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Conjugates of (p, q; r)-Absolutely Summing Operators

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§1. Introduction

By K. Miyazaki [4] a linear operator T from a Banach space E into another Banach space F is said to be $(p, q; r)$-absolutely summing for $1 \leq p, q, r \leq \infty$ if there exists a constant c such that for every finite sequence $\{x_i\}$ in E the inequality

$$\{\sum_i (i^{1/p-1/q}\|Tx_i\|^*)^q\}^{1/q} \leq c \sup_{\|x'\|\leq 1} (\sum_i |<x_i, x'>|^r)^{1/r}$$

is satisfied. Here $\{\|Tx_i\|^*\}$ denotes the non-increasing rearrangement of $\{\|Tx_i\|\}$, and as usual $\{\sum_i(\dots)^q\}^{1/q}$ and $(\sum_i|\dots|^r)^{1/r}$ are supposed to mean sup for $q=\infty$ and $r=\infty$ respectively. Especially, $(p, p; r)$-absolutely summing operators are exactly (p, r)-absolutely summing operators which were defined by B. Mitjagin and A. Pełczyński [3] and $(p, p; p)$-absolutely summing operators coincide with absolutely p-summing operators which are due to A. Pietsch [6]. The conjugates of absolutely p-summing operators have been investigated by several authors and especially characterized by J. S. Cohen [1] as strongly p'-summing operators where $1/p+1/p'=1$. The purpose of this paper is to investigate the conjugates of $(p, q; r)$-absolutely summing operators.

We shall introduce the notion of $(r; p, q)$-strongly summing operators and show that the conjugates of $(p, q; r)$-absolutely summing operators are $(r'; p', q')$-strongly summing operators where $1/p+1/p'=1/q+1/q'=1/r+1/r'=1$ and that the converse holds under a certain assumption. As a consequence of this result, we shall characterize the conjugates of (p, q)-absolutely summing operators.

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§2. Conjugates of (p, q; r)-absolutely summing operators

Let E and F be Banach spaces and let E' and F' be their continuous dual spaces. Let K be the real or complex field.

For $1 \leq p \leq \infty$ a sequence $\{x_i\}$ with values in E is called weakly p-summable provided for any $x' \in E'$ the sequence $\{<x_i, x'>\}$ belongs to l_p. The space $l_p(E)$ of weakly p-summable sequences is a normed space with the norm

$$\varepsilon_p(\{x_i\}) = \begin{cases} \sup_{\|x'\|\le 1} (\sum_{i=1}^{\infty} |<x_i, x'>|^p)^{1/p} & (1 \le p < \infty), \\ \sup_i \|x_i\| & (p = \infty). \end{cases}$$

For $1 \le p \le \infty$ a sequence $\{x_i\}$ with values in E is called strongly p-summable ([1]) provided for every sequence $\{x_i'\} \in l_{p'}(E')(1/p + 1/p' = 1)$ the series $\sum_{i=1}^{\infty} <x_i, x_i'>$ converges. The space $l_p\langle E\rangle$ of strongly p-summable sequences is a normed space with the norm

$$\sigma_p(\{x_i\}) = \sup_{\varepsilon_{p'}(\{x_i'\}) \le 1} \sum_{i=1}^{\infty} |<x_i, x_i'>| .$$

For $1 \le p,\ q \le \infty$ a sequence $\{x_i\}$ with values in E is called (p, q)-absolutely summable provided the sequence $\{\|x_i\|\}$ belongs to $l_{p,q}$. The space $l_{p,q}\{E\}$ of (p, q)-absolutely summable sequences is a quasi-normed space with the quasi-norm

$$\alpha_{p,q}(\{x_i\}) = \begin{cases} (\sum_{i=1}^{\infty} i^{q/p-1} \|x_i\|^{*q})^{1/q} & (1 \le q < \infty), \\ \sup_i i^{1/p} \|x_i\|^* & (q = \infty), \end{cases}$$

where $\{\|x_i\|^*\}$ denotes the non-increasing rearrangement of $\{\|x_i\|\}$.

