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On the Cauchy Problem for Linear Hyperbolic Differential Equations with Multiple Characteristics and Constant Leading Coefficients

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In his paper [9, Theorem 1, p. 667] G. Peyser proved that when P is a hyperbolic differential polynomial with constant coefficients, $P+Q$ is always hyperbolic for an arbitrary differential polynomial Q with constant coefficients of order $<m-d$ $(0\leqq d<m)$ if and only if the degree of degeneracy of $P\leqq d$. Obviously P is strictly hyperbolic if and only if $d=0$. We can extend the result to the case where P is a hyperbolic differential polynomial with variable coefficients $\in\mathscr{B}(R_{n+1})$ having constant leading coefficients and Q is an arbitrary differential polynomial with variable coefficients $\in\mathscr{B}(R_{n+1})$.

The main purpose of this paper is to obtain some refinements of our previous paper [11] by taking into account the degree of degeneracy mentioned above. Our method of approaching the Cauchy problem relies largely upon the L^2-estimates as developed in [11]. Section 1 is devoted to the preliminary discussions by means of which our energy inequalities will be derived as shown in section 2. It is to be noticed that the energy inequalities obtained here coincide with those established in [11] provided that the degree of degeneracy in question equals $m-1$. After recalling the Cauchy problem taken in the sense of M. Itano [3] we shall establish in section 3 with the aid of the energy inequalities obtained above the uniqueness and the existence of the solutions, which present generalizations of the results in [11, Theorem 2.1, p. 453]. The final section deals with a generalization of G. Peyser's result.

1. Preliminaries

Let P be a differential polynomial in R_{n+1} written in the form $P=D_t^m+\sum_{\nu_0=0}^{m-1}\sum_{|\nu|\leqq m}a_\nu(t,x)D^\nu$, $a_\nu\in\mathscr{B}(R_{n+1})$, where $D=(D_t, D_x)$, $D_x=(D_1, D_2,\dots, D_n)$ with $D_t=\frac{1}{i}\frac{\partial}{\partial t}$, $D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$, $j=1, 2, \dots, n$, and ν denotes a multi-index $\nu=(\nu_0, \nu')=(\nu_0, \nu_1,\dots, \nu_n)$. Throughout the present paper we shall assume that the principal part P_m of P has constant coefficients and that P is hyperbolic with respect to t, when each point (t, x) is fixed. For simplicity we shall call "hyperbolic" instead of "hyperbolic with respect to t". Let P_j, $j=0, 1,\dots, m$, be the homogeneous parts of order j of P and $P_m(\tau, \xi)$ be the polynomial associated with P_m, where $(\tau, \xi)=(\tau, \xi_1, \xi_2,\dots, \xi_n)$ is a point of the dual Eucliean space Ξ_{n+1}. We shall then use the following notations:

$$(1)\qquad \begin{aligned} &P_m^{(k)}(\tau, \xi)=\frac{\partial^k}{\partial\tau^k}P_m(\tau, \xi)=m(m-1)\dots(m-k+1)\prod_{j=1}^{m-k}(\tau-\lambda_j^{m-k}(\xi)),\\ &P_{m,j}^{(k)}(\tau, \xi)=P_m^{(k)}(\tau, \xi)/(\tau-\lambda_j^{m-k}(\xi)),\end{aligned}$$

where $\lambda_j^{m-k}(\xi)$, $j=1, 2,\dots, m-k$, stand for the roots of polynomials $P_m^{(k)}(\tau, \xi)$ in τ, which are listed in non-decreasing order. From the definitions of $P_m^{(k)}$ and $P_{m,j}^{(k-1)}$ we have

$$(2)\qquad P_m^{(k)}=\sum_{j=1}^{m-k+1}P_{m,j}^{(k-1)}.$$

Let μ, ν be multi-indices with $|\mu|=\sum_{j=0}^{n}\mu_j=m-1$, $|\nu|=m$, and let $u\in C_0^\infty(R_{n+1})$. We can then write

$$(3)\qquad (D^\nu u\overline{D^\mu u}-D^\mu u\overline{D^\nu u})=D_tG_0(D, \bar{D})u\bar{u}+\sum_{j=1}^{n}D_jG_j(D, \bar{D})u\bar{u},$$

where $G_j(D, \bar{D})u\bar{u}$, $j=0, 1,\dots, n$, are Hermitian differential quadratic forms in the derivatives of order $m-1$ of $u(t, x)$ [1, pp. 74–75, also 2, pp. 187–189]. From the relation (3) we have immediately

$$(4)\qquad -\operatorname{Im}P_m^{(k-1)}u\cdot\overline{P_m^{(k)}u}=\frac{\partial}{\partial t}A_0^{m-k}(u)+\sum_{j=1}^{n}\frac{\partial}{\partial x_j}A_j^{m-k}(u)$$

with Hermitian differential quadratic forms $A_j^{m-k}(u)$, $j=0, 1,\dots, n$, in the derivatives of order $m-k$ of u. Then we can show that

$$\int_{S_{t'}}A_0^{m-k}(u)dx\geqq 0,$$

where $S_{t'}$ is a hyperplane $t=t'$ in R_{n+1}.

In fact, since $A_0^{m-k}(u)$ is written in the form $\sum_{|\nu|=m-k}\sum_{|\mu|=m-k}a_{\nu,\mu}^{m-k}D^\nu u\overline{D^\mu u}$, it follows from Parseval's formula that

$$(5)\qquad \int_{S_{t'}}A_0^{m-k}(u)dx=\frac{1}{(2\pi)^n}\int K^{m-k}(\hat{u})d\xi,$$

where

$$K^{m-k}(\hat{u})=\sum_{|\nu|=m-k}\sum_{|\mu|=m-k}a_{\nu,\mu}^{m-k}\xi^{\nu'+\mu'}D_t^{\nu_0}\hat{u}(t', \xi)\overline{D_t^{\mu_0}\hat{u}(t', \xi)}.$$

Putting $v(t, x)=e^{i<x,\xi>}\hat{u}(t, \xi)$, we have $A_0^{m-k}(v)=K^{m-k}(\hat{u})$. Since $A_j^{m-k}(v)$, $j=1, 2,\dots, n$, are independent of x, the equation (4) means

$$-\mathrm{Im}\, P_m^{(k-1)}v\cdot\overline{P_m^{(k)}v}=\frac{\partial}{\partial t}K^{m-k}(\hat{u}). \tag{6}$$

On the other hand, in view of the relations (1) and (2), we see that

$$-\mathrm{Im}\, P_m^{(k-1)}v\cdot\overline{P_m^{(k)}v}=-\mathrm{Im}\, P_m^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)\cdot\overline{P_m^{(k)}(D_t, \xi)\hat{u}(t, \xi)}$$

$$=-\mathrm{Im}\sum_{j=1}^{m-k+1}(D_t-\lambda_j^{m-k}(\xi))P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)\cdot\overline{P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)}$$

$$=\frac{\partial}{\partial t}|P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)|^2.$$

Consequently, we obtain

$$K^{m-k}(\hat{u})=\sum_{j=1}^{m-k+1}|P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)|^2, \tag{7}$$

which completes the proof.

