HIROSHIMA MATH. J.
3 (1973), 109–120

The Paley-Wiener Theorem for Distributions on Symmetric Spaces

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(Received January 20, 1973)

1. Introduction

Let S be a symmetric space of the non-compact type. In his paper [10], S. Helgason obtained the Paley-Wiener theorem for the Fourier transform of the functions in $C_c^\infty(S)$.

The purpose of this paper is to characterize the Fourier transform of the distributions with compact support.

The crucial points of our proof are as follows. By means of convolution by a Dirac sequence we consider the regularization of tempered distributions (for the characterization of the Fourier transform of tempered distributions, see Theorem 6 in [1]) which, we notice, are the functions in $C_c^\infty(S)$. Then we use the above mentioned Helgason's Paley-Wiener theorem.

2. Notation and Preliminaries

As usual $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the field of real numbers and the field of complex numbers, respectively. Let i denote a square root of -1. If M is a manifold, $C^\infty(M)$ and $C_c^\infty(M)$ denote the set of complex valued C^∞ functions on M and the set of C^∞ functions on M with compact support, respectively. If V is a finite dimensional vector space over $\boldsymbol{R}$, $\mathscr{S}(V)$ denotes the space of rapidly decreasing functions on V([12]) and $\boldsymbol{D}(V)$ denotes the algebra of differential operators with constant coefficients on V.

Let $\boldsymbol{R}^n$ be the n-dimensional Euclidean space, $|x|$ the Euclidean norm of $x \in \boldsymbol{R}^n$ and dx the Euclidean measure on $\boldsymbol{R}^n$. Let ρ be the function on $\boldsymbol{R}^n$ defined by

$$\rho(x)=\begin{cases} a \exp\left\{-\dfrac{1}{1-|x|^2}\right\} & \text{if } |x|<1, \\ 0 & \text{if } |x| \geqq 1, \end{cases}$$

where

$$a^{-1}=\int_{|x|<1}\exp\left\{-\frac{1}{1-|x|^2}\right\}dx.$$

For $\varepsilon>0$, ρ_ε, the function on $\boldsymbol{R}^n$ is defined by $\rho_\varepsilon(x)=\varepsilon^{-n}\rho(x/\varepsilon)$. We call $\{\rho_\varepsilon\}_{\varepsilon>0}$ a Dirac sequence on $\boldsymbol{R}^n$.

Let G be a connected semisimple Lie group with finite center, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{g}_{\boldsymbol{C}}$ the complexification of $\mathfrak{g}$, $\mathscr{B}$ the universal enveloping algebra of $\mathfrak{g}_{\boldsymbol{C}}$ which is identified with the algebra of left invariant differential operators on G and $<\ ,\ >$ the Killing form of $\mathfrak{g}_{\boldsymbol{C}}$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition, θ the corresponding Cartan involution, $\mathfrak{a}_\mathfrak{p}$ a maximal abelian subspace of $\mathfrak{p}$, K and A the analytic subgroups of G with Lie algebras $\mathfrak{k}$ and $\mathfrak{a}_\mathfrak{p}$, respectively. Let $\mathfrak{a}_\mathfrak{p}^*$ denote the dual of $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_{\mathfrak{p}\boldsymbol{C}}^*$ its complexification. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$, which contains $\mathfrak{a}_\mathfrak{p}$, $\mathfrak{h}_{\boldsymbol{C}}$ its complexification and Δ the set of nonzero roots of $\mathfrak{g}_{\boldsymbol{C}}$ with respect to $\mathfrak{h}_{\boldsymbol{C}}$. Put $\mathfrak{h}\cap\mathfrak{k}=\mathfrak{a}_\mathfrak{t}$ and $\mathfrak{h}^*=\mathfrak{a}_\mathfrak{p}+i\mathfrak{a}_\mathfrak{t}$. Select compatible orderings in the dual spaces of $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{h}^*$, respectively. Each root $\alpha\in\Delta$ is real valued on $\mathfrak{h}^*$ and we get an ordering of Δ. Let P and P_+ denote the set of positive roots and the set of positive roots which do not vanish identically on $\mathfrak{a}_\mathfrak{p}$, respectively. Let $\rho=\frac{1}{2}\sum_{\alpha\in P_+}\bar{\alpha}$ and $\mathfrak{n}=(\sum_{\alpha\in P_+}\mathfrak{g}^\alpha)\cap\mathfrak{g}$, where $\bar{\alpha}$ is the restriction of α to $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{g}^\alpha$ is the root space for each α. Let N and $\bar{N}$ denote the analytic subgroups of G with the Lie algebras $\mathfrak{n}$ and $\bar{\mathfrak{n}}=\theta\mathfrak{n}$, respectively. Thus we obtain $G=KAN$ the Iwasawa decomposition corresponding to the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}_\mathfrak{p}+\mathfrak{n}$. Any element $g\in G$ is written uniquely as $g=\kappa(g)\exp H(g)n(g)$, $(\kappa(g)\in K$, $H(g)\in\mathfrak{a}_\mathfrak{p}$, $n(g)\in N)$. For $a\in A$ we write $H(a)=\log a$. If $\lambda,\mu\in\mathfrak{a}_{\mathfrak{p}\boldsymbol{C}}^*$ let $H_\lambda\in\mathfrak{a}_{\mathfrak{p}\boldsymbol{C}}$ be determined by $\lambda(H)=<H_\lambda, H>$ for $H\in\mathfrak{a}_{\mathfrak{p}\boldsymbol{C}}$ and put $<\lambda,\mu>=<H_\lambda,H_\mu>$. Since $<\ ,\ >$ is positive definite on $\mathfrak{p}$ we put $\|\lambda\|=<\lambda,\lambda>^{1/2}$ for $\lambda\in\mathfrak{a}_\mathfrak{p}^*$ and $\|X\|=<X,X>^{1/2}$for $X\in\mathfrak{p}$. For each $\lambda\in\mathfrak{a}_{\mathfrak{p}\boldsymbol{C}}^*$, we put $\lambda=\mathscr{R}\lambda+i\mathscr{I}\lambda$, $\mathscr{R}\lambda$, $\mathscr{I}\lambda\in\mathfrak{a}_\mathfrak{p}^*$, and put $\|\lambda\|=(\|\mathscr{R}\lambda\|^2+\|\|\mathscr{I}\lambda\|^2)^{1/2}$. Let M and M' be the centralizer and normalizer of $\mathfrak{a}_\mathfrak{p}$ in K, respectively, and put $W=M'/M$, which is called the (little) Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{a}_\mathfrak{p}$. $\boldsymbol{D}(K/M)$ denotes the algebra of K-invariant differential operators on K/M.

