HIROSHIMA MATH. J.
6 (1976), 1–6

Oscillation and Existence of Unique Positive Solutions for Nonlinear n-th Order Equations with Forcing Term

Athanassios G. KARTSATOS

(Received February 17, 1975)

1. Introduction

The main purpose of this paper is twofold: it is shown first that under certain assumptions every bounded solution of the equation

$$\text{(I)} \qquad (a_{n-1}(t)(\cdots(a_2(t)(a_1(t)x')'\cdots)')' + P(t)H(x(g(t))) = Q(t)$$

is either oscillatory, or tends to a finite limit as $t \to +\infty$. The proof of this result is much simpler than the one given by Singh and Dahiya [4, Theorem 1], who considered the case $n=2$ under stronger assumptions. Secondly, a result is given according to which an equation of the form

$$\text{(II)} \qquad x^{(n)} + G(t, x) = Q(t)$$

can have at most one positive solution. This result extends to the general case a result of Atkinson [1]. Atkinson considers second order equations and makes use of Sturm's comparison theorem. This theorem does not hold in its full generality for n-th order equations. Here a result is employed from Kartsatos [3, Theorem 2].

A function $f(t)$, $t \in [t_0, +\infty)$, $t_0 \geq 0$ is said to be "oscillatory" if it has an unbounded set of zeros on $[t_0, +\infty)$. By a solution of (I) (or (II)) we mean a function $x(t)$ which satisfies (I) (or (II)) for every t in an infinite interval $[t_0, +\infty)$, $t_0 \geq \alpha$. In what follows, $R = (-\infty, +\infty)$, $R_+ = [0, +\infty)$, $J = [\alpha, +\infty)$, where α is a fixed number.

2. Oscillation and nonoscillation of (I)

THEOREM 1. *Assume that equation* (I) *satisfies the following*:

(i) $a_k: J \to R_+ \backslash \{0\}$ *is continuous, and such that*

$$\int_\alpha^\infty \frac{dt}{a_k(t)} < +\infty, \quad k = 1, 2, \ldots, n-1;$$

(ii) *$P(t) \equiv P_1(t)+P_2(t)$, where $P_1: J \to R_+ \backslash \{0\}$, $P_2: J \to R$ are continuous, and such that*

$$\int_\alpha^\infty |P_2(t)| dt < +\infty;$$

(iii) *$H: R \to R$ is continuous, and $xH(x) > 0$ for every $x \neq 0$;*

(iv) *$g: J \to R_+ \backslash \{0\}$ is continuous, and $\lim_{t\to\infty} g(t) = +\infty$;*

(v) *$Q: J \to R$ is continuous, and*

$$\left| \int_\alpha^\infty Q(t) dt \right| < +\infty$$

Then every bounded nonoscillatory solution of (I) *has a finite limit as $t \to +\infty$.*

PROOF. Let $x(t)$ be a bounded nonoscillatory solution of (I), and assume that $x(t) > 0$, $t \in [\alpha_1, +\infty)$, $\alpha_1 \geq \alpha$ (a corresponding argument holds in case $x(t)$ is assumed to be eventually negative). Then there exists $\alpha_2 \geq \alpha_1$ such that $g(t) \geq \alpha_1$ for $t \geq \alpha_2$. Thus, $x(g(t)) > 0$ for $t \geq \alpha_2$. Now integrating (I) from α_2 to $t \geq \alpha_2$ we obtain

$$(1) \qquad G_{n-1}(t) = G_{n-1}(\alpha_2) - \int_{\alpha_2}^t P_1(s) H(x(g(s))) ds$$
$$- \int_{\alpha_2}^t P_2(s) H(x(g(s))) ds + \int_{\alpha_2}^t Q(s) ds,$$

where $G_k(t) \equiv a_k(t) G_{k-1}(t)$, $G_0(t) \equiv x(t)$, $k = 1, 2, \dots, n-1$.