From the relation (2) it follows that

$$|P_m^{(k)}(D_t, \xi)\hat{u}(t, \xi)|^2\leqq(m-k+1)\sum_{j=1}^{m-k+1}|P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)|^2.$$

Combining this with the relations (5) and (7) yields the following

LEMMA 1. *Let* $u\in C_0^\infty(R_{n+1})$. *Then*

$$\int_{S_{t'}}|P_m^{(k)}(D)u|^2dx\leqq m\int_{S_{t'}}A_0^{m-k}(u)dx, \qquad k=1, 2,\dots, m.$$

Let $Q=D_t^m+\sum_{\nu_0=0}^{m-1}\sum_{|\nu|\leqq m}b_\nu D^\nu$ be a differential polynomial with constant coefficients. Owing to G. Peyser, Q is called properly hyperbolic if

i) the roots of the polynomial $Q_m(\tau, \xi)$ in τ are all real for all $\xi\in\Xi_n$,

ii) $Q_{m-k}(\tau, \xi)$ are expressed as follows:

$$Q_{m-k}(\tau, \xi)=\sum_{j=0}^{m-k+1}b_{k,j}(\xi)Q_{m,j}^{(k-1)}(\tau, \xi), \qquad k=1, 2,\dots, m \tag{8}$$

with bounded functions $b_{k,j}(\xi)$, $\xi\in\Xi_n$, [5, p. 480]. Clearly, a properly hyperbolic operator is hyperbolic. S. L. Svensson has shown that a hyperbolic operator Q is also properly hyperbolic [10, p. 154], which will be used in our later discussion. Let $\mathscr{H}_{(s)}(R_n)$ be a Sobolev space [2, p. 45] with norm $v\to\|v\|_{(s)}$:

$$\|v\|_{(s)}^2=\frac{1}{(2\pi)^n}\int(1+|\xi|^2)^s|\hat{v}(\xi)|^2d\xi.$$

Let $S^s(D_x)$ (or simply S^s) be a convolution operator with a symbol $S^s(\xi)=(1+|\xi|^2)^{s/2}$. Then, for any $b\in\mathscr{B}(R_n)$ and any real σ, $[b]S^s-S^s[b]$ is a bounded operator with norm $\|[b]S^s-S^s[b]\|_{(\sigma\to\sigma-s+1)}$ from $\mathscr{H}_{(\sigma)}(R_n)$ into $\mathscr{H}_{(\sigma-s+1)}(R_n)$, where we have used the notation $[b]$ to denote the multipicator: $\phi\to b\phi$ [6, p. 389].

LEMMA 2. *Let s be an arbitrary real number. Then there exists a constant C_s independent of u such that*

$$\int_{S_{t'}}|S^s(P_{m-k}u)|^2dx\leqq C_s\int_{S_{t'}}A_0^{m-k}(S^su)dx,\quad u\in C_0^\infty(R_{n+1}),\quad k=1, 2,\dots, m.$$

PROOF. Let the operator P be frozen at a point (t_0, x_0) and let Q denote the associated differential polynomial with constant coefficients. Then we have with a constant C independent of u

$$\int_{S_{t'}}|Q_{m-k}u|^2dx\leqq C\int_{S_{t'}}A_0^{m-k}(u)dx. \tag{9}$$

In fact, since Q is hyperbolic and $Q_m=P_m$, it follows from (8) that

$$|Q_{m-k}(D_t, \xi)\hat{u}(t, \xi)|^2\leqq\sum_{j=0}^{m-k+1}|b_{k,j}(\xi)|^2\sum_{j=0}^{m-k+1}|P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)|^2$$

$$\leqq C\sum_{j=0}^{m-k+1}|P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)|^2.$$

From (5), (7) and Parseval's formula, we obtain (9).

Next, we can write with an integer N

$$P_{m-k}(t, x, D)=\sum_{j=1}^{N}c_j(t, x)P_{m-k}(t_j, x_j, D),\qquad c_j\in\mathscr{B}(R_{n+1}),$$

where $P_{m-k}(t_j, x_j, D)$ denotes the operator frozen at (t_j, x_j). Since

$$S^s(P_{m-k}u)=\sum_{j=1}^{N}(S^sc_j(t, x)-c_j(t, x)S^s)P_{m-k}(t_j, x_j, D)u$$

$$+\sum_{j=1}^{N}c_j(t, x)P_{m-k}(t_j, x_j, D)S^su,$$

and $S^sc_j(t, x)-c_j(t, x)S^s$ is a bounded operator from $\mathscr{H}_{(s-1)}(R_n)$ into $\mathscr{H}_{(0)}(R_n)$, it follows that

$$\int_{S_{t'}}|S^s(P_{m-k}u)|^2dx\leqq C_s'\sum_{j=1}^{N}\int_{S_{t'}}|S^{s-1}P_{m-k}(t_j, x_j, D)u|^2dx+$$

$$+C'\sum_{j=1}^{N}\int_{S_{t'}}|P_{m-k}(t_j, x_j, D)S^s u|^2 dx$$

$$\leqq C''_s\sum_{j=1}^{N}\int_{S_{t'}}|P_{m-k}(t_j, x_j, D)S^s u|^2 dx,$$

where C'_s, C' and C''_s are constants such that

$$C'_s=\max_j \sup_t \|S^s[c_j]-[c_j]S^s\|^2_{(s-1\to 0)},$$

$$C'=\max_j \sup_{t,x}|c_j(t, x)|^2,$$

and

$$C''_s=\max(C'_s, C').$$

From this estimate and (9), we have with $C_s=C\cdot C''_s$

$$\int_{S_{t'}}|S({}^sP_{m-k}u)|^2 dx\leqq C_s\int_{S_{t'}} A_0^{m-k}(S^s u)dx,$$

as desired.

For our later need we recall the following (cf. [1, p. 72])

LEMMA 3. *Let $r(t')$ and $\rho(t')$ be two real valued functions defined in the interval $0\leqq t'\leqq T$ and suppose that $r(t')$ is continuous and $\rho(t')$ is non-decreasing. Then the inequality*

$$r(t')\leqq C(\rho(t')+\int_0^{t'} r(t)dt) \qquad (C \text{ is a constant})$$

implies

$$r(t')\leqq C\, e^{ct'}\rho(t').$$

2. Energy inequalities

Let P be a hyperbolic differential polynomial with variable coefficients $\in\mathscr{B}(R_{n+1})$ having constant leading coefficients. We say that the degree of degeneracy of $P\leqq d$ when the polynomial $P_m(\tau, \xi)$ in τ has the highest multiplicity of roots $\leqq d+1$ for any $\xi\in\Xi_n-\{0\}$.

By making use of Lemmas 1, 2 and 3, first we show that the following Proposition 1 which gives an extension of G. Peyser's result [8, Theorem 2.1, p. 484].

PROPOSITION 1. *Let T be an arbitrary positive number. Then there exists a constant C_T independent of u but depending on s such that*

$$\sum_{k=1}^{m}\int_{S_{t'}} A_0^{m-k}(S^s u)dx \leqq C_T\Big(\sum_{k=1}^{m}\int_{S_0} A_0^{m-k}(S^s u)dx$$
$$+\int_0^{t'}\int_{R_n} |S^s(Pu)|^2 dxdt\Big), \qquad 0\leqq t'\leqq T,\ u\in C_0^\infty(R_{m+1}).$$

PROOF. Let $v=S^s u$. Then, from the relation (6) and Parseval's formula, we have

$$\int_{S_{t'}} A_0^{m-k}(v)dx-\int_{S_0} A_0^{m-k}(v)dx=\int_0^{t'}\int_{R_n} -\operatorname{Im} P_m^{(k-1)}v\cdot \overline{P_m^{(k)}v}dxdt.$$

Since

$$\int_0^{t'}\int_{R_n} -\operatorname{Im} P_m^{(k-1)}v\cdot \overline{P_m^{(k)}v}\, dxdt$$
$$\leqq\Big(\int_0^{t'}\int_{R_n} |P_m^{(k-1)}v|^2 dxdt+\int_0^{t'}\int_{R_n} |P_m^{(k)}v|^2 dxdt,$$

it follows that

$$\sum_{k=1}^{m}\int_{S_{t'}} A_0^{m-k}(v)dx\leqq 2\Big(\sum_{k=1}^{m}\int_{S_0} A^{m-k}(v)dx+\int_0^{t'}\int_{R_n} |P_m v|^2 dxdt$$
$$+\sum_{k=1}^{m}\int_0^{t'}\int_{R_n} |P_m^{(k)}v|^2 dxdt\Big).$$

Therefore, owing to Lemma 1 we now obtain

$$\sum_{k=1}^{n}\int_{S_{t'}} A_0^{m-k}(v)dx\leqq 2m\Big(\sum_{k=1}^{m}\int_{S_0} A_0^{m-k}(v)dx \tag{10}$$
$$+\int_0^{t'}\int_{R_n} |P_m v|^2 dxdt+\sum_{k=1}^{m}\int_0^{t'}\int_{R_n} A_0^{m-k}(v)dxdt\Big).$$

