HIROSHIMA MATH. J.
2 (1972), 243–251

Modules which Have No Co-irreducible Submodules

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(Received August 20, 1972)

It is known that, for a ring R, every injective R-module has an indecomposable direct summand if and only if every ideal is an intersection of two ideals, at least one of which is irreducible ([4]). In [1], it is pointed out that the zero ideal of the ring of continuous functions defined on the interval $[0, 1]$ does not satisfy the above condition and there are no other examples as far as the author knows.

The aim of this paper is to present such an example as a domain and investigate the characters of ideals which do not satisfy the condition. Also we shall settle several conjectures by making use of this example.

The author wishes to express his thanks to Prof. M. Nishi for his valuable suggestions and encouragement.

Throughout this paper all rings will be commutative with unit and all modules will be unitary. For an ideal I and an element r of a ring R, $I: r$ means the ideal $\{s \in R; sr \in I\}$. For an element x of an R-module, $0(x)$ means the order ideal of x. We write $x \in S-T$ for $x \in S$ and $x \notin T$.

§1. Co-irreducible modules

Let R be a ring and M an R-module. We shall say that M is co-irreducible if $M \neq 0$ and for any non-zero submodules N_1 and N_2 of M, $N_1 \cap N_2 \neq 0$. If M is a co-irreducible R-module, then non-zero submodules and essential extensions of M are also co-irreducible. Let M be an R-module and N a submodule of M. We shall say that N is irreducible in M if M/N is co-irreducible. In other words, if $N=M_1 \cap M_2$ for submodules M_1 and M_2 of M, then $N=M_1$ or $N=M_2$. For an ideal I of R, we say that I is an irreducible ideal if I is irreducible in R as an R-module. Then prime ideals of R are irreducible.

THEOREM 1.1. *The following conditions in a ring R are equivalent,*

1) *Any non-zero R-module contains a co-irreducible submodule.*

2) *If I is an ideal of R, different from R, then there exists an element r of R such that $I: r$ is an irreducible ideal.*

PROOF. We assume the condition 1). Let $I(\subsetneqq R)$ be an ideal of R. Then the non-zero module $R/I=Rx$ contains a co-irreducible submodule Rrx for some $r \in R$. Since Rrx is isomorphic to $R/0(rx)$ and $0(rx)=I: r$, $I: r$ is irreducible. Conversely we assume the condition 2). It is sufficient to show the condition 1) for a cyclic module $Rx \neq 0$. Now Rx is isomorphic to $R/0(x)$ and by assumption, there exists an element r of R such that $0(x): r$ is irreducible. This implies that Rrx is a co-irreducible submodule of Rx. q.e.d.

We can readily see that the conditions in Theorem 1.1 are originally equivalent to the condition of remark in [4, p 516]. Therefore by R.B. Warfields ([5, p 269]) we have the following:

PROPOSITION 1.2. *If the conditions in Theorem* 1.1 *are fulfilled, then any injective R-module is the injective hull of a direct sum of indecomposable injective R-modules.*

§2. Example

We shall construct an example in which the condition 1) in Theorem 1.1 is not satisfied. To do this we use the technique of construction of a domain by a lattice-ordered group. Let G be an additive abelian group with partial order $\leq$ compatible with the operation in G. G is a lattice-ordered group if a, $b \in G$ implies $\inf(a, b) \in G$. A segment of the lattice-ordered group G is a non-empty subset A of $G^{+}=\{x \in G;\ x \geq 0\}$ such that $a \in A$ and $b \geq a$ imply $b \in A$, and a, $b \in A$ implies $\inf(a, b) \in A$. A is a principal segment if there exists an element a in G^{+} such that $A=\{g \in G^{+};\ g \geq a\}$, and we denote it by (a). For a segment A of G and an element g of G^{+}, A: g means the segment $\{f \in G^{+};\ f+g \in A\}$. A is an irreducible segment if A is not written as an intersection of two segments of G which contain A properly. In [3, p79], P. Jaffard shows that to each lattice-ordered group G there corresponds an integral domain D. Let F be an arbitrary field and R be the group ring of G with respect to F. Then R can be regarded as the set of finite formal sums $\Sigma a_i X^{g_i}$, $a_i \in F$, $g_i \in G$. For an element $\Sigma a_i X^{g_i}$ of $R^{*}=R-\{0\}$, we define a map ϕ of R^{*} onto G by $\phi(\Sigma a_i X^{g_i})=\inf\{g_i\}$. It is known that the group ring R is a domain ([3, p. 12]). Let K be the quotient field of R; the map ϕ may be extended to $K^{*}=K-\{0\}$ by $\phi(r_1/r_2)=\phi(r_1)-\phi(r_2)$. The map ϕ has the following properties:

$$\phi(pq)=\phi(p)+\phi(q)$$

$$\phi(p+q) \geq \inf(\phi(p), \phi(q))$$

Let D be the set $\{0\} \cup \{p \in K^{*};\ \phi(p) \geq 0\}$. In [2], W. Heinzer shows that D is a bezoutian domain. Moreover, it can be easily seen that there is a one-to-one inclusion preserving correspondence between proper segments in G and proper ideals in D. That is, if I is an ideal of D, then $\phi(I-\{0\})$ is a segment of G, and conversely, if A is a segment of G, then $\phi^{-1}(A) \cup \{0\}$ is an ideal of D. And if A is a prime (resp. irreducible) segment, then $\phi^{-1}(A) \cup \{0\}$ is a prime (resp. irreducible) ideal, and conversely.

From now on, we take for G the set of $\boldsymbol{Z}$-valued left continuous step functions on $\boldsymbol{R}$ with at most finitely many points of discontinuity. G is a group under pointwise addition, and is lattice-ordered by the relation $\leq$, where $f\leq g$ if and only if $f(x)\leq g(x)$ for all $x\in\boldsymbol{R}$. Now we shall study segments of G. For any $x_0\in\boldsymbol{R}$, let $\{x_n\}_{n\in\boldsymbol{N}}$ be a monotone decreasing sequence in $\boldsymbol{R}$ which converges to x_0. Define a function f_n by $f_n(x)=1$ on $(x_0, x_n]$ and $f_n(x)=0$ elsewhere. Put $Q_{x_0}=\bigcup_{n=1}^{\infty}(f_n)$. Then Q_{x_0} is independent of a choice of the sequence $\{x_n\}$. Moreover, we define three other types of segments as follows:

$$\begin{aligned}
Q_\infty &=\{f\in G^+;\ \text{there is } r\in\boldsymbol{R} \text{ such that } f \text{ is positive on } (r,\infty)\}\\
Q_{-\infty} &=\{f\in G;\ \text{there is } r\in\boldsymbol{R} \text{ such that } f \text{ is positive on } (-\infty, r]\}\\
P_{x_0} &=\{f\in G^+;\ f(x_0)>0\}
\end{aligned}$$

We can readily see that these are prime segments.

PROPOSITION 2.1. *P_{x_0}, Q_{x_0}, Q_∞ and $Q_{-\infty}$ are the only prime segments of G.*

