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A Remark on Certain Symmetric Stable Processes*

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1. In a series of author's works [2], [3], etc., we try to clarify the relation between the fine topology and the order of divergence of Green functions corresponding to Markov processes. The main result is as follows. There are given Markov processes X_i, $i=1, 2$ on the same "good" state space E which have Green functions $G_i(x, y)$, $i=1, 2$ respectively. If

$$G_1(x, y) \approx G_2(x, y),^{1)}$$

then the fine topologies given by X_i, $i=1, 2$ are equivalent under certain regularity conditions on X_i, $i=1, 2$. Moreover it is shown that a certain order relation of the divergence at the diagonal of $G_i(x, y)$, $i=1, 2$, induces a relation on strength of the fine topologies given by X_i, $i=1, 2$, [2].

In this note we show by examples that

$$G_1(x, y) \leqq \text{Const.}\, G_2(x, y)$$

does not always imply that the fine topology induced by X_2 is stronger than that induced by X_1[2]. In addition our examples show that the fine topologies are not equivalent and the orders of divergence of Green functions are different, even if polar sets (sets of capacity zero) corresponding to X_i, $i=1, 2$ coincide[3].

2. Let X be a symmetric (not necessarily spherically symmetric) stable process on $R^n (n \geqq 3)$. Then it is known that there exists a potential kernel $G(x, y) = g(x-y)$ (we call it Green function of X) such that

$$\int_0^\infty T_t f(x) dt = \int_{R^n} G(x, y) f(y) dy, \qquad G(x, y) > 0,$$

for each continuous function f of compact support, where $\{T_t\}$ is the semi-group of transition operators for X.

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1) It means that for any $x \in E$ there exists a neighborhood U and exist positive constants C_i, $i=1, 2$ such that $C_1 G_1(X, y) \leqq G_2(x, y) \leqq C_2 G_1(x, y)$, $(x, y) \in U \times U$.

2) This means that each X_1-finely open set is a X_2-finely open set.

3) In one dimensional case, it is easily shown that this phenomenon occurs for unsymmetric stable processes of index α, $0 < \alpha < 1$. Our aim of this note is to give examples of symmetric infinitely divisible processes in higher dimensional spaces.

PROPOSITION. *There are given symmetric stable processes X_i, $i=1, 2$ of index α, $0<\alpha<2$, on $R^n(n\geqq 3)$. X_1 is spherically symmetric and X_2 is the infinitely divisible Markov process obtained by assuming that the coordinate processes are independent symmetric stable processes of index α.* We assume that $0<\alpha\leqq\frac{n-1}{2}$ *and $n\geqq 3$. Then the following hold*[4].

i) Each compact set K which has nonzero capacity for X_1(resp. X_2) contains regular points[5] *for K with respect to X_1(resp. X_2).*

ii) Each polar set for X_1 is polar for X_2 and the converse is also valid.

iii) $G_1(x, y)\leqq CG_2(x, y)$[6] *for every x, y where $G_i(x, y)$, $i=1, 2$ are Green functions of X_i, $i=1, 2$, respectively.*

iv) Fine topologies given by X_i, $i=1, 2$ are not equivalent[7].

PROOF. The statement *i)* follows directly from the symmetry of the processes X_i, $i=1, 2$ (see, for example, Blumenthal-Getoor [1]). The statement *ii)* is a consequence of S. Orey's result (Cor. 2.1 in [4]). The statement *iii)* is proved as follows. It is known that $G_1(x, y)=C|x-y|^{\alpha-n}$. Let $\psi(x)=|x_1|^\alpha+\cdots+|x_n|^\alpha$ be the exponent of X_2 and $g(x-y)=G_2(x, y)$. Then $g(x)=g(-x)$ and

$$g(x)=(\rho_\alpha(x))^{1-\alpha/n}F^{-1}\left(\frac{1}{\psi}\right)(x'), \qquad |x'|=1\,,$$

where $\rho_\alpha(x)$ is a positive C^∞-function of $R^n-\{0\}$ uniquely defined by