On the other hand, in view of the relation $P_m v=S^s(Pu)-\sum_{k=1}^{m} S^s(P_{m-k}u)$ together with Lemma 2, we have with a constant C_s.

$$\int_0^{t'}\int_{R_n} |P_m v|^2 dxdt$$
$$\leqq (m+1)\Big(\int_0^{t'}\int_{R_n} |S^s(Pu)|^2 dxdt+\sum_{k=1}^{m}\int_0^{t'}\int_{R_n} |S^s(P_{m-k}u)|^2 dxdt\Big)$$
$$\leqq (m+1)C_s\Big(\int_0^{t'}\int_{R_n} |S^s(Pu)|^2 dxdt+\sum_{k=1}^{m}\int_0^{t'}\int_{R_n} A_0^{m-k}(v)dxdt\Big).$$

Combining this estimate with (10) yields with a constant $C_3=2m(m+1)C_s$

$$\sum_{k=1}^{m}\int_{S_{t'}} A_0^{m-k}(v)dx \leqq C_3\Big(\sum_{k=1}^{m}\int_{S_0} A_0^{m-k}(v)dx+\int_0^{t'}\int_{R_n}|S^s(Pu)|^2dxdt$$
$$+\sum_{k=1}^{n}\int_0^{t'}\int_{R_n} A_0^{m-k}(v)dxdt\Big).$$

If we put $r(t')=\sum_{k=1}^{m}\int_{S_{t'}} A_0^{m-k'}(v)dx$ and $\rho(t')=\sum_{k=1}^{m}\int_{S_0} A_0^{m-k}(v)dx+\int_0^{t'}\int_{R_n}|S^s(Pu)|^2dxdt$, then Lemma 3 shows that our proposition holds.

In the previous paper [11] we established an energy inequality of the form

$$\int_{R_n}|u(t', x)|^2dx \leqq C_T\Big(\sum_{|\nu|\leqq m-1}\int_{R_n}|(D^\nu u)(0, x)|^2dx$$
$$+\int_0^{t'}\int_{R_n}|(Pu)(t, x)|^2dxdt\Big),\ 0\leqq t'\leqq T, \quad u\in C_0^\infty(R_{n+1})$$

for a hyperbolic differential polynomial P with constant coefficients, where C_T denotes a constant independent of u. Now, if we take into account the degree of degeneracy of P, we can derive a more precise estimate. The following is a modification of L. Gårding's lemma [1, p. 76].

PROPOSITION 2. *Let the degree of degeneracy of $P_m \leqq d$. Then there exists a constant $C>0$ independent of u such that*

$$\int_{S_{t'}} A_0^{m-k}(u)dx \geqq C\sum_{|\nu|=m-k}\int_{S_{t'}}|D^\nu u(t', x)|^2dx, \qquad k=d+1, d+2,\dots, m$$

for any $u\in C_0^\infty(R_{n+1})$.

PROOF. Since the degree of degeneracy of $P_m \leqq d$, the polynomials $P_m^{(k-1)}(\tau, \xi)$ in τ, $k=d+1, d+2, .., m$, have simple real zeros for any $\xi\in\Xi_n-\{0\}$, that is, there exists a constant $\delta>0$ such that

$$\text{(11)} \qquad \inf_{|\xi|=1}|\lambda_h^{m-k+1}(\xi)-\lambda_j^{m-k+1}(\xi)|\geqq\delta,\ h\neq j,$$

where $\lambda_j^{m-k+1}(\xi)$, $j=1, 2,\dots, m-k+1$, are roots of the polynomial $P_m^{(k-1)}(\xi)$. Hence, as L. Gårding showed in his paper [1, p. 76]

$$\text{(12)} \qquad D_t^{\nu_0}\hat{u}(t,\xi)=\sum_{j=1}^{m-k+1}\beta_{j,k}^{\nu_0}(\xi)P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi),\ \nu_0=0, 1, \dots, m-k$$

for each $\xi\in\Xi_n-\{0\}$. Here we have written

$$\beta_{j,k}^{\nu_0}(\xi)=|\xi|^{-m+k+\nu_0}(\lambda_j^{m-k+1}(\omega))^{\nu_0}/m(m-1)\dots(m-k+2)\prod_{h\neq j}^{m-k+1}(\lambda_h^{m-k+1}(\omega)$$
$$-\lambda_j^{m-k+1}(\omega)),$$

where $\omega=\xi/|\xi|$. This equality together with (11) yields a constant $C(\delta)>0$ such that

$$|\beta_{j,k}^{\nu_0}(\xi)|\leqq C(\delta)|\xi|^{-m+k+\nu_0}, \qquad j=1, 2,\dots, m-k+1.$$

Consequently, in view of Cauchy-Schwarz's inequality and (12) we have

$$|D_t^{\nu_0}u(t, \xi)|^2\leqq \sum_{j=1}^{m-k+1}|\beta_{j,k}^{\nu_0}(\xi)|^2\sum_{j=1}^{m-k+1}|P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)|^2$$

$$\leqq C(\delta)|\xi|^{2(-m+k+\nu_0)}\sum_{j=1}^{m-k+1}|P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)|^2,$$

and therefore

$$K^{m-k}(\hat{u})\geqq C\sum_{\nu_0=0}^{m-k}|\xi|^{2(m-k-\nu_0)}|D_t^{\nu_0}\hat{u}(t, \xi)|^2$$

with a constant $C=1/C(\delta)$ independent of u. Then, owing to Parseval's formula, we can write

$$\int_{S_{t'}}A_0^{m-k}(u)dx\geqq C\sum_{|\nu|=m-k}\int_{S_{t'}}|(D^\nu u)(t', x)|dx,$$

which was to be proved.

Summing up Propositions 1 and 2, we can state the following.

THEOREM 1. *Let P be a hyperbolic differential polynomial with variable coefficients $\in\mathscr{B}(R_{n+1})$ having constant leading coefficients and the degree of degeneracy of $P\leqq d$. Then there exists a constant C_T independent of u such that*

$$[E_{(s,d)}]\quad \sum_{j=0}^{m-1-d}\|D_t^j u(t', \cdot)\|_{(s+m-1-d-j)}^2\leqq C_T\Big(\sum_{j=0}^{m-1}\|D_t^j u(0, \cdot)\|_{(s+m-1-j)}^2$$

$$+\int_0^{t'}\|(Pu)(t, \cdot)\|_{(s)}^2dt\Big), \qquad 0\leqq t'\leqq T,\ u\in C_0^\infty(R_{n+1}).$$

Here we note that $[E_{(s,d)}]$ gives a more precise estimate than the one obtained in [11, p. 449], as clearly this is nothing less than $[E_{(s,m-1)}]$.