Now we consider the two possible cases:

CASE 1. $$\int_{\alpha_2}^\infty P_1(s) H(x(g(s))) ds = +\infty$$

CASE 2. $$\int_{\alpha_2}^\infty P_1(s) H(x(g(s))) ds < +\infty.$$

In Case 1, using the fact that the last two integrals in (1) converge, we obtain $\lim_{t\to+\infty} G_{n-1}(t) = -\infty$. Since $G_{n-1}(t) \equiv a_{n-1}(t) G'_{n-2}(t)$ and $a_{n-1}(t) > 0$, it follows that $G'_{n-2}(t)$ and $G_{n-2}(t)$ are eventually of constant sign. Continuing the same way we obtain that $x'(t)$ is of constant sign for all large t. Thus $x(t)$ tends monotonically to a finite limit as $t \to +\infty$.

In Case 2, we obtain that $\lim_{t\to\infty} G_{n-1}(t) = \lambda$ exists and is finite. If this limit is positive or negative, then $G_{n-1}(t)$ is respectively positive or negative for all large t. Either case implies immediately that all the functions $G_k(t)$, $k = 1, 2, \dots,$

$n-2$ are eventually of constant sign, which yields the monotonicity of $x(t)$. Now let

$$\lim_{t\to\infty} G_{n-1}(t)=0. \tag{2}$$

Then given $\varepsilon>0$ there exists $\mu(\varepsilon)>0$ such that

$$|a_{n-1}(t')G'_{n-2}(t')|<\varepsilon, \quad \left|\int_{t'}^{t''}\frac{ds}{a_{n-}(s)}\right|<1 \tag{3}$$

for every t', $t''\geq\mu(\varepsilon)$. Consequently, dividing the first of (3) by $a_{n-1}(t')$ and integrating from t' to t'' we obtain

$$|G_{n-2}(t')-G_{n-2}(t'')|<\varepsilon, \qquad t',\ t''>\mu(\varepsilon). \tag{4}$$

This is the Cauchy criterion for functions. It implies that $\lim_{t\to\infty} G_{n-2}(t)$ exists and is finite. If this limit is positive or negative, then $x(t)$ is eventually monotonic. If it is zero, we continue the same way. Thus, we either have $x(t)$ monotonic for all large t, or

$$\lim_{t\to\infty} a_1(t)x'(t)=0, \tag{5}$$

which implies, as above, the existence of the limit $\lim_{t\to\infty} x(t)$. This completes the proof.

Singh and Dahiya [4, Theorem 1] showed the above theorem in the case $n=2$, $g(t)=t-\lambda(t)$, $H(x)=x$ and under the additional assumptions: (i)–(v), $a_1(t)\geq p>0$, $\int_{\alpha_1}^{\infty} P_1(t)dt=+\infty$, $\int_{\alpha_1}^{\infty}|Q(t)|dt<+\infty$.

THEOREM 2. *In addition to* (i)–(v) *of* Theorem 1, *let* $\liminf_{|x|\to\infty} H(x)\neq 0$, $\int_{\alpha}^{\infty} P_1(t)dt=+\infty$, $P_2(t)\equiv 0$, $t\in J$. *Then every nonoscillatory solution of* (I) *is bounded and tends to a finite limit as* $t\to+\infty$.

PROOF. Assume that $x(t)$ is a nonoscillatory solution of (I) with $x(t)>0$, $x(g(t))>0$, $t\geq\alpha_1\geq\alpha$. Consider the two cases of the proof of Theorem 1. In Case 1 the solution $x(t)$ and all the functions $G_k(t)$, $k=1, 2,\ldots, n-2$ are eventually monotonic. If all the G_k are eventually nonpositive, then $x(t)$ is decreasing, thus bounded. If this does not happen, let $G_i(t)$, $0\leq i\leq n-2$ be the last of the $G_k(t)$ with the property $G_i(t)\geq 0$ for all large t. Then $G'_i(t)\leq 0$ for all large t, which implies the existence of a constant $M>0$ such that $|G_i(t)|\leq M$, $t\geq\alpha_2\geq\alpha_1$. Consequently,

$$-M\int_{\alpha_2}^{t}\frac{ds}{a_i(s)}\leq G_{i-1}(t)-G_{i-1}(\alpha_2)\leq M\int_{\alpha_2}^{t}\frac{ds}{a_i(s)}, \tag{6}$$

