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Principal Oriented Bordism Modules of Finite Subgroups of S^3

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Introduction

The principal oriented bordism module $\Omega_*(G)$ of a compact Lie group G is defined to be the module of all equivariant bordism classes of closed principal oriented (smooth) G-manifolds, and is a module over the oriented bordism ring Ω_* of R. Thom (cf. [2]).

Let G be a finite subgroup of the unit sphere S^3 in the quaternion field $\boldsymbol{H}$. Then, it is well known that G is a cyclic group Z_m, a binary dihedral group $D^*(4m)$ or a binary polyhedral group T^*, O^*, or I^* (cf. (1.1)).

The Ω_*-module structure of $\Omega_*(G)$ is determined by P. E. Conner and E. E. Floyd [2, Ch. VII] for $G=Z_{p^k}$ (p: odd prime), and by K. Shibata [7, §§1–4] for $G=Z_2$. Also, it is proved by N. Hassani [3] that there is an isomorphism $\Omega_*(Z_{mm'})=\Omega_*(Z_m)\otimes_{\Omega_*}\Omega_*(Z_{m'})$ if m and m' are relatively prime.

Furthermore, in the recent papers K1 and K2, we have determined $\Omega_*(G)$ for $G=Z_{2^k}$, $k\geqq 2$ (K1-Theorem 2.18), and for $G=H_m=D^*(2^{m+1})$, $m\geqq 2$ (K2-Theorem 8.12). As a continuation to these papers, we study in this paper the Ω_*-module structures of $\Omega_*(G)$ for the remaining finite subgroups G of S^3, that is,

$$G = D^*(2^{m+1}t) \ (t\text{: odd} \geqq 3,\ m \geqq 1), \quad T^*, \quad O^* \quad \text{and} \quad I^*.$$

Our results are stated in Theorems 4.8, 5.9, 6.10 and 7.8 as follows:

$$\tilde{\Omega}_*(D^*(2^{m+1}t)) = D\tilde{\Omega}_*(D^*(2^{m+1})) \oplus D'\mathfrak{Z}_{t,1},$$

$$\tilde{\Omega}_*(T^*) = T(\mathfrak{L}_2 \oplus \mathfrak{W}_2) \oplus T'\tilde{\Omega}_*(Z_3),$$

$$\tilde{\Omega}_*(O^*) = \mathfrak{O} \oplus O(\mathfrak{W}_2 \oplus \mathfrak{Q}_2),$$

$$\tilde{\Omega}_*(I^*) = \mathfrak{I} \oplus I_4\mathfrak{G}_{2,1} \oplus I_3\mathfrak{Z}_{3,1} \oplus I_5\mathfrak{Z}_{5,1}.$$

Here

$$\tilde{\Omega}_*(D^*(2^{m+1})) = \mathfrak{L}_m \oplus \mathfrak{W}_m \oplus \mathfrak{Q}_m \oplus \mathfrak{Q}'_m \quad \text{(cf. K2-Theorem 8.12)},$$

$$\tilde{\Omega}_*(Z_t) = \mathfrak{Z}_{t,0} \oplus \mathfrak{Z}_{t,1} \quad \text{(cf. Theorem 3.8)},$$

$$\tilde{\Omega}_*(Z_4) = \mathfrak{H}_2 \oplus \mathfrak{G}_{2,0} \oplus \mathfrak{G}_{2,1} \quad \text{(cf. Theorem 7.6)},$$

and $\mathfrak{O}$ and $\mathfrak{J}$ are the submodules of $\tilde{\Omega}_*(O^*)$ and $\tilde{\Omega}_*(I^*)$ generated by the classes $[O^*, S^{4n+3}]$ $(n \geqq 0)$ and $3^{n+1}5^{[n/2]+1}[I^*, S^{4n+3}]$ $(n \geqq 0)$, respectively, where $[G, S^{4n+3}]$ is the class of the G-manifold S^{4n+3} with the diagonal G-action. Further, D, D', T, T', O and I_k $(k=3, 4, 5)$ are the extensions induced by the natural inclusions

$$D^*(2^{m+1}) \xrightarrow{D} D^*(2^{m+1}t) \xleftarrow{D'} Z_t, \qquad D^*(8) \xrightarrow{T} T^* \xleftarrow{T'} Z_3$$

$$D^*(8) \xrightarrow{O} O^*, \qquad Z_k \xrightarrow{I_k} I^* \qquad (k = 3, 4, 5),$$

(cf. (4.1), (5.1), (1.9), (7.2)), and these extensions are monomorphic on the submodules written just behind them.

We prepare the results for the homologies of $G=D^*(2^{m+1}t)$, T^*, O^* and I^* in § 1. The unoriented bordism modules $\mathfrak{N}_*(G)$ are determined in § 2 by Theorems 2.4, 2.7 and 2.11. After preparing some preliminary results in § 3, we determine $\Omega_*(G)$ in §§ 4–7. Throughout this paper, we often use the notations and the results of the recent papers K1 and K2.