PROOF. It is evident that there are no inclusion relations between these segments. Suppose A is a proper segment such that $A\not\subset P_{x_0}$, Q_{x_0}, Q_∞ $Q_{-\infty}$ for all $x_0\in\boldsymbol{R}$. Since $A\not\subset Q_\infty$, $Q_{-\infty}$, then there exist an element f in A and x_1, $x_2\in\boldsymbol{R}$, $x_1<x_2$, such that $f(x)=0$ on $(-\infty, x_1]\cup(x_2,\infty)$. Also $A\not\subset P_{x_0}$, Q_{x_0} implies that there exist $f_{x_0}\in A$ and x', $x''\in\boldsymbol{R}$, $x'<x_0<x''$ such that $f_{x_0}(x)=0$ on $I_{x_0}=[x', x'']$. Then the interval $[x_1, x_2]$ is covered by the sets I_{x_0}. Since $[x_1, x_2]$ is compact, then it has a finite covering $I_{y_1}\cup I_{y_2}\cup\cdots\cup I_{y_n}$. Let h_{y_i} be the function corresponding to y_i. Thus, $0=\inf(f, h_{y_1}, h_{y_2},\cdots, h_{y_n})\in A$, which implies that $A=G^+$; this contradicts the assumption on A. Therefore prime segments P_{x_0}, Q_{x_0}, Q_∞ and $Q_{-\infty}$ are maximal. Next we shall show that these are minimal. We shall treat the segment P_{x_0} only and omit the other cases. Let P be a prime segment $P\subset P_{x_0}$; then there exists $g\in P$ such that $g(x_0)=1$, for, if the value k of $f_0\in P$ at x_0 is greater than 1, then there exists $x_1(<x_0)\in\boldsymbol{R}$ such that $f_0(x)=k$ on $(x_1, x_0]$. Define a function $g\in G^+$ by $g(x)=1$ on $(x_1, x_0]$ and $g(x)=f_0(x)$ elsewhere. Then $kg\geq f_0$. Since $f_0\in P$ and P is a prime segment, then $kg\in P$ and $g\in P$. Thus $g(x_0)=1$. Therefore for any $f\in P_{x_0}$, there exists $h\in P$ such that $f(x_0)=h(x_0)$. Put $f_0=\inf(f, h)$. Clearly $f_0\in P_{x_0}$. Moreover $f_0\in P$, because $h=(h-f_0)+f_0\leq(h-f_0)^++f_0\in P$, where $(h-f_0)^+=\sup(h-f_0, 0)$, $(h-f_0)^+\notin P$ and P is prime, thus $f_0\in P$. Hence $f\in P$, this implies $P_{x_0}=P$. q.e.d.

By the proof of Proposition 2.1, we have shown that if A is a segment of G such that $A\not\subset Q_{x_0}$, P_{x_0} for all $x_0\in\boldsymbol{R}$, then for any $x_1<x_2$ in $\boldsymbol{R}$, there exists a function $f\in A$ such that $f(x)=0$ on $(x_1, x_2]$, and moreover if $A\not\subset Q_{-\infty}$, then there exists $g\in A$ such that $g(x)=0$ on $(-\infty, x_1]$. We shall call such a process of constructing f *C-machine.*

Let H be the group of $\boldsymbol{Z}$-valued functions on $\boldsymbol{R}$. H is a lattice-ordered group by defining the order like that of G. If A is a segment of G, we define a function $F(A) \in H^+$ by $F(A)(x) = \min_{f \in A} f(x)$, $x \in \boldsymbol{R}$. For any $f \in H$, let $D(f)$ be the set of points of left discontinuity of f.

PROPOSITION 2.2. *If A is a segment of G, then $F(A)$ has the following properties:*

1) *$F(A)$ is a bounded function.*
2) $\overline{\lim}_{x \to x_0 - 0} F(A)(x) \leq F(A)(x_0)$ *for all* $x_0 \in D(F(A))$.

PROOF. By definition of $F(A)$, $F(A) \leq f$ for all $f \in A$. Since f is a bounded function, the first assertion is clear. For all $x_0 \in D(F(A))$, there exist $g \in A$ and $x_1 \in \boldsymbol{R}$, $x_1 < x_0$, such that $g(x_0) = F(A)(x_0)$ and $g(x) = g(x_0)$ on $(x_1, x_0]$. Hence, $g(x) \geq F(A)(x)$ on $(x_1, x_0]$. Thus the second assertion holds. q.e.d.