3. The Cauchy problem for hyperbolic differential equations

Let us consider the Cauchy problem for P in $R_{n+1}^+=\{(t, x)\in R\times R_n:\ t>0\}$ with initial hyperplane $t=0$. Here the Cauchy problem is understood in the sense of M. Itano [3]: to find a solution $\in\mathscr{D}'(R_{n+1}^+)$ such that

(13) $$Pu=f \quad \text{in } R_{n+1}^+$$

$$\lim_{t\downarrow 0}(u, D_t u, \dots, D_t^{m-1}u)=\vec{\alpha}$$

for preassingned $f\in\mathscr{D}'(R_{n+1}^+)$ and $\vec{\alpha}=(\alpha_0, \alpha_1, \dots, \alpha_{m-1})\in\mathscr{D}'(R_n)\times\mathscr{D}'(R_n)\times\cdots\times\mathscr{D}'(R_n)$, where $\lim_{t\downarrow 0} u$ denotes the distributional boundary value of $u\in\mathscr{D}'(R_{n+1}^+)$ which is defined in [3] according to S. Łojasiewicz [7] as follows: Let $\phi\in C_0^\infty(R_t^+)$ be arbitrarily chosen so that $\phi(t)\geqq 0$ and $\int\phi(t)dt=1$ and let $\phi_\varepsilon(t)=\frac{1}{\varepsilon}\phi\left(\frac{t}{\varepsilon}\right)$, $\varepsilon>0$. If $\lim_{\varepsilon\downarrow 0}\phi_\varepsilon u$ exists in $\mathscr{D}'(R_{n+1})$, then it must be of the form $\delta_t\otimes\alpha$. α is defined to be $\lim_{t\downarrow 0} u$. An important notion "canonical extension" was introduced by M. Itano. Let $\rho(t)=\int_0^t\phi(t')dt'$ and $\rho_{(\varepsilon)}=\rho\left(\frac{t}{\varepsilon}\right)$. If, for $u\in\mathscr{D}'(R_{n+1}^+)$, $\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}u$ exists in $\mathscr{D}'(R_{n+1})$, the limit is called the canonical extension of u over $t=0$ and denoted by $u_\sim$ [3, p. 12]. To fix the idea, let us recall some results obtained there. If there exists a solution $u\in\mathscr{D}'(R_{n+1}^+)$ of (13), then u and f must have the canonical extensions $u_\sim, f_\sim$ and the Cauchy problem (13) is rewritten with $v=u_\sim$ in the form:

$$Pv=f_\sim+\sum_{k=0}^{m-1}D_t^k\delta_t\otimes\gamma_k(0), \tag{14}$$

where

$$\gamma_k(t)=-i\sum_{\nu_0=k+1}^{m}\sum_{j=1}^{\nu_0-k}(-1)^{\nu_0-j-k}\binom{\nu_0-j}{k}D_t^{\nu_0-j-k}a_{\nu_0}(t,x, D_x)\alpha_{j-1}.$$

Here $a_{\nu_0}(t, x, D_x)$ abbreviates $\sum_{|\nu'|\leqq m-\nu_0}a_\nu(t, x)D_x^{\nu'}$ for $\nu_0<m$ and $a_m=1$. Conversely, any solution $v\in\mathring{\mathscr{D}}'(\bar{R}_{n+1}^+)$ of the equation (14) is the canonical extension of a solution of (13).

According to L. Hörmander [2, Chap. 2] we shall use the notations $\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$, $\mathring{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ and the like. Let us denote by $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ the space of distributions $u\in\mathscr{D}'(R_{n+1}^+)$ such that ϕu belongs to $\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ for all $\phi\in C_0^\infty(R)$, and let $\tilde{\mathscr{H}}^*_{(\sigma,s)}(\bar{R}_{n+1}^+)$ be the adjoint space of $\tilde{\mathscr{H}}_{(-\sigma,-s)}(\bar{R}_{n+1}^+)$ with respect to an extension of the sesquilinearform from $\int_0^\infty\int_{R_n}u\bar{v}dxdt$, $u\in C_0^\infty(\bar{R}_{n+1}^+)$, $v\in C_0^\infty(R_{n+1}^+)$. The scalar product between them will be denoted by $(\cdot|\cdot)$. It is to be noticed that the space $\tilde{\mathscr{H}}^*_{(\sigma,s)}(\bar{R}_{n+1}^+)$ consists of all the elements $u\in\mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ with support $\subset[0, T]\times R_n$ for some $T>0$, which may depend on u. As for these spaces, we note that

i) If $\sigma>\frac{1}{2}$, $u\in\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ is considered as a continuous function of $t\in[0, \infty)$ with values in $\mathscr{H}_{(s+\sigma-1/2)}(R_n)$ [4, Proposition 4, p. 410].

ii) If $\sigma>-\frac{1}{2}$, the canonical extension $u_\sim$ does exist for any $u\in\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ [4, Proposition 5, p. 413].

iii) $\mathscr{H}_{(\sigma,s)}(\bar{R}^+_{n+1})$ and $\mathring{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ are identified for $|\sigma|<\frac{1}{2}$, and in this case the canonical extension $u_\sim$ belongs to $\mathring{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ for any $u\in\mathscr{H}_{(\sigma,s)}(\bar{R}^+_{n+1})$ [4, Proposition 7, p. 146].

iv) If k is a positive integer, then $\lim_{t\downarrow 0} u=\lim_{t\downarrow 0} D_t u=\cdots=\lim_{t\downarrow 0} D_t^{k-1}u=0$ for any $u\in\mathring{\mathscr{H}}_{(k,s)}(\bar{R}^+_{n+1})$ [4, Corollary 3, p. 419].
Similary for $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ and $\mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}^+_{n+1})$

From now on we shall write $\mathbf{H}_{(s)}=\mathscr{H}_{(s+m-1)}(R_n)\times\mathscr{H}_{(s+m-2)}(R_n)\times\cdots\times\mathscr{H}_{(s)}(R_n)$ and $\mathbf{H}^{\#}_{(s)}=\mathscr{H}_{(s)}(R_n)\times\mathscr{H}_{(s+1)}(R_n)\times\cdots\times\mathscr{H}_{(s+m-1)}(R_n)$. We shall continue to denote by P a hyperbolic differential polynomial with variable coefficients $\in\mathscr{B}(R_{n+1})$ having constant leading coefficients and assume that the degree of degeneracy of $P\leq d$.

First we show the following.

PROPOSITION 3. *For any given $g\in\tilde{\mathscr{H}}^*_{(0,s)}(\bar{R}^+_{n+1})$ there exists a solution $\in\tilde{\mathscr{H}}^*_{(0,-s+m-d-1)}(\bar{R}^+_{n+1})$ such that $P^*v=g$ in R^+_{n+1}.*

PROOF. Let $s'=s-m+d+1$. Consider a subspace $A\subset\tilde{\mathscr{H}}_{(0,s')}(\bar{R}^+_{n+1})\times\mathbf{H}_{(s')}$ consisting of $(P\phi,\vec{\phi}_0)$, $\phi\in C_0^\infty(\bar{R}^+_{n+1})$, where $\vec{\phi}_0=(\phi(0,x), D_t\phi(0,x),\dots, D_t^{m-1}\phi(0,x))$. Let l be the linear form $A\ni(P\phi,\vec{\phi}_0)\to(\phi|g)$, where $(\phi|g)$ denotes the scalar product between $\tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})$ and $\tilde{\mathscr{H}}^*_{(0,-s)}(\bar{R}^+_{n+1})$. It follows from Theorem 1 that the energy inequality $[E_{(s',d)}]$ holds for P. Hence we see that the map $(P\phi, \vec{\phi}_0)\to\phi$ is continuous from A into $\tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})$, which means that l is continuous. Consequently, owing to Hahn-Banach theorem, l can be extended to a continuous linear form on $\tilde{\mathscr{H}}_{(0,s')}(R_{n+1})\times\mathbf{H}_{(s')}$, and therefore there exist a $v\in\tilde{\mathscr{H}}^*_{(0,-s)}(\bar{R}^+_{n+1})$ and a $\vec{\beta}\in\mathbf{H}^{\#}_{(-s-d)}$ such that

$$(\phi|g)=(P\phi|v)+(\vec{\phi}_0|\vec{\beta}).$$

Letting $\phi\in C_0^\infty(R^+_{n+1})$, we have $\vec{\phi}_0=0$. Hence,

$$(\phi|g)=(P\phi|v)=(\phi|P^*v),$$

which implies that $P^*v=g$ in R^+_{n+1}. This ends the proof.