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§ 1. The homologies of finite subgroups of S^3

Let G be a finite subgroup of the unit sphere S^3 in the quaternion field $\boldsymbol{H}$. Then it is well-known that

(1.1) (J. A. Wolf [10, Th. 2.6.7]) *G is a cyclic group Z_m of order m $(\geqq 1)$, a binary dihedral group $D^*(4m)$ of order $4m$ $(\geqq 8)$, a binary tetrahedral group T^* of order 24, a binary octahedral group O^* of order 48, or a binary icosahedral group I^* of order 120. Further, two finite subgroups of S^3 are isomorphic if and only if they are conjugate in S^3.*

Here, the above groups are given as follows (cf. [10, 5.3, 7.1], [6, § 3]):

$$Z_m = Z_m[x] = [x \colon x^m = 1],$$

$$D^*(4m) = D^*(4m)[x, y] = [x, y \colon x^m = y^2, xyx = y],$$

(1.2) $$T^* = [x, y, z \colon x^3 = y^4 = 1, y^2 = z^2, xyx^{-1} = z, xzx^{-1} = yz, yzy^{-1} = z^{-1}],$$

$$O^* = [x, y, z, t \colon x^3 = y^4 = 1, y^2 = z^2 = t^2, xyx^{-1} = z, xzx^{-1} = yz, yzy^{-1} = z^{-1}, txt^{-1} = x^{-1}, tyt^{-1} = zy, tzt^{-1} = z^{-1}],$$

$$I^* = [x, y: x^2 = (xy)^3 = y^5, x^4 = 1],$$

where $G=G[x_1,\dots, x_n]=[x_1,\dots, x_n: R_1,\dots, R_k]$ means that G is generated by the elements $x_1,\dots, x_n$ with the relations $R_1,\dots, R_k$.

We notice that $D^*(2^{m+1})$ is denoted in K2 by

$$H_m = D^*(2^{m+1}) \qquad (m \geqq 2),$$

which is called the generalized quaternion group.

Now, we consider the (co)homology group of G. According to [6, §2], we see that

(1.3) *$H^*(G; Z)$ has period* 4,

since G acts freely on S^3, and

(1.4) [1, p. 237] $\hat{H}^1(G; Z) = \hat{H}^{-1}(G; Z) = 0,$

$$\hat{H}^0(G; Z) = Z_g \qquad (g = \sharp G), \qquad \hat{H}^{-2}(G; Z) = G/D(G),$$

where $\sharp G$ is the order of G and $D(G)$ is the commutator subgroup of G.

We can see easily by (1.2) that

$$\begin{aligned} &D(D^*(4m)) = Z_m[x^2], \quad D(T^*) = D^*(8)[y, z], \\ &D(O^*) = T^*[x, y, z], \quad D(I^*) = I^*. \end{aligned} \tag{1.5}$$

Therefore, we have the following results by (1.3), (1.4) and (1.5).

Proposition 1.6.

$$\text{(i)} \quad \tilde{H}_n(D^*(4m); Z) = \begin{cases} Z_{4m} & \text{if } n \equiv 3\ (4), \\ Z_2 \oplus Z_2 & \text{if } n \equiv 1\ (4), \quad m\text{: even} \geqq 2, \\ Z_4 & \text{if } n \equiv 1\ (4), \quad m\text{: odd} \geqq 3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(ii)} \quad \tilde{H}_n(T^*; Z) = \begin{cases} Z_{24} & \text{if } n \equiv 3\ (4), \\ Z_3 & \text{if } n \equiv 1\ (4), \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(iii)}\quad \tilde{H}_n(O^*; Z) = \begin{cases} Z_{48} & \text{if}\quad n \equiv 3\ (4), \\ Z_2 & \text{if}\quad n \equiv 1\ (4), \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(iv)}\quad \tilde{H}_n(I^*; Z) = \begin{cases} Z_{120} & \text{if}\quad n \equiv 3\ (4), \\ 0 & \text{otherwise.} \end{cases}$$

Let H be a subgroup of G. For the induced homomorphism

$$i_*: H_n(H; \Lambda) \longrightarrow H_n(G; \Lambda) \qquad (\Lambda = Z \text{ or } Z_2)$$

of the inclusion $i: H \subset G$, we obtain the following properties.

LEMMA 1.7. (i) *For $n=4k+3$ and $\Lambda=Z$, i_* is monomorphic.*
(ii) *For $n=4k$ and $\Lambda=Z_2$, i_* is monomorphic.*

PROOF. (i) We consider the unit sphere S^{4k+3} in $\boldsymbol{H}^{k+1}$ as the S^3-manifold by the diagonal action $q(q_0,\dots, q_k)=(qq_0,\dots, qq_k)$. Then, $\cup_k S^{4k+3}/K$ is the classifying space of a subgroup K of S^3, and S^{4k+3}/K is its $(4k+3)$-skeleton.

Consider the covering space

$$G/H \longrightarrow S^{4k+3}/H \xrightarrow{i} S^{4k+3}/G, \qquad (i: H \subset G).$$

Then there is the commutative diagram

$$\begin{array}{ccc} H_{4k+3}(S^{4k+3}/H; Z) & \xrightarrow{i_*} & H_{4k+3}(S^{4k+3}/G; Z) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j'_*} \\ Z_{\sharp H} = H_{4k+3}(\cup_k S^{4k+3}/H; Z) & \xrightarrow{i_*} & H_{4k+3}(\cup_k S^{4k+3}/G; Z) = Z_{\sharp G}, \end{array}$$

where j and j' are the inclusions. For the upper i_*, we have $i_*(1)=r1$, where 1's are the fundamental classes and $r=\sharp(G/H)$, since i is the r-fold covering. Therefore, we have the desired result since $j_*(1)$ and $j'_*(1)$ are the generators of $Z_{\sharp H}$ and $Z_{\sharp G}$, respectively.

(ii) When $k=0$, (ii) is clear. Assume that $k \geqq 1$ and consider the commutative diagram

$$\begin{array}{ccccc} H_{4k}(H; Z) & \longrightarrow & H_{4k}(H; Z_2) & \xrightarrow{\beta} & H_{4k-1}(H; Z) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ H_{4k}(G; Z) & \longrightarrow & H_{4k}(G; Z_2) & \xrightarrow{\beta} & H_{4k-1}(G; Z) \end{array}$$

of the Bockstein exact sequence. Since $H_{4k}(K; Z)=\hat{H}^{-1}(K; Z)=0$ for $K=G$ and H by (1.4), β's are monomorphic and hence we see (ii) by (i). q.e.d.