PROPOSITION 2.3. *If A is a proper segment of G and $F(A) = 0$, then there exists an element h in $G^+ - A$ such that $A: h$ is equal to Q_{x_0}, Q_∞ or $Q_{-\infty}$.*

PROOF. We first note that $F(A) = 0$ if and only if $A \not\subset P_{x_0}$ for all $x_0 \in \boldsymbol{R}$. We shall treat four cases separately.

Case I) When A is in Q_∞ but not in $Q_{-\infty}$. a) If every segment of type Q_{x_0} does not contain A, then A is irreducible. We shall show this. Let f be an element of A. Then there exists $r \in \boldsymbol{R}$ such that $f(x)$ is constant on (r, ∞). We denote the constant by $P(f)$ and $\min_{f \in A} P(f)$ by $P(A)$. Now we suppose that A is reducible, that is, $A = B \cap C$, for some segment $A \subsetneqq B$, C. Then $P(A) \geq P(B)$, $P(C)$. If $P(A) = P(B)$, for any $b \in B$, there exists $a \in A$ such that $a \leq b$ by *C-machine*. Hence $b \in A$ and this implies $A = B$. Thus $P(A) > P(B)$, $P(C)$. But it is impossible, then A is irreducible. Next we can easily see that by *C-machine*, if $P(A) = 1$, then $A = Q_\infty$, and if $P(A) \neq 1$, then there exists $h \in G^+ - A$ such that $A: h = Q_\infty$.

b) If A is contained in Q_{x_0} for some x_0, A is reducible. In fact, let $x_1 \in \boldsymbol{R}$ be $x_1 > x_0$ and f be in A. Define a function g_f (resp. h_f) by $g_f(x) = f(x)$ (resp. $h_f(x) = 0$) on $(-\infty, x_1]$ and $g_f(x) = 0$ (resp. $h_f(x) = f(x)$) on (x_1, ∞) Then g_f and h_f are not in A. Put $B = \bigvee_{f \in A} (g_f)$ and $C = \bigvee_{f \in A} (h_f)$, then B and C are segments containing A properly and $A = B \cap C$. Thus A is reducible. Next we shall show that there exists $h \in G^+ - A$ such that $A: h = Q_{x_0}$. If $x_0 \in \boldsymbol{R}$ is a unique real number such that $A \subset Q_{x_0}$, then there is $h \in G^+ - A$ such that $A: h = Q_\infty$, because, let f be in A and x_1 be $x_0 < x_1$ and define a function h_0 by $h_0(x) = f(x)$ on $(-\infty, x_1]$ and $h_0(x) = 0$ otherwise, then the segment $A: h$ is in case I. a). We suppose that the set of x_0 such that $A \subset Q_{x_0}$ has more than one element. Take $x_0 < x_1$ in the set and let $\bar{x}_1$ be $\inf\{r; f(x) = 0 \text{ on } (r, x_1] \text{ for some } f \in A\}$, then $x_0 \leq \bar{x}_1$ and $A \subset Q_{\bar{x}_1}$. Let g be in A; then there exists $x_2 (> \bar{x}_1) \in \boldsymbol{R}$ such that $g(x)$ is constant on $(\bar{x}_1, x_2]$. We denote the constant by $E_{\bar{x}_1}(g)$ and $\min_{g \in A}$

$E_{\bar{x}_1}(g)$ by $E_{\bar{x}_1}(A)$. Define $g_0 \in G^+ - A$ by $g_0(x)=0$ on $(\bar{x}_1, x_2]$ and $g_0(x)=f(x)$ elsewhere. Then the proof of the rest is similar to that of case I. a).

Case II) When A is in $Q_{-\infty}$ but not in Q_∞, the proof is similar to that of case I.

Case III) When A is in Q_{x_0} for some x_0 but A is in neither Q_∞ nor $Q_{-\infty}$.
a) If x_0 is a unique real number such that A is in Q_{x_0}, then A is irreducible and for some $h \in G^+ - A$, $A: h = Q_{x_0}$.
b) If the set of x_0 such that A is in Q_{x_0} has more than one element, we can readily see that A is reducible. The proof of the rest is similar to case I.