Let us write $\tilde{\mathscr{H}}_{(-\infty)}(\bar{R}^+_{n+1})=\bigcup_{\sigma,s}\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$. Now we can show the following

PROPOSITION 4. *If $u\in\tilde{\mathscr{H}}_{(-\infty)}(\bar{R}^+_{n+1})$ satisfies $Pu=0$ in R^+_{n+1} with initial data $\lim_{t\downarrow 0}(u, D_tu,\dots, D_t^{m-1}u)=0$, then u must vanish.*

PROOF. From our assumption it follows that $u\in\tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})$ for an s [2, Theorem 4.3.1, p. 107]. Consider an arbitrary $g\in C_0^\infty(R^+_{n+1})$. Owing to Proposition 3 there exists a solution $v\in\tilde{\mathscr{H}}^*_{(0,-s+m)}(\bar{R}^+_{n+1})$ of the equation $P^*v=g$. In a similar way we see that $v \in \tilde{\mathscr{H}}_{(m,-s)}(\bar{R}^+_{n+1})$. Now, $C_0^\infty(\bar{R}^+_{n+1})$ is dense in $\tilde{\mathscr{H}}_{(m,-s)}(\bar{R}^+_{n+1})$, so we can choose a sequence $\{v_j\}$, $v_j \in C_0^\infty(\bar{R}^+_{n+1})$, converging in $\tilde{\mathscr{H}}_{(m,-s)}(\bar{R}^+_{n+1})$ to v. Consequently,

$$(u|g)=(u|P^*v)=\lim_{j\to\infty}(u|P^*v_j)=\lim_{j\to\infty}(Pu|v_j)=0,$$

which means that $u\equiv 0$. Thus the proof is complete.

Let P be a hyperbolic differential polynomial as before. Then the formal adjoint operator P^* is also hyperbolic and has the same degree of degeneracy as P, which will be shown in Proposition 6 below. We shall use the notation $\mathscr{E}_t^0(\mathscr{H}_{(s)})$ to denote the space of continuous functions in t with values in $\mathscr{H}_{(s)}(R_n)$.

PROPOSITION 5. *For any given $f \in \tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})$ and $\vec{\alpha} \in \mathbf{H}_{(s)}$ there exists one and only one solution $u \in \tilde{\mathscr{H}}_{(-\infty)}(\bar{R}^+_{n+1})$ to the Cauchy problem* (13). *In addition, we have*

$$D_t^j u \in \mathscr{E}_t^0(\mathscr{H}_{(s+m-d-1-j)}), \qquad j=0, 1,\dots, m-d-1.$$

Furthermore, if the inequalities

$$\Big|\sum_{|\nu|\leq m}(D_t^j a_\nu(t, x)\xi^{\nu'}\tau^{\nu^0}\Big|\leq C(1+|\xi|^2)^{1/2}(1+\tau^2+|\xi|^2)^{(m-d-1)/2}, \qquad j=1, 2,\dots, k,$$

hold with a constant C for a positive integer k, and if $f \in \tilde{\mathscr{H}}_{(k,s)}(\bar{R}^+_{n+1})$ and $\vec{\alpha} \in \mathbf{H}_{(s+k)}$, then the solution u must satisfy

$$D_t^j u \in \mathscr{E}_t^0(\mathscr{H}_{(s+k+m-d-1-j)}), \qquad j=0, 1,\dots, k+m-d-1.$$

PROOF. First we shall show that the set

$$G=\{(P\phi, \vec{\phi}_0):\ \phi \in C_0^\infty(\bar{R}^+_{n+1})\}$$

is everywhere dense in $\tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})\times \mathbf{H}_{(s)}$, where $\vec{\phi}_0=(\phi(0, x),\ D_t\phi(0, x),\dots, D_t^{m-1}\phi(0, x))$. For this end we let $(v, \vec{\beta}) \in \tilde{\mathscr{H}}^*_{(0,-s)}(\bar{R}^+_{n+1})\times \mathbf{H}^\#_{(-s-m+1)}$ be such that

$$(P\phi|v)+(\vec{\phi}_0|\vec{\beta})=0 \qquad \text{for any } (P\phi, \vec{\phi}_0)\in G.$$

Since $(P\phi|v)=0$ for any $\phi \in C_0^\infty(R^+_{n+1})$, we have $(\phi|P^*v)=0$, which means that

$$P^*v=0 \qquad \text{in } R^+_{n+1}. \tag{15}$$

On the other hand, from $v \in \tilde{\mathscr{H}}^*_{(0,-s)}(\bar{R}^+_{n+1})$ there exists a $T>0$ such that $v=0$ for $t \geq \frac{T}{2}$. Consequently,

$$\lim_{t\uparrow T}(v, D_t v\dots, D_t^{m-1}v)=0. \tag{16}$$

As noted before, P^* is hyperbolic and therefore by the same method as in Proposition 4 the Cauchy problem (15) with (16) is uniquely solvable. Hence we have $v=0$. Since the set $\{\vec{\phi}_0: \phi\in C_0^\infty(\bar{R}_{n+1}^+)\}$ is everywhere dense it follows from $(\vec{\phi}_0|\vec{\beta})=0$ that $\vec{\beta}=0$. Now we can choose a sequence $\{\phi_h\}$, $\phi_h\in C_0^\infty(\bar{R}_{n+1}^+)$, such that

$$P\phi_h\to f \qquad \text{in } \tilde{\mathscr{H}}_{(0,s)}(\bar{R}_{n+1}^+),$$

and

$$(\phi_h(0, x), D_t\phi_h(0, x),\dots, D_t^{m-1}\phi_h(0, x))\to\vec{\alpha} \qquad \text{in } \mathbf{H}_{(s)}.$$

In view of the estimate $[E_{(s,d)}]$ each $\{D_t^j\phi_h\}_{h=1,2,\dots}$, $j=0, 1,\dots, m-d-1$, is a Cauchy sequence in $\mathscr{E}_t^0(\mathscr{H}_{(s+m-d-1-j)})$ and therefore converges there to some v_j. Since v_0 satisfies the Cauchy problem (13), it follows then from the uniqueness of the solution that v_0 must coincide with u, and it is clear that $D_t^ju=v_j$, $j=0, 1,\dots, m-d-1$, which means that

$$D_t^ju\in\mathscr{E}_t^0(\mathscr{H}_{(s+m-d-1-j)}), \qquad j=0, 1,\dots, m-d-1.$$

Thus the first assertion of Proposition 5 holds. Next we shall show the second assertion in the case $k=1$. Since $f\in\tilde{\mathscr{H}}_{(1,s)}(\bar{R}_{n+1}^+)\subset\tilde{\mathscr{H}}_{(0,s+1)}(\bar{R}_{n+1}^+)$, it follows that

$$D_t^ju\in\mathscr{E}_t^0(\mathscr{H}_{(s+m-d-j)}), \qquad j=0, 1,\dots, m-d-1. \tag{17}$$

Put $v=D_tu$. v must satisfy the equation

$$Pv=D_tf-\sum_{|\nu|\leq m}(D_ta_\nu(t, x))D^\nu u \qquad \text{in } R_{n+1}^+.$$

On the other hand, from the relation

$$D_t^mu=f-\sum_{\nu_0=0}^{m-1}\sum_{|\nu|\leq m}a_\nu D^\nu u \quad \text{and} \quad \lim_{t\downarrow 0}f\in\mathscr{H}_{(s+1/2)}(\bar{R}_{n+1}^+)$$

it follows that

$$\lim_{t\downarrow 0}(v, D_tv,\dots, D_t^{m-1}v)\in\mathbf{H}_{(s)}.$$

Consequently, owing to the first part of Proposition 5 we obtain

$$D_t^jv\in\mathscr{E}_t^0(\mathscr{H}_{(s+m-d-1-j)}), \qquad j=0, 1,\dots, m-d-1.$$

Combining this with (17) yields

$$D_t^ju\in\mathscr{E}_t^0(\mathscr{H}_{(s+m-d-1-j)}), \qquad j=0, 1,\dots, m-d,$$

as desired. In the general case, repeating this procedure k times, we shall reach the second assertion of Proposition 5. The proof is complete.

Let $(D_t - i\lambda(D_x))^{-1}$ be a convolution operator associated with symbol $(\tau - i\lambda(\xi))^{-1}$, where $\lambda(\xi) = (1+|\xi|^2)^{1/2}$. It defines an isomorphism between $\mathring{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ and $\mathring{\mathscr{H}}_{(\sigma+1,s)}(\bar{R}^+_{n+1})$ [2, p. 53]. In an obvious fashion we can extend it to an isomorphism between $\mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ and $\mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s)}(\bar{R}^+_{n+1})$.