We notice the following lemma which is proved easily.

LEMMA 1.8. *$\varphi = 0$ if and only if $\varphi^* = 0$, where φ: $Z_4 \to Z_2$ is a homomorphism and φ^*: $Ext(Z_2, Z) \to Ext(Z_4, Z)$ is its induced homomorphism.*

Now, we consider the inclusions

$$
\begin{aligned}
&O\colon D^*(8)[x, y] \longrightarrow O^*[x, y, z, t], \quad O(x) = t, \quad O(y) = z,\\
(1.9)\qquad &Oi_2\colon Z_4 \longrightarrow O^*[x, y, z, t], \quad \operatorname{Im} Oi_2 = Z_4[t],\\
&Oj_2\colon Z_4 \longrightarrow O^*[x, y, z, t], \quad \operatorname{Im} Oj_2 = Z_4[z],
\end{aligned}
$$

(cf. (1.2)), where i_2, j_2: $Z_4 \to D^*(8) = H_2$ are the inclusions of K2–(1.6), (1.9). Then

LEMMA 1.10. *For the induced homomorphisms*

$$(Oi_2)_*, (Oj_2)_*\colon H_{4k+1}(Z_4; \Lambda) \longrightarrow H_{4k+1}(O^*; \Lambda), \quad (\Lambda = Z \text{ or } Z_2),$$

we have $(Oi_2)_* \neq 0$ and $(Oj_2)_* = 0$.

PROOF. Set $O_1 = Oj_2$ and $O_2 = Oi_2$. By [1, Prop. 11.3] and (1.3), if $a \in H^4(O^*; Z)$ is the maximal generator, then so is $a_i = O_{i*}a \in H^4(Z_4; Z)$. Hence, by [1, Prop. 11.1], the universal coefficient theorem and (1.4), we have the commutative diagram

$$
\begin{array}{ccccc}
H^2(O^*; Z) & \xrightarrow[\cong]{\cup a^k} & H^{4k+2}(O^*; Z) & \cong & Ext(H_{4k+1}(O^*; Z), Z)\\
\downarrow{\scriptstyle O_i^*} & & \downarrow{\scriptstyle O_i^*} & & \downarrow{\scriptstyle (O_{i*})^*}\\
H^2(Z_4; Z) & \xrightarrow[\cong]{\cup a_i^k} & H^{4k+2}(Z_4; Z) & \cong & Ext(H_{4k+1}(Z_4; Z), Z).
\end{array}
$$

For $k=0$, we see that $O_{1*}=0$ and $O_{2*} \neq 0$ by (1.4), (1.5) and Lemma 1.8. Therefore we have the desired results for $\Lambda = Z$ by the above diagram and Lemma 1.8. The results for $\Lambda = Z_2$ follow from these results, since the mod 2 reduction $H_{4k+1}(O^*; Z) \to H_{4k+1}(O^*; Z_2)$ is isomorphic by Proposition 1.6 (iii). q.e.d.

LEMMA 1.11. *Consider the induced homomorphisms*

$$
\begin{aligned}
&O_*\colon H_{4n+2}(D^*(8); Z_2) \longrightarrow H_{4n+2}(O^*; Z_2),\\
&k_{3*}\colon H_{4n+2}(D^*(8); Z_2) \longrightarrow H_{4n+2}(D^*(16); Z_2),
\end{aligned}
$$

of O in (1.9) *and $k_3 = k_{2,3}$: $D^*(8) \to D^*(16)$ of* K2–(1.13). *If $a \in H_{4n+2}(D^*(8); Z_2)$ satisfies $a \neq 0$ and $k_{3*}(a) = 0$, then $O_*(a) \neq 0$.*

PROOF. By K2-Lemmas 1.3, 1.7, 1.10 and 1.14, we see that

$$H_{4n+1}(D_*(8); Z) = Z_2[i_{2*}(z)] \oplus Z_2[j_{2*}(z)],$$

$$k_{3*}i_{2*}(z) = 0, \qquad k_{3*}j_{2*}(z) \neq 0,$$

where z is a generator of $H_{4n+1}(Z_4; Z)$. On the other hand,

$$O_*i_{2*}(z) \neq 0, \qquad O_*j_{2*}(z) = 0$$

by the above lemma. Hence we have the lemma using the Bockstein homomorphisms $H_{4n+2}(G; Z_2) \to H_{4n+1}(G; Z)$ for $G = D^*(8)$, O^* and $D^*(16)$. q.e.d.

§2. The unoriented bordism module $\mathfrak{N}_*(G)$

In this section, we study the principal unoriented bordism modules $\mathfrak{N}_*(G)$ for $G = D^*(4m)$, T^*, O^* and I^* of (1.2).

Consider the Thom epimorphism

$$\text{(2.1)} \qquad \mu: \mathfrak{N}_*(G) \longrightarrow H_*(G; Z_2)$$

of [2, (8.1)] and the isomorphism

$$\text{(2.2)} \qquad h: \mathfrak{N}_* \otimes H_*(G; Z_2) \xrightarrow[\approx]{} \mathfrak{N}_*(G)$$

of [2, (19.3)] such that $h(1 \otimes c_i) = C_i$, where $\{c_i\}$ is a basis of $H_*(G; Z_2)$ and $\mu C_i = c_i$. By using μ and h, $\mathfrak{N}_*(G)$ for $G = Z_{2^m}$ and $D^*(2^{m+1}) = H_m$ $(m \geqq 2)$ are studied in K1–§1 and K2–§2.