Case IV) When A is in Q_∞ and $Q_{-\infty}$, replacing A by $A: h$ for suitable h, we can reduce to the case I. q.e.d.

For any element f in H^+, $A(f)$ be the set $\{g \in G^+;\ g \geq f\}$. Then $A(f) \neq \phi$ is equivalent to saying that f is a bounded function. When that is so, $A(f)$ is a segment of G. The following proposition follows immediately from definitions.

Proposition 2.4. *Let f be a bounded function in H^+; then $F(A(f)) \geq f$.*

Proposition 2.5. *When f is a bounded function in H^+, $F(A(f))$ coincides with f if and only if $\overline{\lim}_{x \to x_0 - 0} f(x) \leq f(x_0)$ for all $x_0 \in D(f)$.*

Proof. First suppose that $\overline{\lim}_{x \to x_0 - 0} f(x) \leq f(x_0)$ for all $x_0 \in D(f)$. By the definition of the upper limit, there is a positive number ε such that $f(x) \leq f(x_0)$ on $(x_0 - \varepsilon, x_0]$. Since f is bounded, there is $h \in G^+$ such that $h(x) = f(x_0)$ on $(x_0 - \varepsilon, x_0]$ and $h(x) \geq f(x)$ otherwise. Then $h \in A(f)$ and also $F(A(f))(x_0)$ $f(x_0)$ for all $x_0 \in D(f)$. If x_1 is not in $D(f)$, f is left continuous at x_1. Then $F(A(f))\,(x_1) = f(x_1)$ and this means that $F(A(f)) = f$. The converse is clear. q.e.d.

When a bounded function $f \in H^+$ satisfies the condition in Proposition 2.5, we shall say that f has the property (E). If f and g has the property (E) and $A(f) = A(g)$, then $f = g$. If A is a segment of G, then $F(A)$ has the property (E).

Proposition 2.6. *If A is a segment of G and $h \in G^+$, then $F(A: h) = (F(A) - h)^+$*

Proof. $A: h = \{\bigcup_{g \in A} (g)\} : h = \bigcup_{g \in A} \{(g): h\} = \bigcup_{g \in A} (g - h)^+$. Then the assertion is obvious. q.e.d.

When f has the property (E), we shall say that f is irreducible if f is not represented as $\sup(g, h)$, where $f \neq g, h$ and g, h have the property (E). If f is not irreducible, then we shall say that f is reducible; and if $(f - h)^+$ is reducible for every element $h \in G^+ - A(f)$, we shall say that f is of type II.

PROPOSITION 2.7. *Let f be in H^+. Then f is irreducible if and only if $f=0$ or there exists $r_0 \in \boldsymbol{R}$ such that $f(r_0)>0$ and $f(x)=0$ elsewhere.*

PROOF. Assume that f is irreducible and $f\neq 0$. Then there exists a unique $r_0 \in \boldsymbol{R}$ such that $f(r_0)>0$. In fact, we shall suppose that there exists another $r_1 \in \boldsymbol{R}$ such that $f(r_1)>0$. Define a function f_1 (resp. f_2) by $f_1(x)=0$ (resp. $f_2(x)=f(x)$) on $\left(-\infty, \frac{r_1+r_2}{2}\right]$ and $f_1(x)=f(x)$ (resp. $f_2(x)=0$) elsewhere. Then f_1 and f_2 have the property (E), $f\neq f_1$, f_2 and $f=\sup(f_1, f_2)$. This is a contradiction. The converse is clear. q.e.d.