Let $l=\dim\mathfrak{a}_\mathfrak{p}$. The Killing form induces Euclidean measures on A and $\mathfrak{a}_\mathfrak{p}^*$ respectively; multiplying these by the factor $(2\pi)^{-(1/2)l}$ we obtain invariant measures da and $d\lambda$ respectively, and the inversion formula for the Fourier transform

$$\hat{f}(\lambda)=\int_A f(a)\exp\{-i\lambda(\log a)\}da, \qquad \lambda\in\mathfrak{a}_\mathfrak{p}^*, f\in\mathscr{S}(A),$$

holds without any multiplicative constant:

$$f(a)=\int_{\mathfrak{a}_\mathfrak{p}^*}\hat{f}(\lambda)\exp\{i\lambda(\log a)\}d\lambda, \qquad f\in\mathscr{S}(A),$$

where $\mathscr{S}(A)$ is the Schwartz space in the usual sense. We normalize the Haar measure dk on the compact group K so that the total measure is 1. The Haar measures of the nilpotent groups N, $\bar{N}$ are normalized so that

$$\theta(dn)=d\bar{n}, \quad \int_{\bar{N}} \exp\{-2\rho(H(\bar{n}))\}d\bar{n}=1.$$

The Haar measure dg on G can be normalized so that

$$\int_G f(g)dg=\int_{KAN} f(kan)\exp\{2\rho(\log a)\}dkdadn, \qquad f\in C_c^\infty(G).$$

For $x\in G$, define

$$\Xi(x)=\int_K \exp\{-\rho(H(xk)\}dk.$$

Since any $x\in G$ can be written uniquely as $x=k\cdot\exp X$, $(k\in K,\ X\in\mathfrak{p})$, we put

$$\sigma(x)=\|X\|.$$

Then Ξ and σ are spherical functions.

3. K-Bi-Invariant Case

In this paragraph we define the Fourier transform of the K-bi-invariant distributions on G with compact support, and characterize the image of them.

We summarize some results which are used below.

The space $\mathscr{S}(K\backslash G/K)$ is defined as the set of functions $f\in C^\infty(G)$ satisfying the following conditions (a) and (b) viz.: (a) $f(kxk')=f(x)$, $(x\in G,\ k,\ k'\in K)$. (b) *For each left invariant differential operator D on G and each integer $m\geqq 0$, we have $\tau_{D,m}(f)<+\infty$; here $\tau_{D,m}$ is the seminorm defined by*

$$\tau_{D,m}(f)=\sup_{x\in G}(1+\sigma(x))^m\Xi(x)^{-1}|(Df)(x)|.$$

The space $\mathscr{S}(K\backslash G/K)$ is topologized by the seminorms τ, given by the above with varying D and m. It is a Fréchet space.

The space $\mathscr{S}(K\backslash G/K)$ is preceisely what Harish-Chandra calls $I(G)$ in [6].

Let $\mathscr{S}(\mathfrak{a}_\mathfrak{p}^*)^W$ be the subspace of W-invariants in $\mathscr{S}(\mathfrak{a}_\mathfrak{p}^*)$. These spaces are equipped with their usual topologies.

For any $\lambda\in\mathfrak{a}_{\mathfrak{p}C}^*$, the function defined by

$$\phi_\lambda(x)=\int_K \exp\{(i\lambda-\rho)\ (H(xk))\}dk$$

is called the elementary spherical function corresponding to λ. It is known that

$\phi_{s\lambda}=\phi_{\lambda}$ for any $s\in W$ ([5, Corollary to Lemma 17]).