The following Theorem 2 is a refinement of Proposition 5 and a generalization of a result in the previous paper [11, Theorem 2.1, p. 453].

THEOREM 2. *Let σ be a real number such that $\sigma = \sigma' + k$, where k is a non-negative integer and $-\frac{1}{2} < \sigma' \leqq \frac{1}{2}$. Suppose that the inequalities*

$$|\sum_{|\nu| \leqq m} (D_t^j a_\nu(t,x) \xi^{\nu'} \tau^{\nu_0}| \leqq C(1+|\xi|^2)^{1/2}(1+\tau^2+|\xi|^2)^{(m-d-1)/2},$$
$$j = 1, 2, \dots, k+d+1,$$

hold with a constant C. Then, for any given $f \in \tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ and $\vec{\alpha} \in \mathbf{H}_{(s+\sigma)}$ there exists one and only one solution $u \in \tilde{\mathscr{H}}_{(-\infty)}(\bar{R}^+_{n+1})$. In addition, u must satisfy

$$D_t^j u \in \mathscr{E}_t^0(\mathscr{H}_{(s+\sigma+m-d-1-j)}), \qquad j = 0, 1, \dots, k+m-1.$$

PROOF. First we consider the case $k=0$ and show that

$$D_t^j u \in \mathscr{E}_t^0(\mathscr{H}_{(\sigma+s+m-d-1-j)}), \qquad j = 0, 1, \dots, m-1.$$

Let us take an $\varepsilon > 0$ such that $\sigma' - \frac{1}{2} < \varepsilon < \sigma' + \frac{1}{2}$ and put $\sigma'' = \sigma' - \varepsilon$. From $f \in \tilde{\mathscr{H}}_{(\sigma',s)}(\bar{R}^+_{n+1}) \subset \tilde{\mathscr{H}}_{(\sigma'',s+\varepsilon)}(\bar{R}^+_{n+1})$ it follows that $f_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma'',s+\varepsilon)}(\bar{R}^+_{n+1})$. Now we can write $f_\sim$ in the form

$$f_\sim = D_t^{d+1} f_0 + D_t^d f_1 + \cdots + f_{d+1},$$

where we have defined for $j = 0, 1, \dots, d+1$

$$f_j = \binom{d+1}{j}(-i\lambda(D_x))^j (D_t - i\lambda(D_x))^{-(d+1)} f_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma''+d+1, s+\varepsilon-j)}(\bar{R}^+_{n+1}).$$

Consider the Cauchy problems

$$\begin{cases} Pv_j = f_j & \text{in } R^+_{n+1}, \\ \lim\limits_{t \downarrow 0}(v_j, D_t v_j, \dots, D_t^{m-1} v_j) = 0, & j = 0, 1, \dots, d, \end{cases}$$

and

$$\begin{cases} Pv_{d+1} = f_{d+1} & \text{in } R^+_{n+1}, \\ \lim\limits_{t \downarrow 0}(v_{d+1}, D_t v_{d+1}, \dots, D_t^{m-1} v_{d+1}) = \vec{\alpha}. \end{cases}$$

From Proposition 5, we obtain for $j=0, 1,\dots, d+1$

$$D_t^h v_j \in \mathscr{E}_t^0(\mathscr{H}_{(s+\sigma+m-j-h)}), \qquad h=0, 1,\dots, m-1.$$

Since $f_j \in \mathring{\mathscr{H}}_{(\sigma''+d+1, s+\varepsilon-j)}(\bar{R}_{n+1}^+)$ and $\sigma''+\frac{1}{2}>1$, we have

$$D_t^h f_j \in \mathscr{E}_t^0(\mathscr{H}_{(s+\sigma-1/2-j-h)}), \tag{18}$$

and

$$\lim_{t\downarrow 0} D_t^h f_j = 0 \tag{19}$$

for $h=0, 1,\dots, d$ and $j=0, 1,\dots, d+1$. From (18) and the relations $D_t^m v_j = f_j - \sum_{\nu_0=0}^{m-1} \sum_{|\nu|\leqq m} a_\nu(t, x) D^\nu v_j$,

$$D_t^h v_j \in \mathscr{E}_t^0(\mathscr{H}_{(s+\sigma+m-j-h)}), \qquad h=0, 1,\dots, m+d,$$

for $j=0, 1,\dots, d+1$. Put $u=D_t^{d+1}v_0+D_t^d v_1+\cdots+v_{d+1}+w$. Then,

$$Pw=g \qquad \text{in } R_{n+1}^+, \tag{20}$$

where

$$g=-\sum_{j=0}^{d}\sum_{h=1}^{d+1-j}\binom{d+1-j}{h}\sum_{|\nu|\leqq m}(D_t^h a_\nu(t, x))\, D^\nu D_t^{d+1-j-h} v_j \in \tilde{\mathscr{H}}_{(d+1, s+\sigma-d)}(\bar{R}_{n+1}^+).$$

By virtue of (19) and the relations $D_t^m v_j = f_j - \sum_{\nu_0=0}^{m-1}\sum_{|\nu|\leqq m} a_\nu(t, x)\, D^\nu v_j$, we have

$$\lim_{t\downarrow 0}(v_j, D_t v_j,\dots, D_t^{m+d+1-j}v_j)=0, \qquad j=0, 1,\dots, d.$$

Consequently, from the relation $u=D_t^{d+1}v_0+D_t^d v_1+\cdots+v_0+w$, we see that

$$\lim_{t\downarrow 0}(w, D_t w,\dots, D_t^{m-1}w)=0. \tag{21}$$

Observing that w is the solution to the Cauchy problem (20) with (21), we obtain from Proposition 5

$$D_t^j w \in \mathscr{E}_t^0(\mathscr{H}_{(s+\sigma+m-d-j)}), \qquad j=0, 1,\dots, m.$$

Since $u=D_t^{d+1}v_0+D_t^d v_1+\cdots+v_{d+1}+w$, it follows that

$$D_t^j u \in \mathscr{E}_t^0(\mathscr{H}_{(s+\sigma+m-d-1-j)}), \qquad j=0, 1,\dots, m-1.$$

Next we shall show the general case $\sigma=\sigma'+k$ by induction on k. As the case $k=0$ has been shown, we may assume that $k>0$. Suppose that the assertion of Theorem 2 is true for $k-1$. Since $f\in\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+) \subset \tilde{\mathscr{H}}_{(\sigma-1,s+1)}(\bar{R}_{n+1}^+)$, it follows from the assumption of our induction that

$$D_t^j u \in \mathscr{E}_t^0(\mathscr{H}_{(\sigma+s+m-d-1-j)}), \qquad j=0, 1,\dots, k+m-2. \tag{22}$$

Applying D_t to the both sides of the equation $Pu=f$, we have

$$P(D_t u) = D_t f - \sum_{|\nu|\leqq m} (D_t a_\nu(t, x)) D^\nu u.$$

From $\lim_{t\downarrow 0} f \in \mathscr{H}_{(\sigma+s-1/2)}(R_n)$ and the relation $D_t^m u = f - \sum_{\nu_0=0}^{m-1} \sum_{|\nu|\leqq m} a_\nu D^\nu u$, we obtain

$$\lim_{t\downarrow 0}(D_t u, D_t^2 u,\dots, D_t^m u) \in \mathbf{H}_{(s+\sigma-1)}.$$

Since $D_t f - \sum_{|\nu|\leqq m} (D_t a_\nu(t, x)) D^\nu u \in \tilde{\mathscr{H}}_{(\sigma-1,s)}(\bar{R}_{n+1}^+)$, it follows from the assumption of our induction that

$$D_t^j(D_t u) \in \mathscr{E}_t^0(\mathscr{H}_{(s+\sigma-d-j)}), \qquad j=0, 1,\dots, k+m-2. \tag{23}$$

The relations (22) and (23) show that the assertion of Theorem 2 holds. This competes the proof.