Now, we consider the binary dihedral group

$$D^*(2^{m+1}t), \quad t: \text{odd} \geqq 3, \quad m \geqq 1.$$

By (1.2), we see easily that there exists a split exact sequence

$$\text{(2.3)} \quad 1 \longrightarrow Z_t[x^{2^m}] \xrightarrow{D'} D^*(2^{m+1}t)\,[x, y] \underset{D}{\overset{p}{\rightleftarrows}} D^*(2^{m+1})[x^t, y] \longrightarrow 1.$$

Here, if $m = 1$, then $D^*(4)[x^t, y] = Z_4[y]$.

THEOREM 2.4. *The extension homomorphism*

$$D: \mathfrak{N}_*(D^*(2^{m+1})) \xrightarrow[\approx]{} \mathfrak{N}_*(D^*(2^{m+1}t)) \qquad (t: \mathit{odd} \geqq 3,\ m \geqq 1)$$

of $\mathfrak{N}_*$-modules is isomorphic. The domain $\mathfrak{N}_*(D^*(2^{m+1}))$ $(m \geqq 2)$ or $\mathfrak{N}_*(Z_4)$ $(m = 1)$ is given in K2-*Theorem* 2.3 *or* K1-*Theorem* 1.22, *respectively.*

PROOF. By the exact sequence (2.3), we see that D is monomorphic. Hence we have the theorem by (2.2), since $H_*(D^*(2^{m+1}); Z_2) = H_*(D^*(2^{m+1}t); Z_2)$ by Proposition 1.6 (i). q.e.d.

Consider the principal G-manifolds

$$(2.5)\qquad \begin{aligned} &(G,\ S^{4n+3}),\quad q(q_0,\dots,q_n)=(qq_0,\dots,qq_n)\quad (q,\ q_i\in \boldsymbol{H}),\\ &i(a,\ S^n)\qquad (i\colon Z_2\subset G),\end{aligned}$$

for any finite subgroup G of S^3 of even order, where (a, S^n) is the Z_2-manifold of K1–(1.5) with the antipodal action. Then we have the following

LEMMA 2.6. *For the epimorphism μ of* (2.1), *the μ-images*

$$\mu[G,\ S^{4n+3}]\in H_{4n+3}(G;\ Z_2)=Z_2,\ \ \mu i[a,\ S^{4n}]\in H_{4n}(G;\ Z_2)=Z_2$$

are not zero.

PROOF. Since S^{4n+3}/G is the $(4n+3)$-skeleton of the classifying space $BG=\cup_n S^{4n+3}/G$, we have $\mu[G, S^{4n+3}]\neq 0$ by the definition of μ. Since $\mu[a, S^{4n}]\neq 0$ in $H_{4n}(Z_2; Z_2)=Z_2$ by [2, Th. 23.2], we see $\mu i[a, S^{4n}]=i_*\mu[a, S^{4n}]\neq 0$ by Lemma 1.7 (ii). q.e.d.

For the case $G=T^*$ or I^*, we see that $H_n(G; Z_2)=0$ if $n=4m+1,\ 4m+2$ by Proposition 1.6. Hence we have the following theorem by (2.2) and the above lemma.

THEOREM 2.7. *If $G=T^*$ or I^*, then the principal unoriented bordism module $\mathfrak{N}_*(G)$ is a free $\mathfrak{N}_*$-module with basis $\{[G, S^{4n+3}],\ i[a, S^{4n}]\colon n\geqq 0\}$, where the bordism classes are those of* (2.5).

By considering $Z_m=Z_m[\exp(2\pi i/m)]\subset S^1\subset \boldsymbol{C}$, we have the Z_m-manifold

$$(2.8)\qquad (Z_m,\ S^{2n+1}),\quad c(c_0,\dots,c_n)=(cc_0,\dots,cc_n)\quad (c,\ c_i\in \boldsymbol{C}),$$

which is denoted by (T, S^{2n+1}) in K1–(1.6) or (T_l, S^{2n+1}) in K2–(2.8) for $m=2^l$.

By using this manifold and the $D^*(8)$-manifold $(\beta_2, S^{2n+1}\times S^{2n+1})$ of K2–(2.10), we have the following O^*-manifolds:

$$(2.9)\qquad \begin{aligned} &(O^*,\ S^{4n+3}),\quad i(a,\ S^n)\quad (i\colon Z_2\subset O^*),\quad (\text{in } (2.5)),\\ &Oi_2(Z_4,\ S^{2n+1})\quad (Oi_2\colon Z_4\subset O^* \text{ of } (1.9)),\\ &O(\beta_2,\ S^{2n+1}\times S^{2n+1})\quad (O\colon D^*(8)\subset O^* \text{ of } (1.9)).\end{aligned}$$

LEMMA 2.10. *Consider the epimorphism μ of* (2.1) *for $G=O^*$. Then*

$$\mu Oi_2[Z_4,\ S^{4n+1}]\in H_{4n+1}(O^*;\ Z_2)=Z_2,$$

$$\mu O[\beta_2,\ S^{2n+1}\times S^{2n+1}]\in H_{4n+2}(O^*;\ Z_2)=Z_2$$

are not zero.

PROOF. Since $\mu[Z_4, S^{4n+1}]\neq 0$ in $H_{4n+1}(Z_4; Z_2)=Z_2$, we have $\mu O i_2[Z_4, S^{4n+1}]=(Oi_2)_*\mu[Z_4, S^{4n+1}]\neq 0$ by Lemma 1.10. By K2-Lemmas 2.11 and 2.14, we have $\mu[\beta_2]\neq 0$ in $H_{4n+2}(D^*(8); Z_2)$ and $k_{3*}\mu[\beta_2]=0$, where $[\beta_2]=[\beta_2, S^{2n+1}\times S^{2n+1}]$. These and Lemma 1.11 show that $\mu O[\beta_2]=O_*\mu[\beta_2]\neq 0$. q.e.d.