For $f\in\mathcal{S}(K\backslash G/K)$, its Fourier transform is defined by

$$(\mathcal{F}f)(\lambda)=\tilde{f}(\lambda)=\int_G f(x)\phi_{-\lambda}(x)dx, \qquad (\lambda\in\mathfrak{a}_{\mathfrak{p}}^*).$$

THEOREM 1. (Harish-Chandra [5, 6, 7], Helgason [8]). *The map $f\to\tilde{f}$ is a topological isomorphism of $\mathcal{S}(K\backslash G/K)$ onto $\mathcal{S}(\mathfrak{a}_{\mathfrak{p}}^*)^W$.*

Let $\mathcal{S}'(K\backslash G/K)$ and $\mathcal{S}'(\mathfrak{a}_{\mathfrak{p}}^*)^W$ be the strond duals of $\mathcal{S}(K\backslash G/K)$ and $\mathcal{S}(\mathfrak{a}_{\mathfrak{p}}^*)^W$, respectively. If we consider the transposed inverse $(\mathcal{F}^{-1})^*$ of the Fourier transform, we obtain the following.

COROLLARY. *The map $(\mathcal{F}^{-1})^*$ is a topological isomorphism of $\mathcal{S}'(K\backslash G/K)$ onto $\mathcal{S}'(\mathfrak{a}_{\mathfrak{p}}^*)^W$.*

The space $\mathcal{D}(K\backslash G/K)$ is the set of K-bi-invariant functions $f\in C_c^\infty(G)$. Let $\mathcal{H}_r(\mathfrak{a}_{\mathfrak{p}C}^*)^W$ denote the set of functions $F\in C^\infty(\mathfrak{a}_{\mathfrak{p}}^*)$ satisfying the following conditions: (a) *F can be extended to an entire holomorphic function on $\mathfrak{a}_{\mathfrak{p}C}^*$*, (b) *$F(s\lambda)=F(\lambda)$ for any $s\in W$ and $\lambda\in\mathfrak{a}_{\mathfrak{p}C}^*$*, (c) *there exists a constant $R\geqq 0$ such that for each integer $m\geqq 0$*

$$\sup_{\lambda\in\mathfrak{a}_{\mathfrak{p}C}^*}(1+\|\lambda\|)^m\exp\{-R\|\mathcal{I}\lambda\|\}\,|F(\lambda)|<+\infty.$$

Helgason [9] and Gangolli [2] proved the following Paley-Wiener theorem.

THEOREM 2. *The map $f\to\tilde{f}$ is a bijection of $\mathcal{D}(K\backslash G/K)$ onto $\mathcal{H}_r(\mathfrak{a}_{\mathfrak{p}C}^*)^W$.*

REMARK. The constant R in the above condition (c) is characterized by the fact: $f(\exp H)=0$ if $\|H\|>R$, $H\in\mathfrak{a}_{\mathfrak{p}}$.

Let $\mathcal{E}'(K\backslash G/K)$ denote the dual of the space $\mathcal{E}(K\backslash G/K)$ of K-bi-invariant functions $f\in C^\infty(G)$, elements of which are distributions on G with compact support. We define the Fourier transform of $T\in\mathcal{E}'(K\backslash G/K)$ as follows:

$$(\mathcal{F}T)(\lambda)=\tilde{T}(\lambda)=T(\phi_\lambda), \qquad (\lambda\in\mathfrak{a}_{\mathfrak{p}C}^*),$$

where ϕ_λ is the elementary spherical function.

Now in order to state our theorem we introduce the function space $\mathcal{H}_t(\mathfrak{a}_{\mathfrak{p}C}^*)^W$ on $\mathfrak{a}_{\mathfrak{p}}^*$ which consists of functions $F\in C^\infty(\mathfrak{a}_{\mathfrak{p}}^*)$ satisfying (a), (b) and (c) viz.: (a) *F can be extended to an entire holomorphic function on $\mathfrak{a}_{\mathfrak{p}C}^*$.* (b) *$F(s\lambda)=F(\lambda)$ for any $s\in W$ and $\lambda\in\mathfrak{a}_{\mathfrak{p}C}^*$.* (c) *There exist a constant $R\geqq 0$ and an integer $m\geqq 0$ such that*

$$\sup_{\lambda \in \mathfrak{a}_{\mathfrak{p}C}^*} (1+\|\lambda\|)^{-m} \exp\{-R\|\mathscr{I}\lambda\|\}\,|F(\lambda)| < +\infty.$$

THEOREM 3. *The map $T \to \tilde{T}$ is a bijection of $\mathscr{E}'(K\backslash G/K)$ onto $\mathscr{H}_t(\mathfrak{a}_{\mathfrak{p}C}^*)^W$.*

REMARK. The constant R in the above condition is characterized by the fact: $T(f)=0$ if $f \in \mathscr{D}(K\backslash G/K)$ and $f(\exp H)=0$ for $H \in \mathfrak{a}_{\mathfrak{p}}$, $\|H\|<R$.

PROOF. Let us first verify $\tilde{T} \in \mathscr{H}_t(\mathfrak{a}_{\mathfrak{p}C}^*)^W$. The condition (a) is an immediate consequence if we consider the local expression of T and the fact that $\phi_\lambda(x)$ is an entire function with respect to $\lambda \in \mathfrak{a}_{\mathfrak{p}C}^*$. The second condition is clear from the W-invariance of ϕ_λ. Let Ω be any compact set in G. Then we have

$$\sup_{x\in\Omega} |\phi_\lambda(x)| \leqq c \exp\{R\|\mathscr{I}\lambda\|\},$$

where c is a positive constant and $R=\sup_{x\in\Omega}\sigma(x)$. Since T is a continuous mapping of $\mathscr{E}(K\backslash G/K)$ into $\boldsymbol{C}$ with respect to the topology of $\mathscr{E}(K\backslash G/K)$, there exist $b_1, \ldots, b_r \in \mathscr{B}$ and a compact set Ω in G such that for any $f \in \mathscr{E}(K\backslash G/K)$ and a constant $c>0$

$$|T(f)| \leqq c \sum_{1\leqq i\leqq r} \sup_{x\in\Omega} |(b_i f)(x)|.$$

Hence

$$|\tilde{T}(\lambda)| = |T(\phi_\lambda)| \leqq c \sum_{1\leqq i\leqq r} \sup_{x\in\Omega} |(b_i\phi_\lambda)(x)|.$$

