text{(by (4.10))}\\ &=(-1)^j\sum_{i=0}^{l-1}\binom{l-1}{i}A^i p^{r-2+i}\sigma(1)\sigma^{pl+k-p-i(p-1)} && \text{(by (4.1))}\\ &=(-1)^j p^{r-2}\sigma(1)\sigma^{pl+k-p} && \text{(by (4.3) and the assumption)}\\ &=(-1)^j p^{r-1}A\sigma^{pl+k-p+1} && \text{(by the assumption and (2.8))}\\ &=p^{r-1+jp}A\sigma^{pl+k-p+1-jp(p-1)} && \text{(by (4.5))}\\ &=-p^{r-2+jp}\sigma^{pl+k-jp(p-1)} && \text{(by (4.4)).} \end{aligned}$$

q.e.d.

REMARK. By the same proofs, we have the following equality for $j=0$:

$$p^{r-2}\sigma(1)^l\sigma^k=p^{r-1}A\sigma^{pl+k-p+1}, \qquad \text{if } n<pl+k\leqq n+p-1,\ l\geqq 1.$$

According to this lemma, we consider the following elements of (1.3):

$$\sigma(1,k)=\begin{cases}\sigma(1)\sigma^k+p^{a_1(p-1)}\sigma^{p+k} & (\text{if } b_1\leqq k<b_1+p-1)\\ \sigma(1)\sigma^k+p^{(a_1+1)(p-1)}\sigma^{p+k} & (\text{if } k<b_1-p^2+2p-1)\\ \sigma(1)\sigma^k & (\text{otherwise}),\end{cases}$$

for $0\leqq k\leqq \min(p^2-p-1, n-p)$, where

$$n-p+1=a_1(p^2-p)+b_1,\quad 0\leqq b_1<p^2-p.$$

LEMMA 4.12. $t_{p+k}\sigma(1,k)=0$ *in* $\tilde{K}(L^n(p^r))(r\geqq 2,\ n\geqq p)$, *where*

$$t_{p+k}=p^{r+1+[(n-p-k)/p(p-1)]}=\begin{cases}p^{r-1+a_1} \textit{ for } 0\leqq k<b_1\\ p^{r-2+a_1} \textit{ for } b_1\leqq k<p^2-p\end{cases}$$

is the number of (1.5) *if* $r=2$.

PROOF. For the case $b_1\leqq k<b_1+p-1$ or $k<b_1-p^2+2p-1$, it holds

$$n<p+jp(p-1)+k\leqq n+p-1,$$

where $j=[(n-p-k)/p(p-1)]+1=a_1$ or a_1+1, and $j>0$ since $k\leqq n-p<b_1$ if $a_1=0$. Thus we have the desired equality by the above lemma.

For the other cases, we have $[(n-p-k)/p(p-1)]=[(n-1-k)/p(p-1)]$, and so the desired equality by (4.9). q.e.d.

Now, we are ready to prove Th. 1.4 which gives the additive structure of $\tilde{K}(L^n(p^2))$.

Proof of Theorem 1.4. $\tilde{K}(L^n(p^2))$ is generated additively by the elements σ^k, $1\leqq k\leqq \min(p^2-1, n)$, and the order of σ^k is a power of p, by Prop. 2.6 for $m=p^2$. On the other hand, the integral polynomial $\sigma(1, k-p)$ on σ is $\sum_{i=k-p+1}^{k}\alpha_i\sigma^i$ with $\alpha_k=1$ or $1+p^{j(p-1)}$, $j=a_1$ or a_1+1, and $j>0$ (cf. the proofs of the above lemma). Therefore, we see that $\tilde{K}(L^n(p^2))$ is generated additively by the first n elements of

$$(*)\qquad \sigma,\dots, \sigma^{p-1}, \sigma(1, 0),\dots, \sigma(1, p^2-p-1).$$

Hence, the number of the elements of $\tilde{K}(L^n(p^2))(n\geqq p^2-1)$ is not larger than

$$(p^{2+a_0})^{b_0}(p^{1+a_0})^{p-1-b_0}(p^{1+a_1})^{b_1}(p^{a_1})^{p(p-1)-b_1}=p^{2n}$$

by (3.3) and the above lemma for $r=2$, and is equal to p^{2n} by Prop. 2.6. Thus the theorem is proved for $n\geqq p^2-1$.

Similarly, we have the theorem for the case $n<p^2-1$ considering the first n elements of (*), since

$$(p^{2+a_0})^{b_0}(p^{1+a_0})^{p-1-b_0}(p^{1+a_1})^{b_1}=p^{2n}\qquad \text{if } p-1\leqq n<p^2-1,$$

$$(p^{2+a_0})^{b_0}=p^{2n}\qquad \text{if } n<p-1.$$

q.e.d.