For $1 \le p,\ q,\ r \le \infty$ an operator T mapping E into F is called $(p, q; r)$-absolutely summing in K. Miyazaki [4] provided there exists a constant $c \ge 0$ such that for every finite sequence $\{x_i\}$ in E the inequality

$$\alpha_{p,q}(\{Tx_i\}) \le c\varepsilon_r(\{x_i\})$$

is satisfied. The smallest number c for which the above inequality is satisfied is denoted by $\Pi_{p,q;r}(T)$. In particular, $(p, p; r)$-absolutely summing operators are exactly (p, r)-absolutely summing operators which were defined by B. Mitjagin and A. Pełczyński [3] and $(p, p; p)$-absolutely summing operators coincide with absolutely p-summing operators which are due to A. Pietsch [6].

We now introduce the notion of $(r; p, q)$-strongly summing operators in the following

DEFINITION. *For $1 \le p,\ q,\ r \le \infty$ we call an operator T mapping E into F $(r; p, q)$-strongly summing provided there exists a constant $c \ge 0$ such that for every finite sequence $\{x_i\}$ in E the inequality*

$$\sigma_r(\{Tx_i\}) \le c\,\alpha_{p,q}(\{x_i\})$$

is satisfied. The smallest number c for which the above inequality is satisfied is called the $(r; p, q)$-strongly summing norm of T and denoted by $D_{r;p,q}(T)$.

In particular we say $(r; p, p)$-strongly summing operators to be (r, p)-

strongly summing operators and denote $D_{r;p,p}(T)$ by $D_{r,p}(T)$.

(p, p)-strongly summing operators are exactly strongly p-summing operators which were introduced by J. S. Cohen [1].

Now we suppose that $n \geq 1$ and that N is a norm or a quasi-norm on K^n. If $(x_1, \ldots, x_n) \in E^n$ we write

$$\|(x_1, \ldots, x_n)\|_N = \sup_{\|x'\| \leq 1} N(<x_1, x'>, \ldots, <x_n, x'>),$$

$$\|(x_1, \ldots, x_n)\|^N = \sup_{\|x'_i\| \leq 1} N(<x_1, x'_1>, \ldots, <x_n, x'_n>).$$

Then we have the following two lemmas.

LEMMA 1. *Let M be a quasi-norm on K^n which satisfies*

$$M(\{\lambda_i\}) \leq c_1 \max_{1 \leq i \leq n} |\lambda_i|$$

for a certain positive number c_1 and for every $\{\lambda_i\} \in K^n$. Let c_0 be a positive number which satisfies

$$M(\{\lambda_i + \mu_i\}) \leq c_0[M(\{\lambda_i\}) + M(\{\mu_i\})]$$

for any $\{\lambda_i\}$, $\{\mu_i\} \in K^n$. Then for every $(x'_1, \ldots, x'_n) \in (E')^n$,

$$\|(x'_1, \ldots, x'_n)\|^M \leq c_0 \sup_{\substack{\|x_i\| \leq 1 \\ x_i \in E}} M(<x_1, x'_1>, \ldots, <x_n, x'_n>).$$

PROOF. Let $(x'_1, \ldots, x'_n) \in (E')^n$. Let $(x''_1, \ldots, x''_n) \in (E'')^n$ and $\|x''_i\| \leq 1$. Then for any $\varepsilon > 0$ there exists an element $(x_1, \ldots, x_n) \in E^n$ with $\|x_i\| \leq 1$ such that

$$\max_{1 \leq i \leq n} |<x'_i, x''_i - J_E x_i>| \leq \varepsilon / c_0 c_1$$

where J_E denotes the canonical injection of E into E'', since the canonical image of the unit ball of E is $\sigma(E'', E')$-dense in the unit ball of E''. Therefore we have

$$\begin{aligned}
&M(<x'_1, x''_1>, \ldots, <x'_n, x''_n>) \\
&\quad \leq c_0 M(<x'_1, J_E x_1>, \ldots, <x'_n, J_E x_n>) \\
&\qquad\qquad + c_0 M(<x'_1, x''_1 - J_E x_1>, \ldots, <x'_n, x''_n - J_E x_n>) \\
&\quad \leq c_0 M(<x_1, x'_1>, \ldots, <x_n, x'_n>) + c_0 c_1 \max_{1 \leq i \leq n} |<x'_i, x''_i - J_E x_i>| \\
&\quad \leq c_0 \sup_{\|x_i\| \leq 1} M(<x_1, x'_1>, \ldots, <x_n, x'_n>) + \varepsilon,
\end{aligned}$$

which shows that

$$\|(x'_1,\dots, x'_n)\|^M \le c_0 \sup_{\|x_i\|\le 1} M(<x_1, x'_1>,\dots, <x_n, x'_n>).$$