By Lemma 46 in [5], for each $b \in \mathscr{B}$ there exist an integer $d \geqq 0$ and a constant $a>0$ such that

$$|(b\phi_\lambda)(x)| \leqq a(1+\|\lambda\|)^d \phi_{i\mathscr{I}\lambda}(x), \qquad (\lambda \in \mathfrak{a}_{\mathfrak{p}C}^*).$$

Therefore we can find a positive integer m such that

$$|\tilde{T}(\lambda)| \leqq c(1+\|\lambda\|)^m \exp\{R\|\mathscr{I}\lambda\|\}.$$

Hence $\tilde{T} \in \mathscr{H}_t(\mathfrak{a}_{\mathfrak{p}C}^*)^W$. The injectivity follows from the corollary to Theorem 1.

Next we prove the surjectivity of the map. Let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be the Dirac sequence on A and put $\phi_\varepsilon(a) = \frac{1}{\omega} \sum_{s\in W} \rho_\varepsilon(sa)$, where $a \to sa$ is a canonical action of W on A and ω is the order of W. Then $\phi_\varepsilon \in \mathscr{D}(A)^W$. If we put $\mathscr{F}^{-1}(\hat{\phi}_\varepsilon) = f_\varepsilon$, then $f_\varepsilon \in \mathscr{D}(K\backslash G/K)$ and $\{f_\varepsilon\}_{\varepsilon>0}$ is a Dirac sequence on $K\backslash G/K$. Let $\check{f}_\varepsilon$ be defined by $\check{f}_\varepsilon(x) = f_\varepsilon(x^{-1})$ $(x \in G)$. Let F be a function in $\mathscr{H}_t(\mathfrak{a}_{\mathfrak{p}C}^*)^W$ and R be a constant in the definition of $\mathscr{H}_t(\mathfrak{a}_{\mathfrak{p}C}^*)^W$. Since $\mathscr{H}_t(\mathfrak{a}_{\mathfrak{p}C}^*)^W$ is contained in $\mathscr{S}'(\mathfrak{a}_{\mathfrak{p}}^*)^W$, there exists a distribution $T \in \mathscr{S}'(K\backslash G/K)$ such that $\tilde{T}=F$. Let $T * f_\varepsilon$ denote the convolution of T and f_ε defined by

$$(T*f_\varepsilon)(x)=T_y(f_\varepsilon(y^{-1}x)), \qquad (x,\ y\in G),$$

where the subscript y denotes the argument on which the distribution T acts. If we attend to K-bi-invariance of T and f_ε, it is not difficult to see that

$$(\mathscr{F}(T*f_\varepsilon))(\lambda)=\tilde{f}_\varepsilon(\lambda)F(\lambda).$$

Since $f_\varepsilon\in\mathscr{D}(K\backslash G/K)$, by Theorem 2

$$\sup_{\lambda\in\mathfrak{a}^*_{\mathfrak{p}C}}(1+\|\lambda\|)^n\exp\{-\varepsilon\|\mathscr{I}\lambda\|\}\,|\tilde{f}_\varepsilon(\lambda)|<+\infty$$

for any integer $n\geqq 0$. So we obtain

$$\sup_{\lambda\in\mathfrak{a}^*_{\mathfrak{p}C}}(1+\|\lambda\|)^p\exp\{-(R+\varepsilon)\|\mathscr{I}\lambda\|\}\,|F(\lambda)|\,|\tilde{f}_\varepsilon(\lambda)|<+\infty$$

for any integer $p\geqq 0$. Hence we have

$$(T*f_\varepsilon)(\exp H)=0 \quad \text{if } \|H\|>R+\varepsilon,\ H\in\mathfrak{a}_\mathfrak{p}$$

by the remark of Theorem 2. For any $\varepsilon_0>0$ we can select ε such that $\varepsilon_0>\varepsilon>0$ and

$$(T*f_\varepsilon)(\exp H)=0 \quad \text{if } \|H\|>R+\varepsilon_0,\ H\in\mathfrak{a}_\mathfrak{p}.$$

Now if we take a function $f\in\mathscr{D}(K\backslash G/K)$ such that $f(\exp H)=0$ for $H\in\mathfrak{a}_\mathfrak{p}$, $\|H\|<R+\varepsilon_0$, then $T(f)=0$. In fact,

$$0=(T*f_\varepsilon, f)=(T, \check{f}_\varepsilon*f),$$

and $\check{f}_\varepsilon*f$ tends to f with respect to the topology of $\mathscr{S}$ when ε tends to 0. As ε_0 is arbitrary $T(f)=0$ if $f\in\mathscr{D}(K\backslash G/K)$ and $f(\exp H)=0$ for $H\in\mathfrak{a}_\mathfrak{p}$, $\|H\|<R$.

4. General Case

In this paragraph we consider the general case and take off the condition of K-left-invariance on distributions.