By making use of the similar technique of the proof of Proposition 2.3 and by Proposition 2.7 if A is a segment of G and $F(A)$ is reducible, then A is reducible. Let A and B be irreducible segments of G. We shall say that A and B are equivalent if for some $f \in G^+-A$ and $g \in G^+-B$, $A: f=B: g$. Then this relation is an equivalence relation. By the above remark, we have the following:

PROPOSITION 2.8. *In G, every irreducible segment is equivalent to one of the prime segments P_{x_0}, Q_x, Q_∞ and $Q_{-\infty}$.*

THEOREM 2.9. *If $f\neq 0$ has the property (E), then the following statements are equivalent.*

1) *f is of type* II.
2) $\overline{\lim}_{x\to x_0-0} f(x)=f(x_0)$ *for all* $x_0 \in D(f)$.

PROOF. First we assume the condition 1). If $\overline{\lim}_{x\to x_0-0} f(x)<f(x_0)$ for some x_0 in $D(f)$, then $f(x)<f(x_0)$ on $(x_0-\varepsilon, x_0]$ for suitably chosen $\varepsilon>0$. Since f is bounded, there exists $k \in \boldsymbol{R}$ such that $f(x)\leq k$ on $\boldsymbol{R}$. Define a function $h \in G^+$ by $h(x)=f(x_0)-1$ on $(x_0-\varepsilon, x_0]$ and $f(x)=k$ otherwise, then $(f-h)^+(x_0)=1$ and $(f-h)^+(x)=0$ elsewhere. Therefore by Proposition 2.7, $(f-h)^+$ is irreducible. This is a contradiction. Conversely we assume the condition 2). Let h be any element in $G^+-A(f)$ and $g=(f-h)^+$. Then $g\neq 0$ and also g satisfies the condition 2). Therefore it is sufficient to show that f is reducible. Since $f\neq 0$, there exists $x_0 \in \boldsymbol{R}$ such that $f(x_0)>0$. If x_0 is in $D(f)$, by assumption, for any positive number ε, $f(x)>0$ at infinitely many points x on $(x_0-\varepsilon, x_0]$. On the other hand, if x_0 is not in $D(f)$, f is constant (>0) on $(x_1, x_0]$ for some $x_1(<x_0)$. Hence by Proposition 2.7, f is reducible. q.e.d.

Let A be a segment of G. We shall say that A is of type II if $A: h$ is reducible for all $h \in G^+-A$.

COROLLARY 2.10. *If $f \in H^+$ has the property (E), then f is of type II if and only if $A(f)$ is of type II.*

PROOF. We suppose that f is of type II. Then by Proposition 2.6 and the remark after Proposition 2.7, $A(f)$ is of type II. We shall omit the proof of the converse. q.e.d.

COROLLARY 2.11. *Every principal segment of G is of type II.*

Now we can convert the result obtained above on the lattice-ordered group G to language of the bezoutian domain D. We have determined the type of ideals of D which are irreducible or of type II. Namely Proposition 2.8 means that every irreducible ideal of D is equivalent to some prime ideal, and Corollary 2.11 means that every non-zero proper principal ideal of D is of type II. Thus we know that the condition in Theorem 1.1 does not hold in our bezoutian domain D. Moreover, Proposition 2.9 means that there exist ideals of type II which are not principal. We shall also use this example in §3.

§3. The type of modules

Let R be a ring. We shall say that an R-module M is of type I if any non-zero submodule of M contains a co-irreducible submodule and also say that M is of type II if no submodules of M contain a co-irreducible submodule. Then from definitions, non-zero submodules and essential extensions of a module of type I (resp. type II) are also of type I (resp. type II). Any injective module of type I is the injective hull of a direct sum of indecomposable injective modules. Moreover, any torsion free module over a domain is of type I.