LEMMA 2. *Let M be a quasi-norm on K^n which satisfies*

$$M(\{\lambda_i\}) \le c_1 \max_{1\le i\le n} |\lambda_i|$$

for a certain positive number c_1 and for every $\{\lambda_i\}\in K^n$ and let c_0 be a positive number which satisfies

$$M(\{\lambda_i+\mu_i\}) \le c_0[M(\{\lambda_i\})+M(\{\mu_i\})]$$

for any $\{\lambda_i\}$, $\{\mu_i\}\in K^n$. Let N be a norm on K^n. Let $T: E\to F$ be a linear operator such that for a certain positive number c and for every $(x_1,\dots, x_n)\in E^n$

$$\|(Tx_1,\dots, Tx_n)\|^M \le c\|(x_1,\dots, x_n)\|_N.$$

Then for $T'': E''\to F''$ and for every $(x''_1,\dots, x''_n)\in (E'')^n$,

$$\|(T''x''_1,\dots, T''x''_n)\|^M \le c_0^2 c\|(x''_1,\dots, x''_n)\|_N.$$

PROOF. Let $(y'_1,\dots, y'_n)\in (E')^n$ and $\|y'_i\|\le 1$. Then for any $(x_1,\dots, x_n)\in E^n$ we have

$$\begin{aligned} &M(<T'y'_1, J_E x_1>,\dots, <T'y'_n, J_E x_n>) \\ &= M(<Tx_1, y'_1>,\dots, <Tx_n, y'_n>) \\ &\le \|(Tx_1,\dots, Tx_n)\|^M \\ &\le c\|(x_1,\dots, x_n)\|_N. \end{aligned}$$

Therefore, if $\|(x_1,\dots, x_n)\|_N\le 1$, then

$$M(<T'y'_1, J_E x_1>,\dots, <T'y'_n, J_E x_n>) \le c.$$

Now let $(x''_1,\dots, x''_n)\in (E'')^n$ and $\|(x''_1,\dots, x''_n)\|_N\le 1$. Then for any $\varepsilon>0$ there exists an element $(x_1,\dots, x_n)\in E^n$ with $\|(x_1,\dots, x_n)\|_N\le 1$ such that

$$\max_{1\le i\le n} |<T'y'_i, x''_i - J_E x_i>| \le \varepsilon/c_0 c_1,$$

since the set $\{(J_E x_1,\dots, J_E x_n)\in (E'')^n: \|(x_1,\dots, x_n)\|_N\le 1\}$ is $\sigma((E'')^n, (E')^n)$-dense in the set $\{(x''_1,\dots, x''_n)\in (E'')^n: \|(x''_1,\dots, x''_n)\|_N\le 1\}$ ([7]). Consequently

$$\begin{aligned} &M(<T'y'_1, x''_1>,\dots, <T'y'_n, x''_n>) \\ &\le c_0 M(<T'y'_1, J_E x_1>,\dots, <T'y'_n, J_E x_n>) \\ &\quad + c_0 M(<T'y'_1, x''_1 - J_E x_1>,\dots, <T'y'_n, x''_n - J_E x_n>) \end{aligned}$$

$$\leq c_0c+c_0c_1 \max_{1\leq i\leq n} |<T'y_i', x_i''-J_E x_i>|$$

$$\leq c_0c+\varepsilon.$$

Hence, if $\|(x_1'',\dots, x_n'')\|_N\leq 1$, then

$$M(<y_1', T''x_1''>,\dots, <y_n', T''x_n''>)\leq c_0c,$$

which implies that for any $(x_1'',\dots, x_n'')\in(E'')^n$

$$M(<y_1', T''x_1''>,\dots, <y_n', T''x_n''>)\leq c_0c\|(x_1'',\dots, x_n'')\|_N.$$