By Proposition 1.6 (iii), Lemmas 2.6, 2.10 and (2.2), we have the following

THEOREM 2.11. *The principal unoriented bordism module* $\mathfrak{N}_*(O^*)$ *is a free* $\mathfrak{N}_*$*-module with basis*

$$\{[O^*, S^{4n+3}], i[a, S^{4n}], Oi_2[Z_4, S^{4n+1}], O[\beta_2, S^{2n+1}\times S^{2n+1}]: n \geqq 0\},$$

where the bordism classes are those of (2.9).

Finally, we notice the following

LEMMA 2.12. *Let* $G\supset H$ *be finite subgroups of* S^3 *of even order and* $i: H\subset G$ *be the inclusion. If* $\#(G/H)$ *is odd, then*

$$[G, S^{4n+3}] = i[H, S^{4n+3}] \quad in \quad \mathfrak{N}_{4n+3}(G),$$

where (G, S^{4n+3}) *and* (H, S^{4n+3}) *are the ones in* (2.5).

PROOF. Consider the principal G-bundle $G\times_H S^{4n+3}\to S^{4n+3}/H$. Then its classifying map is the projection $i: S^{4n+3}/H\to S^{4n+3}/G$. Since $\#(G/H)$ is odd, we see that $i_*: H_{4n+3}(S^{4n+3}/H; Z_2)\to H_{4n+3}(S^{4n+3}/G; Z_2)$ is isomorphic, and hence the Stiefel-Whitney number of $i[H, S^{4n+3}]$ coincides with the one of $[G, S^{4n+3}]$. Thus we have the lemma by [2, Th. 17.2]. q.e.d.

§3. Preliminaries to the oriented bordism module $\Omega_*(G)$

Consider the homomorphism

$$\mu: \tilde{\Omega}_n(G) \longrightarrow \tilde{H}_n(G; Z) \tag{3.1}$$

for a finite group G defined in the same way as μ of (2.1) (cf. [3, §6]). By [2, (7.2)], this is the edge homomorphism

$$\tilde{\Omega}_n(G) = \tilde{\Omega}_n(BG) = J_{n,0} \longrightarrow E^{\infty}_{n,0} \subset\cdots\subset E^2_{n,0} = \tilde{H}_n(BG; Z) = \tilde{H}_n(G; Z), \tag{3.2}$$

in the bordism spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}$ for $(BG, *)$. Further, it is known that

(3.3) [2, (14.1)] *The bordism spectral sequence* $\{E^r_{p,q}, d^r_{p,q}\}$ *is trivial* mod $\mathscr{C}$, *that is,* $\mathrm{Im}\, d^r_{p,q}$ *is an odd torsion group, where* $\mathscr{C}$ *is the class of odd tor-*

sion groups.

PROPOSITION 3.4. *Assume that* $\tilde{H}_{2n}(G; Z)=0$ *for all* $n \geqq 0$. *Then*
(i) (cf. [4], [5]) μ *of* (3.1) *is epimorphic.*
(ii) *The bordism spectral sequence for* $(BG, *)$ *is trivial.*

PROOF. We see easily (i) by using (3.2), (3.3) and the theorem of C. T. C. Wall for the structure of Ω_* (cf. K2-Theorem 3.6).
We see (ii) by (i) and [2, Th. 15.1]. q.e.d.

By (1.3), (1.4) and (ii) of this proposition, we have

COROLLARY 3.5. *If* G *is a finite subgroup of* S^3, *then the bordism spectral sequence for* $(BG, *)$ *is trivial.*

REMARK 3.6. *By* [4], G *has periodic cohomology if and only if* $\tilde{H}_{2n}(G; Z)=0$ *for all* $n \geqq 0$.

Further, we use the following

THEOREM 3.7 (cf. Th. 14.2]). *The canonical homomorphism*

$$\theta: \tilde{\Omega}_n(G) \longrightarrow \sum_{p+q=n} \tilde{H}_p(G; \Omega_q)$$

is isomorphic mod $\mathscr{C}$, *where* $\mathscr{C}$ *is the class of odd torsion groups.*

Now, according to [2] and [7], we recall the Ω_*-module structure of $\Omega_*(Z_t)$ (t: odd$\geqq 3$) as follows:

THEOREM 3.8 ([2, Th. 46.3], [7, Th. 6.3]). (i) *Consider the submodule* $\mathfrak{Z}_{t,\varepsilon}$ $(\varepsilon=0, 1)$ *of* $\tilde{\Omega}_*(Z_t)$ *generated by the elements*

$$\alpha_{4n+2\varepsilon+1}^{(t)} = [Z_t, S^{4n+2\varepsilon+1}] \quad (in\ (2.8)), \quad n \geqq 0.$$

Then, $\tilde{\Omega}_*(Z_t)$, t: *odd*$\geqq 3$, *is the direct sum*

$$\tilde{\Omega}_*(Z_t) = \mathfrak{Z}_{t,0} \oplus \mathfrak{Z}_{t,1},$$

and $\mathfrak{Z}_{t,\varepsilon}$ *is the quotient module of the free* Ω_*-*module* $\Omega_*\{\{\alpha_{4n+2\varepsilon+1}^{(t)}: n \geqq 0\}\}$ *by the submodule generated by the elements*

$$\beta_{4n+2\varepsilon+1}^{(t)} = \sum_{j=0}^{n} V_{2(n-j)}^{(t)} \alpha_{4j+2\varepsilon+1}^{(t)}, \quad n \geqq 0,$$

where the coefficients $V_{2l}^{(t)} \in \Omega_{4l}$ *are given in* [7, Prop. 6.7].
(ii) *Let* $t=\prod_i p_i^{k_i}$ $(k_i \geqq 1)$ *be the prime decomposition of* t. *Then the order* $a_{2n+1}^{(t)}$ *of* $\alpha_{2n+1}^{(t)}$ *in* $\tilde{\Omega}_*(Z_t)$ *is given by*

$$a_{2n+1}^{(t)} = \prod_i p_i^{k_i+l_i}, \qquad l_i = [(2n+1)/2(p_i-1)].$$

Let G be any finite subgroup of S^3. Consider the orbit manifold S^{4n+3}/G of the G-manifold (G, S^{4n+3}) in (2.5), the principal G-bundle $\xi=(S^{4n+3}, \pi, S^{4n+3}/G)$ and its associated 4-plane bundle $\hat{\xi}$ by the inclusion $G\subset S^3\subset O(4)$. Then, the following is known.