$$\sum_{j=1}^{n}\frac{x_j^2}{\rho_\alpha(x)^{2/\alpha}}=1$$

and F^{-1} denotes the inverse Fourier transform in distribution sense. (Note that $g(x)=F^{-1}(1/\psi)(x)$.) Since $C_2|x|^\alpha\leqq\rho_\alpha(x)\leqq C_1|x|^\alpha$, $C_1\geqq C_2>0$ and $g(x)>0$, the statement (*iii*) follows at once. It may be worthwhile to remark that $g(x)$ is infinite identically on each coordinate axis in case $n\geqq 5$ or $n\geqq 3$ and $0<\alpha\leqq(n-1)/2$ (For the proof see [5] or [2].) For the proof of *iv*) we need the following estimate obtained by S. J. Taylor and W. E. Pruit [5]. Let Q be the sphere of radius r centered at the point $(d+r, 0,\ldots, 0)$ and $d\geqq 2r$. Then

$$P_0^2(\sigma_Q<+\infty)^{8)}\geqq C\left(\frac{r}{d}\right)^{1+\alpha} \quad \text{if } \alpha<\frac{n-1}{2} \tag{2.1}$$

4) The statements *i*), *ii*) are direct consequences of the known results (see the proof).

5) A point x is called a regular point for K with respect to X, if $P_x(\sigma_K=0)=1$, where $\sigma_K=\inf(t>0, x_t\in K)$.

6) In the following we denote various, absolute positive constants by C.

7) In the proof we construct an open set K and a point x such that x is a regular point for K with respect to X_2 but not regular with respect to X_1.

8) P_x^i, $i=1, 2$, denotes the probability concerning paths of X_i, $i=1, 2$ starting from the point x respectively.

and

$$P_0^2(\sigma_Q<+\infty)\geqq C\left(\frac{r}{d}\right)^{1+\alpha}\log\left(\frac{d}{r}\right) \text{ if } \alpha=\frac{n-1}{2}. \tag{2.2}$$

Let Q' be a sphere of radius r and x is at a distance d from Q' with $r\leqq d$. Then

$$P_x^2(\sigma_{Q'}<\infty)\leqq C\left(\frac{r}{d}\right)^{1+\alpha}, \qquad \text{if } \alpha<\frac{n-1}{2}, \tag{2.3}$$

and

$$P_x^2(\sigma_{Q'}<\infty)\leqq C\left(\frac{r}{d}\right)^{n-\alpha}\left(1+\log\frac{d}{r}\right), \qquad \text{if } \alpha=\frac{n-1}{2}. \tag{2.4}$$

We prove *iv)* in case $n=3$ or 4 and $\alpha=(n-1)/2$. Let us choose a sequence of open spheres Q_k of radius r_k centered at the point $(d_k, 0,\dots, 0)$. Further we choose $\{r_k\}$ and $\{d_k\}$ such that

$$d_{k+1}=\frac{1}{8}d_k, \qquad r_k=\left(\frac{1}{k(\log k)^\gamma}\right)^{1/(1+\alpha)}d_k, \qquad 2>\gamma>1. \tag{2.5}$$

Then, using (2.4) and $n=2\alpha+1$, we have for $m>l$ (l is sufficiently large),

$$\sup_{x\in Q_m} P_x^2(\sigma_{Q_l}<+\infty)\leqq C\left(\frac{r_l}{d_l}\right)^{1+\alpha}\left(\frac{d_l}{d_l-d_m-(r_m+r_l)}\right)^{1+\alpha}\log\frac{d_l}{r_l}.$$

Since it holds by (2.5) that

$$\left(\frac{d_l}{d_l-d_m-(r_m+r_l)}\right)^{1+\alpha}\leqq\left(\frac{d_l}{7/8d_l-2r_l}\right)^{1+\alpha}\leqq C,$$

we have

$$\sup_{x\in Q_m} P_x^2(\sigma_{Q_l}<+\infty)\leqq C\cdot\left(\left(\frac{r_l}{d_l}\right)^{1+\alpha}\log\frac{d_l}{r_l}\right). \tag{2.6}$$

In the same way we have

$$\sup_{x\in Q_l} P_x^2(\sigma_{Q_m}<+\infty)\leqq C\cdot\left(\frac{r_m}{d_l}\right)^{1+\alpha}\log\frac{d_l}{r_m}.$$