which proves the boundedness of $G_{i-1}(t)$. Similarly we can show the boundedness of $G_k(t)$ for every $k=0, 1, 2,\ldots, i-2$. This proves the theorem in Case 1. Case 2 can only happen (because of the assumption $\liminf_{|x|\to\infty} H(x)\neq 0$) if $\liminf_{t\to\infty} x(t) =0$. Since $\lim_{t\to\infty} G_{n-1}(t)=\lambda$ exists and is finite, $x(t)$ will be eventually monotonic if $\lambda\gtrless 0$, and since $\liminf_{t\to\infty} x(t)=0$, we must have $\lim_{t\to\infty} x(t)=0$. If $\lambda=0$, then we obtain from (4) that $\lim_{t\to\infty} G_{n-2}(t)=\mu$ exists and is finite. Arguing in the same way as above we deduce that $x(t)$ is eventually monotonic and tends to zero for $\mu\gtrless 0$ otherwise $\lim_{t\to\infty} G_{n-3}(t)=\mu_1$ exists and is finite. Continuing the same way we get that $\lim_{t\to\infty} \{G_0(t)\equiv x(t)\}$ exists and is finite. This completes the proof.

The above theorem extends to the general case Theorem 2 in [4].

THEOREM 3. *Let the assumptions of* Theorem 2 *be satisfied except the integrability of $Q(t)$ and let $Q(t)\equiv G^{0\prime}_{n-1}(t)$ with $G^0_k(t)\equiv a_k(t)\cdot G^{0\prime}_{k-1}(t)$, $G^0_0(t)\equiv M(t)$, where $M(t)$ satisfies*

$$\limsup_{t\to\infty} M(t) = +\infty, \qquad \liminf_{t\to\infty} M(t) = -\infty.$$

Then every solution of (I) *is oscillatory.*

PROOF. Let $x(t)$ be a solution of (I) with $x(t)>0$, $x(g(t))>0$, $t\geq\alpha_1\geq\alpha$. Let $x(t)-M(t)\equiv u(t)$, $t\geq\alpha_1$. Then $M(t)+u(t)$, $M(g(t))+u(g(t))$ are positive for $t\geq\alpha_1$ and $u(t)$ is a solution of the equation

$$G^{*\prime}_{n-1}(t)+P(t)H(u(g(t))+M(g(t))) = 0, \tag{7}$$

where $G^*_k(t)\equiv a_k(t)G^{*\prime}_{k-1}(t)$, $k=1, 2,\ldots, n-1$, $G^*_0(t)\equiv u(t)$. Now we distinguish three cases:

CASE 1. $u(t)$ is positive for all large t;

CASE 2. $u(t)$ is negative for all large t;

CASE 3. $u(t)$ is oscillatory.

In Case 1, it follows, as in the proofs of Theorems 1, 2, that $u(t)$ is bounded. This implies the oscillation of $x(t)$, a contradiction. In Case 2, $x(t)<M(t)$ for all large t, a contradiction to the positivity of $x(t)$. In the third case

$$\int^{\infty} P_1(t)H(u(g(t))+M(g(t)))dt < +\infty \tag{8}$$

otherwise $G^*_{n-1}(t)\to -\infty$, which implies the monotonicity of $u(t)$, a contradiction to its oscillatory character. Inequality (8) implies now that $\lim_{t\to\infty} G^*_{n-1}(t)$ exists and equals zero, otherwise $u(t)$ would be monotonic, a contradiction as above.

Thus, as in Theorem 2, every $G_k^*(t)$ is bounded, and in particular $G_0^*(t) \equiv u(t)$ is bounded, which implies the oscillation of $x(t)$ as in Case 1 above. This completes the proof.

3. Equations possessing at most one positive solution

The following theorem extends to the general case Theorem 6 in Atkinson's paper [1].