THEOREM 3.9 [8, Th. 3.1]. *The tangent bundle $\tau(S^{4n+3}/G)$ of S^{4n+3}/G satisfies*

$$\tau(S^{4n+3}/G) \oplus \theta^1 = (n+1)\hat{\xi},$$

where θ^1 is the trivial line bundle.

LEMMA 3.10 [8, Lemma 7.1]. *Let $w=\sum w_j \in H^*(BO(4); Z_2)$ be the universal Stiefel-Whitney class, and j: $S^3\subset O(4)$ be the inclusion. Then*

$$j^*w = 1+u,$$

where u is the generator of $H^4(BS^3; Z_2) = Z_2$.

LEMMA 3.11. *Let G be a finite subgroup of S^3 of even order, and i: $BG \to BS^3$ be the induced map of the inclusion $G\subset S^3$. Then i^*: $H^*(BS^3; Z_2)\to H^*(BG; Z_2)$ is monomorphic.*

PROOF. Let $\{E_r^{p,q}\}$ be the Serre cohomology spectral sequence of the bundle

$$S^3/G \longrightarrow BG \longrightarrow BS^3.$$

Then, by the easy calculations, we see that $E_\infty^{4n,0}=H^{4n}(BG; Z_2)$ and $E_2^{4n,0}=E_\infty^{4n,0}$. Thus we have the lemma. q.e.d.

By Theorem 3.9 and the above lemmas, we have

LEMMA 3.12. *The Stiefel-Whitney class $w(\tau(S^{4n+3}/G))$ of the tangent bundle $\tau(S^{4n+3}/G)$ is given by*

$$w(\tau(S^{4n+3}/G)) = (1+v)^{n+1} \quad \text{if} \quad n \geqq 1, \quad = 1 \quad \text{if} \quad n = 0,$$

where $v\in H^4(S^{4n+3}/G; Z_2) = Z_2$ is the generator.

PROPOSITION 3.13. *Let G be any finite subgroup of S^3 of even order. Then*

$$[S^{4n+3}/G] = 0 \quad in \quad \Omega_{4n+3}.$$

PROOF. Since $\dim(S^{4n+3}/G)$ is odd, all the Stiefel-Whitney and Pontrjagin numbers of S^{4n+3}/G are zero by the above lemma. Hence we have the proposition by [9, §8, Cor. 1]. q.e.d.

§4. $\Omega_*(D^*(4m))$

In K1 and K2, we have already studied the Ω_*-module structure of $\Omega_*(G)$ for $G=Z_{2^k}$ $(k\geqq 2)$ and $D^*(2^{m+1})$ $(m\geqq 2)$. In this section, we study the case that G is the binary dihedral group

$$D^*(2^{m+1}t) = D^*(2^{m+1}t)[x, y], \quad x = \exp(\pi i/2^{m-1}t), \quad y = j,$$

for $m\geqq 1$ and t: odd$\geqq 3$, by using the split exact sequence

$$(4.1) \qquad 1 \longrightarrow Z_t[x^{2^m}] \xrightarrow{D'} D^*(2^{m+1}) \underset{D}{\overset{p}{\rightleftarrows}} D^*(2^{m+1})[x^t, y] \longrightarrow 1$$

of (2.3), where $D^*(4)[x^t, y]=Z_4[y]$ if $m=1$.

For a finite abelian group A, denote by A_{even} and A_{odd} the 2-primary and odd components of A, respectively.

Lemma 4.2. (i) $\#\tilde{\Omega}_n(D^*(2^{m+1}t))_{\text{even}}=\#\tilde{\Omega}_n(D^*(2^{m+1}))$.

(ii) $$\#\tilde{\Omega}_n(D^*(2^{m+1}t))_{\text{odd}} = \begin{cases} \#\tilde{\Omega}_n(Z_t) & \text{if } n \equiv 3\ (4), \\ 1 & \text{otherwise.} \end{cases}$$

Proof. By the structure of Ω_* (cf. K2-Theorem 3.6) and Proposition 1.6, we see easily that E^2-terms of the bordism spectral sequence of $\tilde{\Omega}_*(BD^*(2^{m+1}t))$ satisfy

$$\tilde{H}_p(D^*(2^{m+1}t);\,\Omega_q)_{\text{even}} = \tilde{H}_p(D^*(2^{m+1});\,\Omega_q),$$

$$\tilde{H}_p(D^*(2^{m+1}t);\,\Omega_q)_{\text{odd}} = \begin{cases} \tilde{H}_p(Z_t;\,\Omega_q) & \text{if } p \equiv 3\ (4), \quad q \equiv 0\ (4), \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have the desired results by Corollary 3.5. q.e.d.

Proposition 4.3. *The extension homomorphism*

$$D\colon \tilde{\Omega}_*(D^*(2^{m+1})) \longrightarrow \tilde{\Omega}_*(D^*(2^{m+1}t)),$$

induced by the inclusion D in (4.1), *is monomorphic and*

$$\operatorname{Im} D = \tilde{\Omega}_*(D^*(2^{m+1}t))_{\text{even}}.$$

Proof. The former is clear by the split exact sequence of (4.1), and the latter is shown by this result and the above lemma. q.e.d.