The space $\mathscr{S}(G/K)$ is defined as the set of functions $f\in C^\infty(G)$ satisfying the following conditions (a) and (b) viz.: (a) $f(xk)=f(x)$, $x\in G$, $k\in K$. (b) *For each left invariant differential operator D on G and each integer $m\geqq 0$, we have $\mu_{D,m}(f)<+\infty$; here $\mu_{D,m}$ is the seminorm defined by*

$$\mu_{D,m}(f)=\sup_{x\in G}(1+\sigma(x))^m\Xi(x)^{-1}|(Df)(x)|.$$

The space $\mathscr{S}(G/K)$ is topologized by the seminorms μ, given by the above with varying D and m. It is a Fréchet space.

Let $\mathscr{S}(\mathfrak{a}_\mathfrak{p}^* \times (K/M))$ denote the set of functions $F \in C^\infty(\mathfrak{a}_\mathfrak{p}^* \times (K/M))$ which satisfy the following condition: for each $E \in \boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^*)$, $u \in \boldsymbol{D}(K/M)$ and each integer $r \geqq 0$,

$$\nu_{E,u,r}(F) = \sup_{(\lambda, kM) \in \mathfrak{a}_\mathfrak{p}^* \times (K/M)} (1 + \|\lambda\|)^r |(E_\lambda u_k F)(\lambda, kM)| < +\infty,$$

where the subscripts λ and k denote the arguments on which E and u act respectively.

Then the collection of these seminorms $\nu_{E,u,r}$ given by the above with varying E, u and r, topologize $\mathscr{S}(\mathfrak{a}_\mathfrak{p}^* \times (K/M))$ so that $\mathscr{S}(\mathfrak{a}_\mathfrak{p}^* \times (K/M))$ becomes a Fréchet space.

For any continuous complex valued function F on $\mathfrak{a}_\mathfrak{p}^* \times (K/M)$ let us extend the domain of F to all of $\mathfrak{a}_\mathfrak{p}^* \times G$ by defining

$$F_\lambda(x) = F(\lambda, x) = \exp\{(i\lambda - \rho)(H(x))\} F(\lambda, \kappa(x)M), \quad (x \in G).$$

Define the function $\mathscr{P}F_\lambda$ on G by

$$(\mathscr{P}F_\lambda)(x) = \int_K F_\lambda(xk)dk, \qquad (x \in G).$$

For any function $f \in \mathscr{C}(S)$, its Fourier transform $\tilde{f}$ is defined by

$$(\mathscr{F}f)(\lambda, kM) = \tilde{f}(\lambda, kM) = \int_{AN} f(kan) \exp\{(-i\lambda + \rho)(\log a)\} \, dadn$$

for $\lambda \in \mathfrak{a}_\mathfrak{p}^*$, $kM \in K/M$.

Now let $\mathscr{S}(\mathfrak{a}_\mathfrak{p}^* \times (K/M))^W$ denote the subspace of $\mathscr{S}(\mathfrak{a}_\mathfrak{p}^* \times (K/M))$ consisting of all $F \in \mathscr{S}(\mathfrak{a}_\mathfrak{p}^* \times (K/M))$ which satisfy the following functional equation:

$$\mathscr{P}F_{s\lambda} = \mathscr{P}F_\lambda, \qquad (\lambda \in \mathfrak{a}_\mathfrak{p}^* \text{ and } s \in W).$$

The spaces $\mathscr{S}'(G/K)$ and $\mathscr{S}'(\mathfrak{a}_\mathfrak{p}^* \times (K/M))^W$ are the strong duals of the topological vector spaces $\mathscr{S}(G/K)$ and $\mathscr{S}(\mathfrak{a}_\mathfrak{p}^* \times (K/M))^W$, respectively. Let $(\mathscr{F}^{-1})^*$ denote the transposed inverse of the Fourier transform.

THEOREM 4. (Eguchi and Okamoto [1. Theorem 6]). *The map* $(\mathscr{F}^{-1})^*$ *is a topological isomorphism of* $\mathscr{S}'(G/K)$ *onto* $\mathscr{S}'(\mathfrak{a}_\mathfrak{p}^* \times (K/M))^W$.

We need the Paley-Wiener type theorem. In order to state the theorem by Helgason [10] we denote by $\mathscr{H}_r(\mathfrak{a}_{\mathfrak{p}C}^* \times (K/M))^W$ the set of functions $F \in C^\infty(\mathfrak{a}_\mathfrak{p}^* \times (K/M))$ which satisfy the following conditions (a), (b) and (c) viz.: (a) *For each* kM *in* K/M *the function on* $\mathfrak{a}_\mathfrak{p}^*$ *defined by* $\lambda \to F(\lambda, kM)$ *can be extended to an entire holomorphic function on* $\mathfrak{a}_{\mathfrak{p}C}^*$. (b) $\mathscr{P}F_{s\lambda} = \mathscr{P}F_\lambda$, ($\lambda \in \mathfrak{a}_{\mathfrak{p}C}^*$ *and* $s \in W$). (c)

There exists a constant $R>0$ such that for each integer $m \geqq 0$

$$\sup_{(\lambda, kM) \in \mathfrak{a}_{\mathfrak{p}C}^* \times (K/M)} (1+\|\lambda\|)^m \exp\{-R\|\mathscr{I}\lambda\|\} |F(\lambda, kM)| < +\infty.$$

The space $\mathscr{D}(G/K)$ is the set of K-right-invariant functions $f \in C^\infty(G)$ with compact support.