Hence, using the strong Markov property, we get

$$\begin{aligned}P_0^2(\sigma_{Q_l}<+\infty,\ \sigma_{Q_m}<+\infty)\leqq{}& C\cdot\left(\frac{r_l}{d_l}\right)^{1+\alpha}\log\frac{d_l}{r_l}P_0^2(\sigma_{Q_m}<+\infty)\\ &+C\cdot\left(\frac{r_m}{d_l}\right)^{1+\alpha}\log\frac{d_l}{r_m}P_0^2(\sigma_{Q_l}<+\infty)=I_1+I_2\end{aligned}$$

It follows easily from (2.2) that

$$I_1 \leqq C \cdot P_0^2(\sigma_{Q_l} < +\infty)\, P_0^2(\sigma_{Q_m} < +\infty)$$

and

$$\begin{aligned} I_2 &\leqq C \cdot \left(\frac{r_m}{d_m}\right)^{1+\alpha}\left(\frac{d_m}{d_l}\right)^{1+\alpha}\left(\log\frac{d_m}{r_m}+\log\frac{d_l}{d_m}\right)P_0^2(\sigma_{Q_l} < +\infty) \\ &\leqq C \cdot P_0^2(\sigma_{Q_l} < +\infty)\Big\{C \cdot P_0^2(\sigma_{Q_m} < +\infty) \\ &\quad + \left(\frac{r_m}{d_m}\right)^{1+\alpha}\left(\log\frac{d_m}{r_m}\right)\left(\frac{d_m}{d_l}\right)^{1+\alpha} \cdot \left(\log\frac{d_l}{d_m}\right)\left(\log\frac{d_m}{r_m}\right)^{-1}\Big\}. \end{aligned}$$

Since $m > l$ are sufficiently large, we have

$$\left(\frac{d_m}{d_l}\right)^{1+\alpha}\log\frac{d_l}{d_m} = \left(\frac{1}{8}\right)^{m-l}\log 8^{m-l} \leqq C$$

and $\log d_m/r_m > 1$. Hence it follows from (2.2) that

$$I_2 \leqq CP_0^2(\sigma_{Q_l} < +\infty)P_0^2(\sigma_{Q_m} < +\infty).$$

Consequently we have

$$P_0(\sigma_{Q_l} < +\infty,\ \sigma_{Q_m} < +\infty) \leqq CP_0(\sigma_{Q_l} < +\infty)P_0(\sigma_{Q_m} < +\infty). \tag{2.7}$$

Now since

$$\sum_k P_0^2(\sigma_{Q_k} < +\infty) \geqq C\sum_k\left(\frac{r_k}{d_k}\right)^{1+\alpha}\log\frac{d_k}{r_k} \geqq C\sum_k \frac{1}{k(\log k)^{\gamma-1}}$$

by (2.2)and (2.5), we have

$$\sum_k P_0^2(\sigma_{Q_k} < +\infty) = +\infty \tag{2.8}$$

for $2 > \gamma > 1$. Combining (2.8) with (2.6), we can conclude that

$$P_0^2(\overline{\lim_{k\to\infty}}\{\sigma_{Q_k} < +\infty\}) > 0$$

by Lamperti's lemma. Hence the point 0 is a regular point for $K \equiv \bigcup_k Q_k$ with respect to X_2. On the other hand we can easily see that

$$\sum_k P_0^1(\sigma_{Q_k} < +\infty) \leqq C \cdot \sum_k\left(\frac{r_k}{d_k}\right)^{1+\alpha} = C \cdot \sum_k\left(\frac{1}{k(\log k)^{\gamma}}\right).$$

Therefore

$$\sum_k P_0^1(\sigma_{Q_k} < +\infty) < +\infty$$

for $2>\gamma>1$, which implies that the point 0 is not a regular point for K with respect to X_1 as is easily seen using Borel-Cantelli lemma. Consequently we have proved that there exist an open set K and a point x such that x is a regular point for K with respect to X_2 but not a regular point for K with respect to X_1. Now the statement *iv*) is obvious in case $\alpha=(n-1)/2$ and $n=3$ or 4. When $n\geqq 5$ and $\alpha<(n-1)/2$, we can prove that there exist an open set K and a point x of the above type in the same way and the proof is rather easy. Hence we omit the proof here.

3. If we could prove that $G_1(x, y)\approx G_2(x, y)$ follows from the equivalence of fine topologies of two given Markov processes with Green functions $G_i(x, y)$, $i=1, 2$ respectively, then the statement *iv*) of the above proposition is trivial. This converse problem of the fine topology is still open except for a special case (see [3]).

References

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