Therefore by Lemma 1 we have the desired inequality

$$\|(T''x_1'',\dots, T''x_n'')\|^M\leq c_0^2c\|(x_1'',\dots, x_n'')\|_N.$$

THEOREM 1. *Let $1\leq p, q, r\leq\infty$. If an operator $T: E\rightarrow F$ is $(p, q; r)$-absolutely summing, then $T'': E''\rightarrow F''$ is $(p, q; r)$-absolutely summing and*

$$\Pi_{p,q;r}(T'')\leq\max(2^{2/p}, 2^{2/q})\Pi_{p,q;r}(T).$$

PROOF. If $n\geq 1$, we define M and N respectively by

$$M(\lambda_1,\dots, \lambda_n)=(\sum_{i=1}^n i^{q/p-1}|\lambda_i|^{*q})^{1/q}$$

and

$$N(\lambda_1,\dots, \lambda_n)=(\sum_{i=1}^n|\lambda_i|^r)^{1/r}$$

in the case where $1\leq q, r<\infty$. Then, since

$$M(\{\lambda_i+\mu_i\})\leq\max(2^{1/p}, 2^{1/q})[M(\{\lambda_i\})+M(\{\mu_i\})],$$

$$M(\{\lambda_i\})\leq(\sum_{i=1}^n i^{q/p-1})^{1/q}\max_{1\leq i\leq n}|\lambda_i|,$$

$$\alpha_{p,q}(\{x_i\}_{1\leq i\leq n})=\|(x_1,\dots, x_n)\|^M$$

and

$$\varepsilon_r(\{x_i\}_{1\leq i\leq n})=\|(x_1,\dots, x_n)\|_N,$$

the statement in this case is an immediate consequence of Lemma 2. The other cases can be shown similarly.

THEOREM 2. *Let $1\leq p, q, r\leq\infty$. If an operator $T: E\rightarrow F$ is $(p, q; r)$-absolutely summing, then its conjugate operator $T': F'\rightarrow E'$ is $(r'; p', q')$-strongly* *summing and*

$$D_{r';p',q'}(T') \leq \max (2^{2/p}, 2^{2/q})\Pi_{p,q;r}(T)$$

where $1/p+1/p'=1/q+1/q'=1/r+1/r'=1$.

PROOF. Let $T\in\Pi_{p,q;r}(E, F)$. Then by the previous theorem $T''\in\Pi_{p,q;r}(E'', F'')$ and $\Pi_{p,q;r}(T'')\leq\max(2^{2/p}, 2^{2/q})\Pi_{p,q;r}(T)$. Hence, if $\{y'_i\}_{1\leq i\leq n}$ is a finite sequence in F' and if $\{x''_j\}\in l_r(E'')$, we have

$$\begin{aligned}\left|\sum_{i=1}^{n} <T'y'_i, x''_i>\right| &\leq \sum_{i=1}^{n} \|T''x''_i\| \|y'_i\| \\ &\leq \alpha_{p,q}(\{T''x''_i\}_{1\leq i\leq n})\alpha_{p',q'}(\{y'_i\}_{1\leq i\leq n}) \\ &\leq \Pi_{p,q;r}(T'')\varepsilon_r(\{x''_j\})\alpha_{p',q'}(\{y'_i\}) \\ &\leq \max (2^{2/p}, 2^{2/q})\Pi_{p,q;r}(T)\varepsilon_r(\{x''_j\})\alpha_{p',q'}(\{y'_i\}),\end{aligned}$$

since

$$\|\{\xi_i\eta_i\}\|_{l_1} \leq \|\{\xi_i\}\|_{l_{p,q}}\cdot\|\{\eta_i\}\|_{l_{p',q'}}$$

for $\{\xi_i\}\in l_{p,q}$ and $\{\eta_i\}\in l_{p',q'}$ ([5]). Therefore,

$$\sigma_{r'}(\{T'y'_i\}) \leq \max(2^{2/p}, 2^{2/q})\Pi_{p,q;r}(T)\alpha_{p',q'}(\{y'_i\}),$$

which shows that $T'\in D_{r';p',q'}(F', E')$ and $D_{r';p',q'}(T')\leq\max(2^{2/p}, 2^{2/q})\Pi_{p,q;r}(T)$.

REMARK 1. The converse of Theorem 2 holds under the assumption that $l_{p,q}\{F\}'$ and $l_{p',q'}\{F'\}$ are topologically isomorphic.