THEOREM 5. (Helgason [10]). *The map $f \to \tilde{f}$ is a bijection of $\mathscr{D}(G/K)$ onto $\mathscr{H}_r(\mathfrak{a}_{\mathfrak{p}C}^* \times (K/M))^W$.*

REMARK. The constant R in the above condition (c) is characterized by the fact: $f(gK)=0$ if $\sigma(g)>R$, $g \in G$.

Let $\mathscr{E}'(G/K)$ denote the strong dual of the space $\mathscr{E}(G/K)$ of K-right-invariant functions $f \in C^\infty(G)$, elements of which are distributions on G with compact support. For each distribution $T \in \mathscr{E}'(G/K)$ we define the Fourier transform of T by

$$(\mathscr{F}T)(\lambda, kM) = \tilde{T}(\lambda, kM) = T_x(\exp\{(-i\lambda-\rho)(H(x^{-1}k))\}),$$

$$(\lambda \in \mathfrak{a}_{\mathfrak{p}}^* \text{ and } kM \in K/M),$$

where the subscript x denotes the argument on which the distribution T acts.

The space $\mathscr{H}_t(\mathfrak{a}_{\mathfrak{p}C}^* \times (K/M))^W$ is the set of functions $F \in C^\infty(\mathfrak{a}_{\mathfrak{p}}^* \times (K/M))$ satisfying the following conditions (a), (b) and (c) viz.: (a) *For each $kM \in K/M$, the function on $\mathfrak{a}_{\mathfrak{p}}^*$ defined by $\lambda \to F(\lambda, kM)$ can be extended to an entire holomorphic function on $\mathfrak{a}_{\mathfrak{p}C}^*$.* (b) $\mathscr{P}F_{s\lambda} = \mathscr{P}F_\lambda$, *($\lambda \in \mathfrak{a}_{\mathfrak{p}C}^*$ and $s \in W$).* (c) *There exist an integer $m \geqq 0$ and a constant $R>0$ such that*

$$\sup_{(\lambda, kM) \in \mathfrak{a}_{\mathfrak{p}C}^* \times (K/M)} (1+\|\lambda\|)^{-m} \exp\{-R\|\mathscr{I}\lambda\|\} |F(\lambda, kM)| < +\infty.$$

THEOREM 6. *The map $T \to \tilde{T}$ is a bijection of $\mathscr{E}'(G/K)$ onto $\mathscr{H}_t(\mathfrak{a}_{\mathfrak{p}C}^* \times (K/M))^W$.*

PROOF. Let us first verify $\tilde{T} \in \mathscr{H}_t(\mathfrak{a}_{\mathfrak{p}C}^* \times (K/M))^W$. The condition (a) is an immediate consequence if we consider the local expression of T and the fact that $\exp\{(-i\lambda-\rho)(H(x^{-1}k))\}$ is an entire function with respect to $\lambda \in \mathfrak{a}_{\mathfrak{p}C}^*$. The second condition is clear from

$$(\mathscr{P}\tilde{T})(\lambda, x) = T_y\Big(\int_K \exp\{(-i\lambda-\rho)(H(y^{-1}xk))\} dk\Big) = T_y(\phi_\lambda(y^{-1}x)).$$

To prove the third condition we use the following

LEMMA. *Let Ω be a compact set in G. Put $\psi_k(\lambda : x) = \exp\{(-i\lambda-\rho)$*

$(H(x^{-1}k))\}$ *for* $\lambda \in \mathfrak{a}_{\mathfrak{p}}^*$, $x \in G$ *and* $k \in K$. *Then for any* $b \in \mathscr{B}$, *we can select an integer* $d \geqq 0$ *and a positive constant* c *such that*

$$|(b_x\psi)(\lambda : x)| \leqq c(1+\|\lambda\|)^d \exp\{\|\mathscr{I}\lambda\|\sigma(x)\}.$$

PROOF OF THE LEMMA. We can prove this lemma by arguments similar to the proof of Lemma 46 in [5]. Let $\beta_1, \ldots, \beta_r$ be a fundamental system of roots in P. Select linear functions $\Lambda_1, \ldots, \Lambda_r$ on $\mathfrak{h}$ such that $\Lambda_i(H_{\beta_j}) = 2\beta_j(H_{\beta_j})\delta_{ij}$ $(1 \leqq i, j \leqq r)$. Then by Theorem 1 of [3], there exists an irreducible representation π_i of $\mathfrak{g}$ on a finite-dimensional space V_i with the highest weight Λ_i. Select a unit vector ξ_i in V_i belonging to Λ_i. Extend λ and ρ to linear functions on $\mathfrak{h}$ by defining them to be zero on $\mathfrak{a}_{\mathfrak{k}}$. Then $\lambda = \sum_{1 \leqq i \leqq r} \lambda_i \Lambda_i$ and $\rho = \sum_{1 \leqq i \leqq r} \rho_i \Lambda_i$ where $\lambda_i \in \boldsymbol{C}$ and $\rho_i \in \boldsymbol{R}$. We have $|\pi_i(x^{-1}k)\xi_i| = \exp\{\Lambda_i(H(x^{-1}k))\}$. Let $X_1, \ldots, X_n$ be a base for $\mathfrak{g}$ over $\boldsymbol{R}$ and for any $x \in G$ and $t = (t_1, \ldots, t_n) \in \boldsymbol{R}^n$, put $x_t = x \exp(t_1X_1 + \cdots + t_nX_n)$. Let $M = (m_1, \ldots, m_n)$ denote an n-tuple of nonnegative integers and write $|M| = m_1 + \cdots + m_n$, $t^M = t_1^{m_1} \ldots t_n^{m_n}$ and denote by $X(M)$ the coefficient (in $\mathscr{B}$) of t^M in $(|M|!)^{-1}(t_1X_1 + \cdots + t_nX_n)^{|M|}$. Also put $|t| = \max_{1 \leqq i \leqq n} |t_i|$ and $M + M' = (m_1 + m_1', \ldots, m_n + m_n')$ if $M' = (m_1', \ldots, m_n')$. Let E_j be the space of all linear endomorphisms of V_j $(1 \leqq j \leqq r)$. For any $T \in E_j$, put $|T| = \sup_{|v| \leqq 1} |Tv|$. Then E_j is a Banach space under this norm and