In connection to Th. 1.1 (i) and (4.9), we have

Proposition 4.13. *The order of* $\sigma(1)^l\sigma^k$ *of* $\tilde{K}(L^n(p^2))$ *is equal to* p^{1+j}, $j=[(n+p-1-pl-k)/p(p-1)]$, *for* $l\geqq 1$, $k\geqq 0$, $pl+k<n+p$; *and* $\sigma(1)^l\sigma^k=0$ *if* $pl+k\geqq n+p$.

Proof. Assume $p^j\sigma(1)^l\sigma^k=0$ for some l and k. Since $k'=n+p-1-jp(p-1)-pl\geqq k$, we have $p^j\sigma(1)^l\sigma^{k'}=0$. On the other hand,

$$p^j\sigma(1)^l\sigma^{k'}=-p^{jp}\sigma^{pl+k'}\quad \text{if } j>0$$

by Lemma 4.11 for $r=2$, and the order of $\sigma^{pl+k'}$ is equal to p^{jp+1} by Th. 1.1 (i), which is a contradiction. If $j=0$,

$$\sigma(1)^l\sigma^{k'}=pA\sigma^{pl+k'-p+1}=pA\sigma^n=p\sigma^n\neq 0$$

by Remark after Lemma 4.11 and Th. 1.1 (i) for $r=2$, which is a contradiction. Therefore, we have the desired results using (4.9) for $r=2$. q.e.d.

Concerning with $L^n(p^r)(r\geqq 3)$, the above proofs based on Prop. 2.6, (3.3) and Lemma 4.12 are efficient for the special case $n<p^2$, and we have

THEOREM 4.14. *Let p be a prime, $r\geqq 3$ and $1\leqq n<p^2$. Then*

$$\tilde{K}(L^n(p^r))\cong\sum_{k=1}^{n} Z_{t_k}\quad (\textit{direct sum})$$

where $t_k=p^{r-1}$ if $p\leqq k\leqq n$, $=p^{r+[(n-k)/(p-1)]}$ if $1\leqq k<p$. Also the k-th summand Z_{t_k} is generated by

$$\sigma^k\ (\textit{if } 1\leqq k<p),\quad \sigma(1,k-p)\ (\textit{if } p\leqq k\leqq n),$$

where $\sigma(1,k-p)$ is the element of (1.3), *i.e.,*

$$\sigma(1,k-p)=\begin{cases}\sigma(1)\sigma^{k-p}+p^{p-1}\sigma^k & (\textit{if } p\leqq k<n-p^2+2p)\\ \sigma(1)\sigma^{k-p} & (\textit{if } n-p^2+2p<k\leqq n).\end{cases}$$

§5. Proof of Theorem 1.7

Now, let $p=2q+1$ be an odd prime.

Using the element $\bar{\sigma}=r\sigma$ of (2.10), we define the element

$$\bar{\sigma}(1)=\sum_{i=1}^{q+1}\frac{p}{2i-1}\binom{q+i-1}{2i-2}\bar{\sigma}^i \tag{5.1}$$

of $\widetilde{KO}(L^n(p^r))$ or $\widetilde{KO}(L_0^n(p^r))$.

LEMMA 5.2. *For the complexification c,*

$$c\bar{\sigma}(1)=((1+\sigma)^p-1)\sigma/(1+\sigma)^{q+1}=\sigma(1)\sigma/(1+\sigma)^{q+1}.$$

PROOF. By (2.12),

$$c\bar{\sigma}(1)=\sum_{i=1}^{q+1}\frac{p}{2i-1}\binom{q+i-1}{2i-2}\frac{\sigma^{2i}}{(1+\sigma)^i}$$

$$=\frac{1}{(1+\sigma)^{q+1}}\sum_{j=2}^{p+1}\left\{\sum_{i=1}^{j-1}\frac{p}{2i-1}\binom{q+i-1}{2i-2}\binom{q+1-i}{j-2i}\right\}\sigma^j$$

$$=\frac{1}{(1+\sigma)^{q+1}}\sum_{j=2}^{p+1}\frac{p}{j-1}\left\{\sum_{i=1}^{j-1}\binom{q+i-1}{j-2}\binom{j-1}{2i-1}\right\}\sigma^j$$

$$=\frac{1}{(1+\sigma)^{q+1}}\sum_{j=2}^{p+1}\binom{p}{j-1}\sigma^j=\frac{\sigma(1)\sigma}{(1+\sigma)^{q+1}},$$

using the lemma due to T. Kambe [3, Lemma (3.7)]. q.e.d.

LEMMA 5.3. *In* $\widetilde{KO}(L_0^n(p^2))$ *and* $\widetilde{KO}(L^n(p^2))$, *it holds*

$$\bar{\sigma}^{q(p+1)+1}=\sum_{i=1}^{q(p+1)}-\frac{p^2}{2i-1}\binom{q(p+1)+i-1}{2i-2}\bar{\sigma}^i.$$

PROOF. We can show that the c-image of the left hand side minus the right hand side is equal to $((1+\sigma)^{p^2}-1)\sigma/(1+\sigma)^{q(p+1)+1}$ similarly as the proofs of the above lemma. Thus we have the lemma by (2.7) and Prop. 2.11 for $m=p^2$. q.e.d.