PROPOSITION 3.1. *Any direct sum of a family of modules of type I is also of type I.*

PROOF. If a direct sum $\bigoplus M_j$, M_j being of type I, is not of type I, then we can find a non-zero submodule N such that no submodules of N are co-irreducible. Let $x \neq 0$ be an element of N; then we can write $x = x_1 + x_2 + \cdots + x_n$ for $1 \leq i \leq n < \infty$, $x_i \in M_{j_i}$. Hence $0(x) = 0(x_1) \cap 0(x_2) \cap \cdots \cap 0(x_n)$. We may assume that this intersection is irredundant. If $n=1$, then there exists $r \in R$ such that $0(x_1): r$ is irreducible. On the other hand, $0(x): r$ is not irreducible, and this is a contradiction. If $n > 1$, then there exists an element r in $\bigcap_{i \geq 2} 0(x_i)$ but not in $0(x_1)$. Then $0(x): r = 0(x_1): r$. Since $0(x): r$ is of type II but $0(x_1): r$ is not of type II, this is also a contradiction. q.e.d.

PROPOSITION 3.2. *Any direct sum of a family of modules of sype II is also of type II.*

The proof is similar to that of Proposition 3.1.

Example 3.3. *An infinite direct product of modules of type I is not necessarily of type I.*

Let $P_x(x \in \boldsymbol{R})$ be prime segments of the lattice-ordered group G in §2 and be $\bigcap_{x \in \boldsymbol{R}} P_x$; then is the set of all $f \in G^+$ such that $f(x) \geq 1$ on $\boldsymbol{R}$. Therefore I is a principal segment. By Corollary 2.11, I is of type II. If we use the same notion for ideals of D corresponding to segments of G, the module D/P_x is co-irreducible (i.e. of type I) and the module D/I is of type II. Consider a natural isomorphism of D/I into $\prod_{x \in \boldsymbol{R}} D/P_x$, then the assertion is clear.

Example 3.4. *An infinite direct product of modules of type II is not necessarily of type II.*

For any natural number n, define a function $f_n \in G^+$ by $f_n(x)=n$ on $(-1, 0]$ and $f_n(x)=0$ elsewhere. Then principal segments (f_n) are of type II. Put $I=\bigcap_{n \in \boldsymbol{N}} (f_n)$, then is an empty set. We note that to an empty set of G, there corresponds zero in D. And the module $D/(f_n)$ is of type II and $D/0$ is coirreducible, because D is a domain. Consider a natural isomorphism of D into $\prod_{n \in \boldsymbol{N}} D/(f_n)$, then the assertion is clear.

The following proposition is not essentially new ([1, p 329]), however we can give another approach.

Proposition 3.5. *Let E be an injective R-module. Then there exists a maximal submodule of type II; and furthermore it is injective and unique up to isomorphism.*

Proof. If E has not a submodule of type II, then E is of type I. When E has a submodule of type II, we can find a maximal submodule N of type II by Zorn's lemma. Then N is an injective module, because an R-module of type II is closed under an essential extension. Then there exists a submodule M such that $E=M \oplus N$. We can readily see that M is of type I. Let $E=M' \oplus N'$ be the second decomposition, where M' is of type I and N' is of type II. Then we shall show that $E=M' \oplus N$. In fact, clearly $M' \cap N=0$ and also $M' \oplus N$ is an injective module, if $M' \oplus N \neq E$, then there exists a non-zero submodule L such that $M' \oplus N \oplus L=E$. Thus $N \oplus L$ is isomorphic to N'. Since N' is of type II, L is of type II, and this contradicts maximality of N. Therefore $E= M' \oplus N$. From this, $N \cong N'$. q.e.d.

References

[1] Manabu Harada and Yousin Sai, On categories of indecomposable modules I, Osaka J. Math. **7** (1970), 323–344.

[2] W. Heinzer, J-noetherian integral domain with 1 in the stable range, Proc. Amer. Math. Soc. **19** (1968), 1369–1372.
[3] P. Jaffard, Les systémes d'idéaux, Travaux et Recherches Mathématique Dunod, Paris (1960).
[4] E. Matlis, Injective modules over noetherian rings, Pacific J. Math. **8** (1958), 511–528.
[5] R. B. Warfields, Decompositions of injective modules, Pacific J. Math. **31** (1969), 263–276.

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