$$\pi_j(\exp\{-(t_1X_1 + \cdots + t_nX_n)\}) = \sum_M t^M(-1)^{|M|}\pi_j(X(M)),$$

the series converging absolutely and uniformly in E_j (see [4, §5]) provided $|t|$ remains bounded. Let $\tilde{\theta}$ denote the conjugation of $\mathfrak{g}_{\boldsymbol{C}}$ with respect to the compact real form $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ and $b \to b^*$ $(b \in \mathscr{B})$ be the anti-automorphism of $\mathscr{B}$ over $\boldsymbol{R}$ which coincides with $-\tilde{\theta}$ on $\mathfrak{g}_{\boldsymbol{C}}$. Then it is clear that $\pi_j(b^*)$ is the adjoint of $\pi_j(b)$ (in the sense of Hilbert space theory). Put

$$b_M = \sum_{M_1 + M_2 = M} (-1)^{|M|}(X(M_1))^* X(M_2)$$

for any M. Then it is obvious that

$$|\pi_j(x_t^{-1}k)\xi_j|^2 = |\pi_j(x^{-1}k)\xi_j|^2 + \sum_{|M| \geqq 1} t^M(\pi_j(x_t^{-1}k)\xi_j, \pi_j(b_M)\pi_j(x^{-1}k)\xi_j)$$

for all k, x and t $(1 \leqq j \leqq r)$. Put

$$\Psi_{M,j}(x) = |\pi_j(x)\xi_j|^{-2}(\pi_j(x)\xi_j, \pi_j(b_M)\pi_j(x)\xi_j), \qquad (x \in G).$$

Then $|\Psi_{M,j}(x)| \leqq |\pi_j(b_M)|$. Hence $\Psi_{M,j}$ is a bounded analytic function on G and

$$|\pi_j(x_t^{-1}k)\xi_j|^2 = |\pi_j(x^{-1}k)\xi_j|^2\{1 + \sum_{|M| \geqq 1} t^M \Psi_{M,j}(x^{-1}k)\}.$$

Obviously this series converges uniformly with respect to x, k and t provided x

varies within a compact subset of G and $|t|$ remains bounded. Therefore, if we put $\nu_j = i\lambda_j + \rho_j$, by the binomial theorem,

$$\exp\{(-i\lambda-\rho)\,(H(x_t^{-1}k))\} = \prod_{1\leqq j\leqq r} |\pi_j(x_t^{-1}k)\xi_j|^{-\nu_j}$$
$$= \exp\{(-i\lambda-\rho)\,(H(x^{-1}k))\} \sum_M t^M \Psi_M(x^{-1}k\colon \lambda)$$

provided $|t|$ is sufficiently small. Here $\Psi_M(x\colon\lambda)$ is a function on $G\times\mathfrak{a}^*_{\mathfrak{p}C}$ which can be written as a polynomial in ν_j and $\Psi_{M',j}$ $(1\leqq j\leqq r,\ |M'|\leqq|M|)$ with constant coefficients. Therefore it is clear that there exist a positive number a_M and an integer $d_M\geqq 0$ such that

$$|\Psi_M(x\colon\lambda)|\leqq a_M(1+\|\lambda\|)^{d_M}, \qquad (x\in G \text{ and } \lambda\in\mathfrak{a}^*_{\mathfrak{p}C}).$$

From this our lemma follows easily.

Now we return to the proof of the theorem. Since T is a continuous map of $\mathscr{E}(G/K)$ into $\boldsymbol{C}$ with respect to the topology of $\mathscr{E}(G/K)$, we can select $b_1,\dots,b_r \in\mathscr{B}$ and a compact set Ω in G such that for any $f\in\mathscr{E}(G/K)$ and a constant $c>0$

$$|T(f)|\leqq c\cdot \sum_{1\leqq j\leqq r} \sup_{x\in\Omega}|(b_j f)\,(x)|.$$

Hence

$$|(\mathscr{F}T)(\lambda,\,kM)| = |T_x(\exp\{(-i\lambda-\rho)\,(H(x^{-1}k))\})|$$
$$\leqq c\cdot \sum_{1\leqq j\leqq r} \sup_{x\in\Omega}|(b_{jx}\Psi_k)\,(\lambda\colon x)|.$$

Therefore, by the above lemma, we can select a positive integer m and a positive constant R such that

$$|(\mathscr{F}T)(\lambda,\,kM)|\leqq c(1+\|\lambda\|)^m \exp\{R\|\mathscr{I}\lambda\|\}.$$

Hence $\mathscr{F}T\in\mathscr{H}_t(\mathfrak{a}^*_{\mathfrak{p}C}\times(K/M))^W$. The injectivity follows from Theorem 4.

