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On the Oscillation Problem of Nonlinear Equations

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1. Introduction

In this paper we consider, among others, equations of the form

$$(\mathrm{I}) \qquad [p(t)x^{(n-1)}]' + H(t, x(g(t))) = Q(t), \quad n \geqq 2.$$

Our main purpose here is to present a theorem which considerably improves a corresponding result of Singh [11, Theorem 1]. Our proof is also much simpler than the one given by Singh in a special case of (I). A corollary to our result is also given and improves the corresponding result of Singh. In Theorem 2 we consider a small forcing $Q(t)$, in Theorem 3 a homogeneous equation with damping, and Theorem 4 deals with the case of a damping treated as a small perturbation.

The reader is referred to a survey paper of the author [6] for several results concerning n-th order equations. Equations with damping have been considered also by Kartsatos and Onose [7], Naito [9] and Sficas [10]. Singh's main result in [11] is related to a result of Hammett [3], but the former does not contain the latter because of an integral condition on $p(t)$. For a natural extension of Hammett's result in the n-th order case, and for $p(t)=1$, the reader is referred to Kartsatos [5]. For other extensions to Hammett's results, relative references are those of Atkinson [1] and Grimmer [2]. For oscillation results concerning forced functional equations the reader is also referred to, for example, Kusano and Onose [8], or Staikos and Sficas [12].

In what follows, $R=(-\infty, \infty)$, $R_+=[0, \infty)$, $R_+^0=(0, \infty)$ and $R_T=[T, \infty)$ for some fixed finite T. Moreover, $n \geqq 2$, and the functions $p\colon R_T \to R_+^0$, $g\colon R_T \to R_+$, $Q\colon R_T \to R$, $H\colon R_T \times R \to R$ will be assumed continuous on their respective domains. Furthermore, $H(t, u)$ will be assumed increasing in u and such that $uH(t, u) > 0$ for every $u \neq 0$. For the function $g(t)$ we merely assume that $\lim_{t\to\infty} g(t) = +\infty$. By a solution of (I) we mean any real function which is n times continuously differentiable and satisfies (I) on an infinite subinterval of $[T, \infty)$. A solution of (I) is said to be "oscillatory" if it has an unbounded set of zeros in its domain of existence. A solution $x(t)$ of (I) is "bounded" if $|x(t)| \leqq K$ for all t in the domain of $x(t)$, where K is a positive constant,

2. Main results

THEOREM 1. *Consider* (I) *under the following assumptions:*

$$\int_T^\infty [1/p(t)]dt < +00, \quad \left|\int_T^\infty Q(t)dt\right| < +00, \quad \int_T^\infty H(t, \pm k)dt = \pm 00$$

for any constant $k>0$. *Then if* $x(t)$ *is a nonoscillatory solution of* (/), $x^{(n-2)}(t)$ *tends to a finite limit as* $t\to\infty$.

PROOF. Let $x(t)$ be a nonoscillatory solution of (I) and assume that $x(t)>0$ for all large t. Then there exists $t_1 \geqq T$ such that $x(t)>0$, $x(g(t))>0$ for all $t\geqq t_1$. Now integrating (I) from t_1 to $t\geqq t_1$, we have

$$p(t)x^{(n-1)}(t) = C-\int_{t_1}^t H(s, x(g(s)))ds+\int_{t_1}^t Q(s)ds, \tag{2.1}$$

where C is a constant.

Since $H(t, x(g(t)))>0$ for $t\geqq t_1$, we consider the following two possible cases:

Case 1. $$\int_{t_1}^\infty H(s, x(g(s)))ds= +\infty,$$

Case 2. $$\int_{t_1}^\infty H(s, x(g(s)))ds < +\text{oo}.$$

Case 1 implies $\lim_{t\to\infty} p(t)x^{(n-1)}(t)=-\infty$ Thus, $x^{(n-1)}(t)<0$ for all large t. Consequently, $x^{(n-2)}(t)$ is monotonic and positive for all large t, otherwise we would obtain the contradiction $\lim_{t\to\infty} x(t)=-\infty$. It follows that the assertion of the theorem is true in Case 1. In Case 2 we must have $\lim_{t\to\infty} p(t)x^{(n-1)}(t)=\mu$ exists and is finite. Let $\mu>0$. Then $x^{(n-1)}(t)>0$ eventually. Now there are two possibilities: either $x^{(n-2)}(t)>0$ or $x^{(n-2)}(t)<0$ for all large t. The second possibility proves our assertion. If the first one is true, then we must have $x(t)\to+\infty$ as $t\to\infty$. It follows that $x(g(t))\geqq\lambda>0$ and $H(t, x(g(t)))\geqq H(t,\lambda)>0$ for all $t\geqq(\text{some})t_2\geqq t_1$. The integral condition on H takes us back to Case 1, a contradiction. A completely analogous situation holds in the case $\mu<0$. Now let $\mu=0$. Then given $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$-\varepsilon < p(t')x^{(n-1)}(t') < \varepsilon, \quad \left|\int_{t'}^{t''}[1/p(t)]dt\right| < 1 \tag{2.2}$$

for every t', $t''\geqq\delta(\varepsilon)$. Dividing the first of (2.2) by $p(t')$ and integrating from t' to t'' we obtain

$$|x^{(n-2)}(t')-x^{(n-2)}(t'')| < \varepsilon, \quad t', t'' > \delta(\varepsilon). \tag{2 3}$$

By the Cauchy criterion for functions, we get that $\lim_{t\to\infty} x^{(n-2)}(t)$ exists and is finite. This completes the proof for $x(t)$ eventually positive. Similarly one can show the assertion for an eventually negative $x(t)$.

Singh considered in [11] the case $H(t, u)=a(t)u$, $g(t)=t-\tau(t)$, where $\tau(t)$ is bounded above,

$$\int_T^{\infty}|Q(t)|dt< +\infty, \quad \text{and} \quad \lim_{n\to\infty}\int_{a_n}^{b_n} a(t)dt= +\infty$$

for any sequences $\{a_n\}$, $\{b_n\}$, $b_n \geqq a_n \geqq T$, with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = +\infty$, and $\lim_{n\to\infty}(b_n - a_n) = +\infty$.

COROLLARY 1. *Let $n=2$ in Theorem 1. Then all nonoscillatory solutions of* (I) *tend to zero as $t\to\infty$ if H satisfies the additional assumption*

$$\lim_{t\to\infty}[1/p(t)]\int_T^t H(s, \pm k)ds = \pm\infty,$$

and $p(t)\geqq\lambda>0$ for $t\geqq T$, where λ is constant.

PROOF. Let $x(t)$ be a nonoscillatory solution such that $x(t)>0$ and $x(g(t))>0$ for $t\geqq t_1\geqq T$. From Theorem 1 we obtain that $\lim_{t\to\infty} x(t)=A$ exists and is finite. Let $A>0$. Then given ε with $0<\varepsilon<A$ there exists $t_2\geqq t_1$ such that

$$-\varepsilon < x(g(t))-A<\varepsilon, \qquad t\geqq t_2.$$

Consequently, $H(t, x(g(t)))\geqq H(t, A-\varepsilon)\geqq 0$ for every $t\geqq t_2$. Integrating now (I) from t_2 to $t\geqq t_2$ and dividing by $p(t)$ we obtain

$$(2.4) \quad x'(t) \leqq -[1/p(t)]\int_{t_2}^t H(s, A-\varepsilon)ds + (1/\lambda)\left|\int_{t_2}^t Q(s)ds\right| + (1/\lambda)p(t_2)|x'(t_2)|.$$

Thus, $x'(t)\to -\infty$ as $t\to\infty$, a contradiction to the positiveness of $x(t)$. It follows that $\lim_{t\to\infty} x(t)=0$ for $x(t)$ eventually positive, and an analogous proof covers the case for $x(t)$ eventually negative.

Singh obtained the conclusion of the above corollary in [11] from Theorem 1 there without any additional assumptions. Singh's Theorem 1 only ensures that $p(t)x'(t)$ tends to $-\infty$ as $t\to\infty$, but this fact is not enough to imply $\lim_{t\to\infty} x(t)=0$. Consequently, Singh needs additional assumptions to conclude Case 1 of Theorem 2 in [11].