Hereafter in this section we shall assume that P is a hyperbolic differential polynomial with constant coefficients. Consider the convex cone $\Gamma^*(P, N)$ which was introduced by L. Hörmander [2, p. 137], $N=(1, 0,\dots, 0) \in \Xi_{n+1}$.

Owing to his results obtained there [2, Corollary 5.3.3, p. 130] we see that if (t_0, x_0) is an arbitrary point of R_{n+1}^+, and if $\in \mathscr{D}'(R_{n+1}^+)$ satisfies the conditions:

$$Pu=0 \text{ in the interior of } ((t_0, x_0)-\Gamma^*(P, N)) \cap R_{n+1}^+,$$

and

$$\lim_{t\downarrow 0}(u, D_t u,\dots, D_t^{m-1} u)=0 \text{ on } ((t_0, x_0)-\Gamma^*(P, N)) \cap \{t=0\},$$

then u vanishes in the interior of $((t_0, x_0)-\Gamma^*(P, N)) \cap R_{n+1}^+$. Now the assumption on the coefficients $a_\nu(t, x)$ stated in Theorem 2 clearly holds.

With these and the partition of unity in mind, we can rewrite Theorem 2 as the following

Corollary 1. *Let P be a hyperbolic differential polynomial with constant coefficients having the degree of degeneracy $\leqq d$. Let $\sigma=\sigma'+k$ be chosen as in Theorem 2. For any given $f \in \mathscr{H}_{(\sigma,s)}^{\mathrm{loc}}(\bar{R}_{n+1}^+)$ and $\vec{\alpha} \in \mathbf{H}_{(\sigma+s)}^{\mathrm{loc}}$ there exists one and only one solution $u \in \mathscr{H}_{(-\infty)}^{\mathrm{loc}}(\bar{R}_{n+1}^+) = \bigcup_{\sigma,s} \mathscr{H}_{(\sigma,s)}^{\mathrm{loc}}(\bar{R}_{n+1}^+)$ to the Cauchy problem* (13). *In addition, u must satisfy*

$$D_t^j u \in \mathscr{E}_t^0(\mathscr{H}_{(\sigma+s+m-d-1-j)}^{\mathrm{loc}}), \qquad j=0, 1,\dots, k+m-1.$$

For non-negative integers k, j, let us denote by $C^{k,j}(R_{n+1})$ the space of functions u which are continuous with their partial derivatives $D^{\nu}u$, $\nu_0 \leqq k$, $|\nu| \leqq k+j$.

As a refinement of Corollary 2.1 [11, p. 455], we can state the following

COROLLARY 2. *Let $P(D)$ be a hyperbolic differential polynomial stated in Corollary 1. Let $r = \left[\frac{n}{2}\right]+1$. For any given $f \in C^{0,r+d+1}(\bar{R}^{+}_{n+1})$ and $\vec{\alpha} \in C^{r+d+m}(R_n) \times C^{r+d+m-1}(R_n) \times \cdots \times C^{r+d}(R_n)$, the solution u exists in $C^m(\bar{R}^{+}_{n+1})$.*

PROOF. In view of Sobolev's lemma (cf. [2, Theorem 2.3.7, p. 44]) we have

$$C^{k+r}(R_n) \subset \mathscr{H}^{\text{loc}}_{(r+k)}(R_n) \subset C^k(R_n), \qquad k=0, 1,\dots.$$

Combining this with Corollary 1, we obtain the conclusion of Corollary 2.

4. Some remarks on hyperbolic differential polynomial with multiple characteristics

This section is devoted to give some comments on the hyperbolic differential polynomial P considered as before. First let us extend G. Peyser's result [9, Theorem 1, p. 667], as stated in the introduction. Here we shall use the following result [2, Theorem 5.5.7, p. 134, also 10, Theorem 1.3, p. 151]: Let $L(D)$ be a differential polynomial with constant coefficients, and assume that the principal part L_m of L is hyperbolic. Then L is hyperbolic if and only if L is weaker than L_m, that is, there exists a constant C such that $|L(\tau, \xi)| \leqq C\tilde{L}(\tau, \xi)$.

THEOREM 3. *Let d be a non-negative integer $<m$. Then $P+Q$ is hyperbolic for an arbitrary differential polynomial Q with variable coefficients of $\mathscr{B}(R_{n+1})$ of order $<m-d$ if and only if one of the following conditions is satisfied.*

i) *The degree of degeneracy of the hyperbolic differential polynomial $P \leqq d$,*

ii) *$P_m^{(d)}(D)$ is strictly hyperbolic.*

PROOF. First we note that conditions i) and ii) are equivalent, which is immediately verified from the definitions of the strict hyperbolicity and the degree of degeneracy. Now let us prove the "if" part.

Since P is hyperbolic, the principal part P_m is also hyperbolic. We shall then show that there exists a constant C such that $|P(t, x, \tau, \xi)| < C\tilde{P}_m(\tau, \xi)$, which means that $P+Q$ is hyperbolic when each point (t, x) is fixed. Let us write with an integer N

$$P(t, x, D) = \sum_{j=1}^{N} b_j(t, x) P(t_j, x_j, D), \qquad b_j \in \mathscr{B}(R_{n+1}), \tag{24}$$

where $P(t_j, x_j, D)$ is the operator frozen at (t_j, x_j). Since $P(t_j, x_j, D)$, $j=1, 2,\dots$,

N, are hyperbolic differential polynomials with the principal part $P_m(D)$, there exists a constant C' such that

$$|P(t_j, x_j, \tau, \xi)| \leqq C' \tilde{P}_m(\tau, \xi), \qquad j=1, 2, \dots, N.$$

Taking into account the relation (24), we obtain with a constant C''

$$|P(t, x, \tau, \xi)| \leqq \Big(\sum_{j=1}^{N} |b_j(t, x)|^2\Big)^{1/2} \Big(\sum_{j=1}^{N} |P(t_j, x_j, \tau, \xi)|^2\Big)^{1/2}$$
$$\leqq C'' \tilde{P}_m(\tau, \xi).$$

On the other hand, since $P_m^{(d)}(D)$ is strictly hyperbolic, $P_m^{(d)} + Q$ is hyperbolic. By the same reasoning as above, we have

$$|P_m^{(d)}(\tau, \xi) + Q(t, x, \tau, \xi)| \leqq C''' \tilde{P}_m^{(d)}(\tau, \xi) \leqq C''' \tilde{P}_m(\tau, \xi),$$

where C''' is a constant. Consequently,

$$|P(t, x, \tau, \xi) + Q(t, x, \tau, \xi)|$$
$$\leqq |P(t, x, \tau, \xi)| + |P_m^{(d)}(\tau, \xi)| + |P_m^{(d)}(\tau, \xi) + Q(t, x, \tau, \xi)|$$
$$\leqq (1 + C'' + C''') \tilde{P}_m(\tau, \xi),$$

as desired.