Next we prove the surjectivity of the map. Let F be any function in $\mathscr{H}_t(\mathfrak{a}^*_{\mathfrak{p}C}\times(K/M))^W$ and assume

$$\sup_{(\lambda,kM)\in\mathfrak{a}^*_{\mathfrak{p}C}\times(K/M)} (1+\|\lambda\|)^{-m}\exp\{-R\|\mathscr{I}\lambda\|\}|F(\lambda,\,kM)|<+\infty$$

for a positive number R and a positive integer m. Since $\mathscr{H}_t(\mathfrak{a}^*_{\mathfrak{p}C}\times(K/M))^W\subset\mathscr{S}'(\mathfrak{a}^*_{\mathfrak{p}}\times(K/M))^W$ there exists a unique distribution T in $\mathscr{S}'(G/K)$ such that $\mathscr{F}T=F$. Let $\{f_\varepsilon\}_{\varepsilon>0}$ be the Dirac sequence which was defined in §3. The convolution of T and f_ε is defined by $(T*f_\varepsilon)\,(x)=T_y(f_\varepsilon(y^{-1}x))$. Then for $\lambda\in\mathfrak{a}^*_p$ and $uM\in K/M$

$$\begin{aligned}(\mathscr{F}(T*f_\varepsilon))\,(\lambda,\,uM) &= (T*f_\varepsilon)_x(\exp\{(-i\lambda-\rho)\,(H(x^{-1}u))\}) \\ &= T_y\Big(\int_G f_\varepsilon(y^{-1}x)\exp\{(-i\lambda-\rho)\ (H(x^{-1}u))\}\,dx\Big) \\ &= T_y\Big(\int_G f_\varepsilon(x)\exp\{(-i\lambda-\rho)\ (H(x^{-1}y^{-1}u))\,dx\}\Big).\end{aligned}$$

Since $x^{-1}y^{-1}u=\kappa(x^{-1}\kappa(y^{-1}u))\exp\{H(x^{-1}\kappa(y^{-1}u))+H(y^{-1}u)\}n$, by Iwasawa decomposition, the last formula equals to

$$\begin{aligned}&T_y\Big(\int_G f_\varepsilon(x)\exp\{(-i\lambda-\rho)\,(H(x^{-1}\kappa(y^{-1}u))\}dx\cdot\exp\{(-i\lambda-\rho)\,(H(y^{-1}u))\}\Big) \\ &\quad = T_y(\tilde{f}_\varepsilon(\lambda,\ \kappa(y^{-1}u)\exp\{(-i\lambda-\rho)\ (H(y^{-1}u))\}) \\ &\quad = \tilde{f}_\varepsilon(\lambda)T_y(\exp\{(-i\lambda-\rho)\ (H(y^{-1}u))\}) \\ &\quad = \tilde{f}_\varepsilon(\lambda)\cdot F(\lambda,\ uM),\end{aligned}$$

where we used that f_ε is K-bi-invariant. From this for any integer $n\geqq 0$ and $u\in \boldsymbol{D}(K/M)$ we can select constants c_1 and c_2 such that

$$\begin{aligned}&(1+\|\lambda\|)^n|(u_k(\mathscr{F}(T*f_\varepsilon)))\ (\lambda,\ kM)| \\ &\quad = \{(1+\|\lambda\|)^{n+m}|\tilde{f}_\varepsilon(\lambda)|\}\cdot\{(1+\|\lambda\|)^{-m}(u_kF)\ (\lambda,\ kM)\} \\ &\quad \leqq c_1c_2\exp\{(R+\varepsilon)\ \|\mathscr{I}\lambda\|\}.\end{aligned}$$

So,

$$\sup_{(\lambda,kM)\in\mathfrak{a}^*_{\mathfrak{p}C}\times(K/M)} (1+\|\lambda\|)^n\exp\{-(R+\varepsilon)\|\mathscr{I}\lambda\|\}|(u_k(\mathscr{F}(T*f_\varepsilon)))(\lambda,\ kM)| < +\infty.$$

Hence, by Theorem 5 $(T*f_\varepsilon)\ (x)=0$ for $x\in G$, $\sigma(x)>R+\varepsilon$. Therefore, for any $\varepsilon_0>0$ if we select ε as $0<\varepsilon<\varepsilon_0$, then $(T*f_\varepsilon)\ (x)=0$ for $x\in G$, $\sigma(x)>R+\varepsilon_0$. Now if we take a function $f\in\mathscr{D}(G/K)$ such that $f(x)=0$ for $x\in G$, $\sigma(x)<R+\varepsilon_0$, then $T(f)=0$. In fact,

$$0=(T*f_\varepsilon,f)=(T,f*\check{f}_\varepsilon)$$

and $f*\check{f}_\varepsilon$ tends to f with respect to the topology of $\mathscr{S}$ when ε tends to 0. As ε_0 is arbitrary $T(f)=0$ if $f\in\mathscr{D}(G/K)$ and $f(x)=0$ for $x\in G$, $\sigma(x)<R$. This proves our theorem.

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