THEOREM 4. *Assume that the equation*

$$x^{(n)} + H(t, x) = 0, \qquad (n\text{: even}) \tag{9}$$

has no eventually positive solutions. Moreover, assume that:

(i) *P is defined, positive and n times continuously differentiable on $[0, \infty)$ with $P^{(n)}(t) \equiv Q(t)$, $t \in [0, \infty)$. Moreover,* $\liminf_{t\to\infty} P(t) = 0$;

(ii) *$H(t, u)$ is defined and continuous on $[0, \infty) \times R$, and is continuously differentiable there w.r.t. u, and $H_2(t, u) \equiv (\partial/\partial u)H(t, u)$ is nonnegative and increasing w.r.t. u. Moreover, $uH(t, u) > 0$ for $u > 0$ and*

$$\int_0^\infty t^{n-1} H_2(t, P(t))dt < \infty.$$

Then the equation

$$x^{(n)} + H(t, x) = Q(t) \tag{10}$$

can have at most one eventually positive solution.

PROOF. Assume that U, V are two solutions of (10) such that $U(t) > 0$, $V(t) > 0$, $t \in [t_0, \infty)$, $t_0 \geq 0$. Then it follows from Theorem 3.1 in [2] that

$$\begin{gathered} 0 < U(t) < P(t), \quad 0 < V(t) < P(t), \quad t \geq (\text{some})t_1 \geq t_0, \\ \lim_{t\to\infty} [U(t) - V(t)] = 0 \end{gathered} \tag{11}$$

Now we show that $W(t) \equiv U(t) - V(t)$ is oscillatory. In fact, assume that there is $t_2 \geq t_1$ such that $W(t) > 0$ for $t \geq t_2$. Then from (16) we obtain

$$W^{(n)}(t) = -[H(t, U(t)) - H(t, V(t))] \leq 0 \tag{12}$$

for every $t \geq t_2$ because H is increasing w.r.t. the second variable. This implies that all the derivatives $W^{(i)}(t)$, $i = 1, 2, \dots, n-1$ are of constant sign for all large t, and that no two consecutive derivatives can be of the same sign for all large t because of the boundedness of $W(t)$. Consequently $W'(t) > 0$ for all large t, a contradiction to the fact that $W(t)$ is positive for all large t and $\lim_{t\to\infty} W(t)=0$.*) Thus, $W(t)$ oscillates. Now, by the mean value theorem, we have

$$H(t, U(t))-H(t, V(t)) = H_2(t, \lambda(t))W(t), \qquad t \geq t_1, \tag{13}$$

where $\lambda(t)$ is a continuous function lying between $U(t)$ and $V(t)$. Thus, $\lambda(t) < P(t)$, $t \geq t_1$.

Consequently, $W(t)$, $t \geq t_1$ is an oscillatory solution of the equation

$$W^{(n)}+H_2(t, \lambda(t))W = 0. \tag{14}$$

Since, however, $H_2(t, \lambda(t)) \leq H_2(t, P(t))$, we have

$$\int_{t_1}^{\infty} t^{n-1}H_2(t, \lambda(t))dt < \infty. \tag{15}$$

An application of Theorem 2 in Kartsatos [3] shows that $W(t)\equiv 0$ for all large t, and this proves our assertion.

As an example, consider the equation

$$x^{(6)}+(1/t^5)x^3 = 720t^{-7}, \qquad t \geq 1. \tag{16}$$

Here we have $P(t)=t^{-1}$, and $H_2(t, P(t))=(3/t^5)P^2(t)=3/t^7$. Moreover the unperturbed equation oscillates (cf., for example, Kartsatos [8, Example]).

References

[1] F. V. Atkinson, On second-order differential inequalities, Proc. Royal Soc. Edinburgh, **72** (1972/73), 109-127.

[2] A. G. Kartsatos, On n-th order differential inequalities, J. Math. Anal. Appl., **47** (1975), 1-9.

[3] A. G. Kartsatos, Maintenance of oscillations under the effect of a periodic forcing term, Proc. Amer. Math. Soc., **33** (1972), 377–383.

[4] B. Singh, R. S. Dahiya, On oscillation of second-order retarded equations, J. Math. Anal. Appl., **47** (1974), 504–512.

Department of Mathematics,
University of South Florida,
Tampa, Florida

*) An analogous situation appears if we assume $W(t)<0$, $t \geq t_2$.