PROOF OF THEOREM 1.7. By Prop. 2.11 and the above lemma, $\widetilde{KO}(L_0^n(p^2))$ is generated additively by the elements $\bar{\sigma}^k$, $1\leqq k\leqq \min(q(p+1), [n/2])$, and the order of $\bar{\sigma}^k$ is a power of p. On the other hand $\bar{\sigma}(1, k-q-1)$ of (1.6) is $\sum_{i=k-q}^{k}\beta_i\bar{\sigma}^i$ with $\beta_k=1+p^{j(p-1)}$, $j=a_1$ or a_1+1, and $j>0$ by definition. Therefore, $\widetilde{KO}(L_0^n(p^2))$ is generated additively by the first $[n/2]$ elements of

$$\bar{\sigma},\cdots,\bar{\sigma}^q,\ \bar{\sigma}(1, 0),\cdots,\bar{\sigma}(1, pq-1).$$

Now, we see that

$$c\bar{\sigma}(1, k)=\sigma(1, 2k+1)/(1+\sigma)^{q+k-1}$$

by (2.12), Lemma 5.2, (1.5) and (1.3), and hence the order of $\bar{\sigma}(1, k)$ is

$$p^{1+a_1}\ (\text{if } 0\leqq k<[b_1/2]),\quad p^{a_1}\ (\text{if } [b_1/2]\leqq k<pq),$$

by Th. 1.4 and Prop. 2.11 (i). Also, the order of $\bar{\sigma}^k$ is equal to

$$p^{2+a_0}\ (\text{if } 1\leqq k\leqq[b_0/2]),\quad p^{1+a_0}\ (\text{if } [b_0/2]<k\leqq q)$$

by Th. 1.1 (ii). Therefore, the theorem follows from these facts, Prop. 2.11 and

$$(p^{2+a_0})^{[b_0/2]}(p^{1+a_0})^{q-[b_0/2]}(p^{1+a_1})^{[b_1/2]}(p^{a_1})^{pq-[b_1/2]}=p^{2[n/2]}$$

if $[n/2]\geqq q(p+1)$,

$$(p^{2+a_0})^{[b_0/2]}(p^{1+a_0})^{q-[b_0/2]}(p^{1+a_1})^{[b_1/2]}=p^{2[n/2]}$$

if $q\leqq[n/2]<q(p+1)$, and $(p^{2+a_0})^{[b_0/2]}=p^{2[n/2]}$ if $[n/2]<q$, together with Lemma 5.3. q.e.d.

The following result follows immediately from Prop. 2.11, Lemma 5.2

and Prop. 4.13.

PROPOSITION 5.4. *For an odd prime p, the element $\bar{\sigma}(1)^l\bar{\sigma}^k$ of $\widetilde{KO}(L^n(p^2))$ or $\widetilde{KO}(L_0^n(p^2))$ is of order p^{1+j}, $j=[(n+p-1-(pl+l+2k))/(p^2-p)]$, for $l\geqq 1$, $k\geqq 0$, $pl+l+2k<n+p$; and $\bar{\sigma}(1)^l\bar{\sigma}^k=0$ if $pl+l+2k>n+p$.*

We notice that the above proofs are valid for $L^n(p^r)(r\geqq 3)$ with $n<p^2$ according to Th. 4.14, and we have

THEOREM 5.5. *Let p be an odd prime, $p=2q+1$, $r\geqq 3$ and $1\leqq n<p^2$. Then*

$$\widetilde{KO}(L^n(p^r))\cong\begin{cases}\sum_{k=1}^{[n/2]} Z_{s_k} & (\text{if } n\not\equiv 0 \bmod 4)\\ \sum_{k=1}^{[n/2]} Z_{s_k}\oplus Z_2 & (\text{if } n\equiv 0 \bmod 4),\end{cases}$$

where $s_k=p^{r-1}$ if $q<k\leqq[n/2]$ and

$$s_k=t_{2k}=p^{r+[(n-2k)/(p-1)]}\quad \text{if } 1\leqq k\leqq q.$$

Also, the k-th summand Z_{s_k} is generated by

$$\bar{\sigma}^k\ (\text{if } 1\leqq k\leqq q),\quad \bar{\sigma}(1,\ k-q-1)\ (\text{if } q<k\leqq[n/2])$$

where

$$\bar{\sigma}(1,\ k-q-1)=\begin{cases}\bar{\sigma}(1)\bar{\sigma}^{k-q-1}+p^{p-1}\bar{\sigma}^k\ (\text{if } q<k\leqq[b_1/2]-2q^2+q)\\ \bar{\sigma}(1)\bar{\sigma}^{k-q-1}\ (\text{if } [b_1/2]-2q^2+q<k\leqq[n/2]).\end{cases}$$

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