THEOREM 2. *Consider* (I) *with the following assumptions:*

$$\lim_{t\to\infty}[1/p(t)]\int_T^t H(s, +k)ds = \pm\infty, \quad p(t) \geqq \lambda > 0,$$

$$\lim_{t\to\infty}\int_T^t\int_{u_n}^\infty\cdots\int_{u_3}^\infty [1/p(u_2)]\int_{u_2}^\infty Q(u_1)du_1du_2du_3\cdots du_n$$

exists and is finite,

where k is an arbitrary positive constant. Then every nonoscillatory solution of (I) *tends to zero as* $t\to\infty$.

PROOF. Let $x(t)$ be an eventually positive solution of (I) and assume that $\liminf_{t\to\infty} x(t) > 0$. Then there exists a constant $K>0$ such that $x(g(t))>K$ for every t (say) $\geq t_1 \geqq T$. Consequently, $H(t, x(g(t))) \geqq H(t,K)>0$ for $t \geq t_1$, and, by integration of (I) from t_1 to $t \geqq t_1$, we get

(2.5) $x^{(n-1)}(t)$

$$\leqq -[1/p(t)]\int_{t_1}^t H(s, K)ds + (1/\lambda)\left|\int_{t_1}^t Q(s)ds\right| + (1/\lambda)p(t_1)|x^{(n-1)}(t_1)|.$$

Thus, we obtain a contradiction by taking the limits of both sides as $t\to\infty$. It follows that $\liminf_{t\to\infty} x(t) = 0$. Now let

$$P(t) = \int_t^\infty\int_{u_n}^\infty\cdots\int_{u_3}^\infty [1/p(u_2)]\int_{u_2}^\infty Q(u_1)du_1du_2du_3\ldots du_n$$

for all $t \geqq t_1$, with t_1 chosen so that $x(t)>0$, $x(g(t))>0$, $t \geqq t_1$. Then letting $w(t) = x(t) - P(t)$ we get

$$[p(t)w^{(n-1)}(t)]' + H(t, w(g(t)) + P(g(t))) = 0. \tag{2.6}$$

Since $x(g(t)) = w(g(t)) + P(g(t)) > 0$ for $t \geqq t_1$, it follows that $p(t)w^{(n-1)}(t)$ is decreasing for $t \geqq t_1$. This implies that $w^{(n-1)}(t)$ is of fixed sign for all large t. Thus, $w(t)$ is monotonic for all large t. Since $x(t) = w(t) + P(t)$ and $\lim_{t\to\infty} P(t) = 0$, if follows that $\lim_{t\to\infty} x(t) = L$ exists and must equal zero because $\liminf_{t\to\infty} x(t) = 0$. A similar proof covers the case of an eventually negative $x(t)$.

It should be noted here that the integral condition on the function $Q(t)$ can be replaced by the condition that $P(t)$ be a solution of the equation

$$[p(t)u^{(n-1)}(t)]' = Q(t), \qquad t \geqq T$$

such that $\lim_{t\to\infty} P(t) = 0$, and $\left|\int_T^t Q(s)ds\right| \leqq K$ (constant). The above theorem does not contain, for $n=3$, Theorem 3 in Singh's paper. However, it does contain a special case of that theorem; namely, when the integral condition on the forcing term $Q(t)$ as above holds. No integrability assumption was made here on the function $1/p(t)$

In the following theorem we consider the equation

$$\text{(II)} \qquad x^{(n)} + q(t)x^{(n-1)} + H(t, x(g(t))) = 0$$

with $q(t) \leqq 0$. This equation was studied by Kartsatos and Onose [7] with $g(t) = t$, and by Naito [9] and Sficas [10]. None of the results of these papers contains the following because of the assumptions on $q(t)$.

Theorem 3. *Consider* (II) *with* $q: R_T \to (-\infty, 0]$ *and continuous. Then every bounded solution of* (II) *is oscillatory for n even, and oscillatory or tending monotonically to zero as $t \to \infty$ for n odd, if*

$$q(t) \geqq -M/t, \qquad \int_T^\infty t^{n-1} H(t, \pm\lambda)dt = \pm\infty$$

for some constant $M > 0$, *any constant* $\lambda > 0$, *and every* $t \geqq T$.