Now, consider the $D^*(2^{m+1}t)$-bordism classes

$$\delta_{4n+3}^{m,t} = [D^*(2^{m+1}t),\ S^{4n+3}] \qquad (n \geqq 0), \tag{4.4}$$

where $(D^*(2^{m+1}t), S^{4n+3})$ are the ones in (2.5). Also, let

$$t_{D'}\colon \tilde{\Omega}_*(D^*(2^{m+1}t)) \longrightarrow \tilde{\Omega}_*(Z_t) \tag{4.5}$$

be the transfer homomorphism induced by the inclusion $D'\colon Z_t \subset D^*(2^{m+1}t)$ in (4.1), i.e., the homomorphism defined by restricting $D^*(2^{m+1}t)$-actions to Z_t.
Then, we see immediately the following

LEMMA 4.6. $\qquad t_{D'}\delta_{4n+3}^{m,t} = \alpha_{4n+3}^{(t)}$,
where $\alpha_{4n+3}^{(t)} = [Z_t, S^{4n+3}]$ *is the class in Theorem* 3.8 (i).

PROPOSITION 4.7. (i) *Let* $\mathfrak{D}_{m,t}$ *be the* Ω_**-submodule of* $\tilde{\Omega}_*(D^*(2^{m+1}t))$ *generated by* $2^{m+1}\delta_{4n+3}^{m,t}$, $n \geqq 0$, *where* $\delta_{4n+3}^{m,t}$ *is the class of* (4.4). *Then*

$$\mathfrak{D}_{m,t} = \tilde{\Omega}_*(D^*(2^{m+1}t))_{\mathrm{odd}} \approx \mathfrak{Z}_{t,1} \quad by \quad t_{D'} \quad of \quad (4.5).$$

Here $\mathfrak{Z}_{t,1}$ *is the submodule of* $\tilde{\Omega}_*(Z_t)$ *in Theorem* 3.8, *and so* $\mathfrak{D}_{m,t}$ *is the quotient module of the free* Ω_**-module* $\Omega_*\{\{2^{m+1}\delta_{4n+3}^{m,t}\colon n \geqq 0\}\}$ *by the submodule generated by the elements*

$$2^{m+1}\sum_{j=0}^{n} V_{2(n-j)}^{(t)}\delta_{4j+3}^{m,t}, \quad n \geqq 0,$$

where the coefficients $V_{2i}^{(t)}$ *are those in Theorem* 3.8 (i).
(ii) *Consider the extension*

$$D'\colon \tilde{\Omega}_*(Z_t) \longrightarrow \tilde{\Omega}_*(D^*(2^{m+1}t))$$

induced by the inclusion D' *in* (4.1). *Then*

$$D'\alpha_{4n+3}^{(t)} = 2^{m+1}\delta_{4n+3}^{m,t},$$

and so D' *maps* $\mathfrak{Z}_{t,1}$ *isomorphically onto* $\mathfrak{D}_{m,t}$.

PROOF. (i) We notice that $2^{m+1}\delta_{4n+3}^{m,t} \in \tilde{\Omega}_*(D^*(2^{m+1}t))_{\mathrm{odd}}$ by Theorem 3.7 and Proposition 1.6. Hence we see that $t_{D'}\tilde{\Omega}_{4n+3}(D^*(2^{m+1}t))_{\mathrm{odd}} = \mathfrak{Z}_{t,1} \cap \tilde{\Omega}_{4n+3}(Z_t)$ by the above lemma and the definition of $\mathfrak{Z}_{t,1}$. Thus we have (i) by Lemma 4.2 (ii)

(ii) We see easily by definition that the class $t_{D'}D'\alpha_{4n+3}^{(t)} \in \tilde{\Omega}_*(Z_t)$ is represented by Z_t $(=Z_t[x^{2^m}])$-manifold

$$D^*(2^{m+1}t) \times_{Z_t} S^{4n+3} = D^*(2^{m+1})[x^t, y] \times S^{4n+3}$$

with the Z_t-action given as follows:

$$a(x^{tl}y^{\varepsilon}, (q_0, .., q_n)) = (x^{tl}y^{\varepsilon}, a^{(1-2\varepsilon)}(q_0, ..., q_n)) \qquad (a = x^{2^m}),$$

for $0 \leqq l < 2^m$, $\varepsilon = 0, 1$. Also the equivariant diffeomorphism of $(Z_t[x^{-2^m}], S^{4n+3})$ onto (Z_t, S^{4n+3}) is defined by multiplying j. These show that $t_{D'}D'\alpha^{(t)}_{4n+3} = 2^{m+1}\alpha^{(t)}_{4n+3}$, and we obtain (ii) by the above lemma and (i). q.e.d.

Now, we have immediately the following main theorem of this section, by Propositions 4.3 and 4.7.

THEOREM 4.8. *The principal oriented bordism module $\tilde{\Omega}_*(D^*(2^{m+1}t))$ of the binary dihedral group $D^*(2^{m+1}t)$ ($m \geqq 1$, t: odd $\geqq 3$) is the direct sum*

$$\tilde{\Omega}_*(D^*(2^{m+1}t)) = D\tilde{\Omega}_*(D^*(2^{m+1})) \oplus \mathfrak{D}_{m,t},$$

where the summands are the 2-primary and odd components and are given in Propositions 4.3 and 4.7, respectively.

§5. $\Omega_*(T^*)$

In this section, we study the binary tetrahedral group T^*.