We now proceed to prove the "only if" part. Let T be an arbitrary positive number. We shall then derive the following inequality with a constant C_T independent of u:

$$\sum_{j=0}^{m-d-1} \|D_t^j u(t', \cdot)\|^2_{(m-d-1-j)} \leqq C_T \Big(\sum_{j=0}^{m-d-1} \|D_t^j u(0, \cdot)\|^2_{(m-d-1-j)} +$$
$$+ \int_0^{t'} \|P_m^{(d)} u(t, \cdot)\|^2_{(0)} dt \Big), \; 0 \leqq t' \leqq T, \; u \in C_0^\infty(R_{n+1}),$$

which means that $P_m^{(d)}$ is strictly hyperbolic [5, p. 101]. From the relation (6) and Parseval's formula we have

$$\int_{S_{t'}} A_0^{m-k}(u) dx - \int_{S_0} A_0^{m-k}(u) dx = \int_0^{t'} \int_{R_n} -\mathrm{Im} P_m^{(k-1)} u \cdot \overline{P_m^{(k)} u} \, dx dt.$$

It follows from Lemma 1 that

$$\int_0^{t'} \int_{R_n} -\mathrm{Im} P_m^{(k-1)} u \cdot \overline{P_m^{(k)} u} \, dx dt$$
$$\leqq \Big(\int_0^{t'} \int_{R_n} |P_m^{(k-1)} u|^2 dx dt + \int_0^{t'} \int_{R_n} |P_m^{(k)} u|^2 dx dt \Big)$$

$$\leqq m\Big(\int_0^{t'}\int_{R_n}|P_m^{(k-1)}u|^2dxdt+\int_0^{t'}\int_{R_n}A_0^{m-k}(u)dxdt\Big).$$

Consequently, we obtain

$$\sum_{k=d+1}^m\int_{S_t}A_0^{m-k}(u)dx\leqq m\Big(\sum_{k=d+1}^m\int_{S_0}A_0^{m-k}(u)dx+\sum_{k=d+1}^m\int_0^{t'}\int_{R_n}|P_m^{(k-1)}u|^2dxdt+$$
$$+\sum_{k=d+1}^m\int_0^{t'}\int_{R_n}A_0^{m-k}(u)dxdt\Big).$$

Put $r(t')=\sum_{k=d+1}^m\int_{S_{t'}}A_0^{m-k}(u)dx$ and $\rho(t')=\sum_{k=d+1}^m\int_{S_0}A_0^{m-k}(u)dx+$ $+\sum_{k=d+1}^m\int_0^{t'}\int_{R_n}|P_m^{)k-1)}u|^2dxdt$. From Lemma 3, we have with a constant $C'_T=me^{mT}$

$$\text{(25)}\quad \sum_{k=d+1}^m\int_{S_{t'}}A_0^{m-k}(u)dx\leqq C'_T\Big(\sum_{k=d+1}^m\int_{S_0}A_0^{m-k}(u)dx+$$
$$+\sum_{k=d+1}^m\int_0^{t'}\int_{R_n}|P_m^{(k-1)}u|^2dxdt\Big),\ 0\leqq t'\leqq T.$$

Let $P+Q$ be temporarily frozen at a point (t_0, x_0) and let us denote it by $M(D)$. Since M is a hyperbolic differential polynomial with the principal part $P_m(D)$, we can write with bounded functions $b_{k,j}(\xi)$, $\xi\in\Xi_n$,

$$M_{m-k}(\tau, \xi)=\sum_{j=1}^{m-k+1}b_{k,j}(\xi)P_{m,j}^{(k-1)}(\tau, \xi),\qquad k=1, 2, \dots, m,$$

where $M_{m-k}(D)$ is a homogeneous part of $M(D)$. Owing to Cauchy-Schwarz's inequality and (7), we obtain

$$|M_{m-k}(D_t, \xi)\hat{u}(t, \xi)|^2\leqq\sum_{j=1}^{m-k+1}|b_{k,j}(\xi)|^2\sum_{j=1}^{m-k+1}|P_{m,j}^{(k-1)}(D_t, \xi)\hat{u}(t, \xi)|^2$$
$$\leqq C'K^{m-k}(\hat{u}),$$

where C' is a constant independent of u. Now we shall write with an integer N

$$P_{m-k}+Q_{m-k}=\sum_{j=1}^N c_j(t, x)(P_{m-k}(t_j, x_j, D)+Q_{m-k}(t_j, x_j, D)),\ c_j\in\mathscr{B}(R_{n+1}).$$

with an operator $P_{m-k}(t_j, x_j, D)+Q_{m-k}(t_j, x_j, D)$ frozen at (t_j, x_j). From the considerations just above, it follows that

$$\text{(26)}\qquad \int_{S_{t'}}|(P_{m-k}+Q_{m-k})u|^2dx\leqq$$

$$\leqq C'' \sum_{j=1}^{N} \int_{S_{t'}} |(P_{m-k}(t_j, x_j, D) + Q_{m-k}(t_j, x_j, D) u|^2 dx$$

$$\leqq C''' \int K(\hat{u}) d\xi = C''' \int_{S_{t'}} A_0^{m-k}(u) dx.$$

with constants C'', $C''' \geqq 1$ independent of u. Since Q is an arbitrary differential polynomial of order $< m-d$, Q can be chosen so that

$$(P_{m-k} + Q_{m-k}) u = D^{\nu} u, \qquad |\nu| = m-k. \tag{27}$$

From (26) and (27), we obtain

$$\sum_{j=0}^{m-d-1} \|D_t^j u(t, \cdot)\|_{(m-d-1-j)}^2 \leqq C''' \sum_{k=d+1}^{m} \int_{S_{t'}} A_0^{m-k}(u) dx. \tag{28}$$

In view of (25) and (28), we have with $C_T'' = C''' C_T'$

$$\sum_{j=0}^{m-d-1} \|D_t^j u(t', \cdot)\|_{(m-d-1-j)}^2 \leqq C_T'' \Big(\sum_{j=0}^{m-d-1} \|D_t^j u(0, \cdot)\|_{(m-d-1-j)}^2$$

$$+ \int_0^{t'} \|(P_m^{(d)} u)(t, \cdot)\|_{(0)}^2 dt + \sum_{j=0}^{m-d-1} \int_0^{t'} \|D_t^j u(t, \cdot)\|_{(m-d-1-j)}^2 dt \Big), \; 0 \leqq t' \leqq T.$$

Consequently, it follows from Lemma 3 that

$$\sum_{j=0}^{m-d-1} \|D_t^j u(t', \cdot)\|_{(m-d-1-j)}^2 \leqq C_T \Big(\sum_{j=0}^{m-d-1} \|D_t^j u(0, \cdot)\|_{(m-d-1-j)}^2$$

$$+ \int_0^{t'} \|(P_m^{(d)} u)(t, \cdot)\|_{(0)}^2 dt \Big), \; 0 \leqq t' \leqq T$$

witn a constant C_T independent of u, which completes the proof.

Finaly we shall show the following

PROPOSITION 6. *Let P be a hyperbolic differential polynomial as before. Then the formal adjoint operator P^* is also hyperbolic and has the same degree of degeneracy as P.*

PROOF. First, we shall write with an integer N

$$P = \sum_{j=1}^{N} c_j(t, x) P(t_j, x_j, D), \qquad c_j \in \mathscr{B}(R_{n+1}).$$

Here $P(t_j, x_j, D)$ denotes the operator frozen at (t_j, x_j). Consequently,

$$P^* = \sum_{j=1}^{N} P^*(t_j, x_j, D) \overline{c_j(t, x)}$$

$$= \sum_{j=1}^{N} \sum_{|\nu| \leqq m} (D^{\nu} \bar{c}_j / \nu!) P^{*(\nu)}(t_j, x_j, D),$$

where $P^*(t_j, x_j, D)$ stand for the formal adjoint operator of $P(t_j, x_j, D)$ and we put

$$P^{*(\nu)}(t_j, x_j, \tau, \xi)=\partial^{|\nu|}P^*(t_j, x_j, \tau, \xi)/\partial\tau^{\nu_0}\partial\xi_1^{\nu_1}\dots\partial\xi_n^{\nu_n}.$$

Owing to Cauchy-Schwartz's inequality, we have,

$$(29)\qquad |P^*(t, x, \tau, \xi)|\leqq \sum_{j=1}^{N} (\sum_{|\nu|\leqq m}|(D^\nu\bar{c}_j/\nu!)|^2)^{\frac{1}{2}}(\sum_{|\nu|\leqq m}|P^{*(\nu)}(t_j, x_j, \tau, \xi)|^2)^{\frac{1}{2}}.$$

Since $P^*(t_j, x_j, D)$ is hyperbolic and its principal part is equal to P_m, it follows that

$$(\sum_{|\nu|\leqq m}|P^{*(\nu)}(t_j, x_j, \tau, \xi)|^2)^{1/2}\leqq C'\tilde{P}_m.$$

From this inequality and (29), we obtain with a constant C

$$|P^*(t, x, \tau, \xi)|\leqq C\tilde{P}_m,$$

which implies that P^* is hyperbolic.

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