PROOF. Let $x(t)$ be such that $x(t) > 0$, $x(g(t)) > 0$ for all $t \geqq T$, and bounded. Then it follows from the Lemma in [7] (cf. also Naito [9]) that $x^{(n-1)}(t) > 0$ for $t \geqq t_1$. Now let n be even. Then $x'(t) > 0$ for $t \geqq t_1$. Let $t_2 \geqq t_1$ be such that $x(g(t)) > K > 0$ for $t \geqq t_2$, and some constant K. Now consider the function $t^{n-1}x^{(n-1)}(t), t \geqq t_2$. Differentiation of this function, and then integration from t_2 to t, taking into consideration (II), yields

$$\text{(2.7)} \qquad t^{n-1}x^{(n-1)}(t) - (n-1)\int_{t_2}^t s^{n-2}x^{(n-1)}(s)ds + \int_{t_2}^t s^{n-1}q(s)x^{(n-1)}(s)ds \leqq t_2^{n-1}x^{(n-1)}(t_2) - \int_{t_2}^t s^{n-1}H(s, K)ds.$$

The first member of this equation is bounded below by

$$\text{(2.8)} \qquad -(n-1+M)\int_{t_2}^t s^{n-2}x^{(n-1)}(s)ds.$$

Taking limits as $t \to \infty$ in (2.7), we get

$$\lim_{t\to\infty}\int_{t_2}^t s^{n-2}x^{(n-1)}(s)ds = +\infty.$$

The rest of the proof for n even follows now as in Theorem 1 in [4]. Similar arguments cover the case n odd and negative solutions.

This theorem can be extended to cover larger classes of functions H; for example, functions of the forms considered in [4]. It can be easily shown now that the conclusion of the above theorem holds for all solutions of (II), if $q(t) \geqq -k$ (for some positive constant k), $t \geqq T$, and

$$\int_T^{\infty} H(s, \pm\lambda)ds = \pm\infty$$

for every constant $\lambda > 0$.

In the following result, the damping $q(t)x^{(n-1)}(t)$ is treated as a "small" perturbation.

THEOREM 4. *Assume that* $q: R_T \to (-\infty, 0]$ *is continuous. Moreover, let*

$$-\int_T^{\infty} t^{n-1} q(t)dt < +\infty.$$

Furthermore, assume that all solutions of

$$\text{(III)} \qquad x^{(n)} + H(t, x(g(t))) = 0, \qquad n \text{ even},$$

oscillate. Then for every nonoscillatory solution $x(t)$ *of* (II) *we have* $\lim_{t\to\infty} x(t) = 0$.

PROOF. Let $x(t), x(g(t)) > 0, t \geqq t_1 \geqq T$. Let $H(t, x(g(t))) = f(t)$, $x^{(n-1)}(t) = y(t), t \geqq t_1$. Then we have

$$\text{(2.9)} \qquad y' + q(t)y + f(t) = 0.$$

Solving this equation we obtain

$$\text{(2.10)} \qquad y(t) = \exp\left[-\int_{t_1}^{t} q(s)ds\right]\left[y(t_1) - \int_{t_1}^{t} f(u)\exp\left[\int_{t_1}^{u} q(s)ds\right]du\right]$$

$$\leqq y(t_1)\exp\left[-\int_{t_1}^{t} q(s)ds\right] \leqq y(t_1)\exp\left[-\int_{t_1}^{\infty} q(t)dt\right].$$

Since, again by the Lemma in [7], $x^{(n-1)}(t) > 0$, it follows that $x^{(n-1)}(t)$ is bounded. Thus, the equation

$$u^{(n)}(t) = -q(t)x^{(n-1)}(t)$$

has a solution $u(t)$ with $\lim_{t\to\infty} u(t) = 0$. In fact, this solution is the function

$$u_0(t) \equiv \int_t^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} q(s)x^{(n-1)}(s)ds.$$

Now we can consider the transformation $w(t) = x(t) - u_0(t)$, which takes (II) into

$$w^{(n)}+H(t,\ w(g(t))+u_0(g(t)))=\circ \tag{2.11}$$

It is easy to show now (cf. Kartsatos [6]) that the existence of a positive solution of (2.11) implies the existence of a positive solution to (III) for all large t, a contradiction to our assumption. Similarly for a negative solution $x(t)$. Consequently, if $x(t)$ is positive (negative), $w(t)=x(t)-u_0(t)$ is negative (positive) for all large t. This implies in both cases: $\lim_{t\to\infty} x(t)=0$.

The above theorem remains true for all bounded solutions of (II), if we assume, in addition to the integral condition on $q(t)$, that all bounded solutions of (III) are oscillatory. This last result improves Theorem 1 in [7].

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