PROOF. Assume that

$$\alpha'_{p,q}(\{y'_j\})\leq\alpha_{p',q'}(\{y'_j\})\leq M\alpha'_{p,q}(\{y'_j\})$$

for a certain positive number M and for all $\{y'_j\}\in l_{p',q'}\{F'\}=l_{p,q}\{F\}'$ where $\alpha'_{p,q}$ denotes the norm of the dual space $l_{p,q}\{F\}'$ of $l_{p,q}\{F\}$. Let $T'\in D_{r';p',q'}(F', E')$. Then for an arbitrary finite sequence $\{x_i\}_{1\leq i\leq n}$ in E and for any $\{y'_j\}\in l_{p',q'}\{F'\}$ we have

$$\begin{aligned}\left|\sum_{i=1}^{n} <Tx_i, y'_i>\right| = \left|\sum_{i=1}^{n} <x_i, T'y'_i>\right| &\leq\sigma_{r'}(\{T'y'_i\}_{1\leq i\leq n})\varepsilon_r(\{x_i\}_{1\leq i\leq n}) \\ &\leq D_{r';p',q'}(T')\alpha_{p',q'}(\{y'_j\})\varepsilon_r(\{x_i\}).\end{aligned}$$

Therefore, by our assumption we have

$$\alpha_{p,q}(\{Tx_i\}) \leq \sup_{\alpha_{p',q'}(\{y'_j\}) \leq M} \left| \sum_{i=1}^{n} < Tx_i, y'_i > \right|$$

$$\leq MD_{r';p',q'}(T')\varepsilon_r(\{x_i\}),$$

which shows that $T \in \Pi_{p,q;r}(E, F)$ and

$$\Pi_{p,q;r}(T) \leq MD_{r';p',q'}(T')$$

as was asserted.

Since $l_p\{F\}'$ and $l_{p'}\{F'\}$ are isometrically isomorphic for $1 \leq p < \infty$, as an immediate consequence of Theorem 2 and Remark 1 we have the following

THEOREM 3. *Let $1 \leq p, q < \infty$. An operator $T: E \to F$ is (p, q)-absolutely summing if and only if its conjugate operator $T': F' \to E'$ is (q', p')-strongly summing. In this case $\Pi_{p,q}(T) = D_{q',p'}(T')$.*

EXAMPLES. (1) Since for $1 \leq q < r < p < \infty$ the identity operator I from $C[0, 1]$ into $L_q(0, 1)$ is $(p, q; r)$-absolutely summing and not (q, r)-absolutely summing ([4]), its conjugate operator I' from $L_{q'}(0, 1)$ into $M[0, 1]$ is $(r'; p', q')$-strongly summing and not (r', q')-strongly summing by Theorems 2 and 3. Here $M[0, 1]$ denotes the Banach space of complex regular Borel measures on $[0, 1]$.

(2) Since for $1 < r \leq p < \infty$ the identity operator I from $C[0, 1]$ into $L_p(0, 1)$ is (p, r)-absolutely summing ([4]), its conjugate operator I' from $L_{p'}(0, 1)$ into $M[0, 1]$ is (r', p')-strongly summing.

(3) It is known ([2]) that the identity operator in l_1 is $(2, 1)$-absolutely summing. The operator is not absolutely p-summing for $1 \leq p < \infty$, which is a consequence of Dvoretzky-Rogers Theorem ([6]). Therefore, the identity operator in l_∞ is $(\infty, 2)$-strongly summing, but not strongly 2-summing by Theorem 3.

§3. Operators whose conjugates are $(p, q; r)$-absolutely summing

The properties of operators whose conjugates are $(p, q; r)$-absolutely summing can be developed by similar discussions as in Section 2.

THEOREM 4. *Let $1 \leq p, q, r \leq \infty$. If $T: E \to F$ is an operator whose conjugate $T': F' \to E'$ is $(p, q; r)$-absolutely summing, then T is $(r'; p', q')$-strongly summing and $D_{r';p',q'}(T) \leq \Pi_{p,q;r}(T')$.*

The proof is similar to that of Theorem 2.

REMARK 2. The converse of Theorem 4 holds under the assumption that $l_{p',q'}\{E\}'$ and $l_{p,q}\{E'\}$ are topologically isomorphic.

This can be shown as in Remark 1.

As an immediate consequence of Theorem 4 and Remark 2 we have

THEOREM 5. *Let* $1<p<\infty$ *and* $1\leq q<\infty$. *An operator* $T: E\rightarrow F$ *is* (q', p')*-strongly summing if and only if its conjugate operator* $T': F'\rightarrow E'$ *is* (p, q)*-absolutely summing. In this case* $D_{q',p'}(T)=\Pi_{p,q}(T')$.

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