It is easy to see by (1.2) that there exists the split exact sequence

(5.1) $$1 \longrightarrow D^*(8)[y, z] \xrightarrow{T} T^*[x, y, z] \underset{T'}{\overset{p}{\rightleftarrows}} Z_3[x] \longrightarrow 1.$$

Now, we use the direct summands

(5.2) $\mathfrak{L}_2$, generated by $[D^*(8), S^{4n+3}]$, $n \geqq 0$,

and

(5.3) $\mathfrak{W}_2$, generated by $lE^{4n+3}W(\omega)$, $n \geqq 0$, $\omega \in \pi$, $(l: Z_2 \subset D^*(8))$,

of $\tilde{\Omega}_*(D^*(8))$, which are given in K2-Theorem 4.8 (i) and (iii). The r-images of the classes in (5.3) satisfy the following equalities by K2-Lemma 4.5:

(5.4) $$rlE^{4n+3}W(\omega) = rg(\omega)\textstyle\sum_{j=0}^{2n+2} a_{2j}l[a, S^{4(n+1)-2j}], \qquad a_0 = 1,$$

where $r: \Omega_*(\ ) \to \mathfrak{N}_*(\ )$ is the orientation ignoring homomorphism.

LEMMA 5.5. (i) $\qquad \sharp\tilde{\Omega}_n(T^*)_{\text{odd}} = \sharp\tilde{\Omega}_n(Z_3)$.

(ii) $\qquad \sharp\tilde{\Omega}_n(T^*)_{\text{even}} = \sharp((\mathfrak{L}_2 \oplus \mathfrak{W}_2) \cap \tilde{\Omega}_n(D^*(8)))$.

PROOF. By Corollary 3.5, the bordism spectral sequence of $\tilde{\Omega}_*(G)$ is trivial for $G = T^*$, Z_3 or $D^*(8)$. Also, by the structure of Ω_* (cf. K2-Theorem 3.6) and Proposition 1.6, we see that

$$\tilde{H}_p(T^*;\ \Omega_q)_{\text{odd}} = \tilde{H}_p(Z_3;\ \Omega_q),$$

$$\tilde{H}_p(T^*;\ \Omega_q)_{\text{even}} = \begin{cases} \tilde{H}_p(D^*(8);\ \Omega_q) & \text{if} \quad p \equiv 3 \quad \text{or} \quad 0\ (4), \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have (i) and $\#\tilde{\Omega}_n(T^*)_{\text{even}}$ is equal to

$$\#(\textstyle\sum_{k\geqq 0} H_{4k+3}(D^*(8);\ \Omega_{n-4k-3}) \oplus \sum_{k>0} H_{4k}(D^*(8);\ \Omega_{n-4k}))$$
$$= \#((\mathfrak{L}_2 \cap \tilde{\Omega}_n(D^*(8))) \oplus (\mathfrak{W}_2 \cap \tilde{\Omega}_n(D^*(8))))$$

by K2-Proposition 4.1 (ii). q. e. d.

PROPOSITION 5.6. *The extension*

$$T\colon \tilde{\Omega}_*(D^*(8)) \longrightarrow \tilde{\Omega}_*(T^*),$$

induced by the inclusion T in (5.1), *is monomorphic on the submodule* $\mathfrak{L}_2 \oplus \mathfrak{W}_2$ *of* (5.2) *and* (5.3).

PROOF. (i) In the first, we prove that T is monomorphic on $\mathfrak{L}_2$. Consider the Thom homomorphism $\mu\colon \tilde{\Omega}_*(T^*) \to \tilde{H}_*(T^*;\ Z)$ of (3.1). Then $\mu T[D^*(8), S^{4n+3}]$ is a generator of the 2-primary component Z_8 of $H_{4n+3}(T^*;\ Z) = Z_{24}$ by ($*$) in the proof of K2-Lemma 4.3 (i), Proposition 1.6 and Lemma 1.7. Hence, $T[D^*(8), S^{4n+3}]$ is of order 8 by K2-Theorem 4.8 (i), and we see that

$$(*) \qquad xT[D^*(8), S^{4n+3}] = 0 \quad \textit{if and only if} \quad x \in 8\Omega_*,$$

in the same way as the proof of K2-Lemma 4.3 (ii).

Consider the Smith homomorphism

$$(5.7) \qquad \Delta\colon \Omega_n(G) \longrightarrow \Omega_{n-4}(G),$$

which is an Ω_*-module homomorphism, defined as follows (cf. K2–(4.6)): For a principal oriented G-manifold (G, M^n), we can take a differentiable equivariant map $\varphi\colon (G, M^n) \to (G, S^{4N+3})$ to the G-manifold (G, S^{4n+3}) in (2.5) for $4N+3 > n$, which is transverse regular on S^{4N-1}, since S^{4N+3}/G is the $(4N+3)$-skeleton of BG. Then Δ is defined by

$$\Delta[G, M^n] = [G, \varphi^{-1}(S^{4N-1})].$$

It is easy to see that

$$\Delta T[D^*(8), S^{4j+3}] = T[D^*(8), S^{4j-1}] \quad \text{if} \quad j \geqq 1, \ = 0 \quad \text{if} \quad j = 0.$$

Assume that

$$\sum_{j=0}^{n} x_j T[D^*(8), S^{4j+3}] = 0 \qquad (x_j \in \Omega_*).$$

Then, by applying $\Delta^n=\Delta\circ\cdots\circ\Delta$ (n-times), we have $x_n T[D^*(8), S^3]=0$ and so $x_n \in 8\Omega_*$ by $(*)$. Thus $x_j \in 8\Omega_*$ for $0\leqq j\leqq n$, and we see that T is monomorphic on $\mathfrak{L}_2$ by K2-Theorem 4.8 (i).

(ii) Since r: $\tilde{\Omega}_*(D^*(8))\to\mathfrak{N}_*(D^*(8))$ is monomorphic on $\mathfrak{W}_2$ by K2-Lemma 4.5, we see that $T\circ r=r\circ T$: $\tilde{\Omega}_*(D^*(8))\to\mathfrak{N}_*(T^*)$ is also monomorphic on $\mathfrak{W}_2$ by (5.4), Theorem 2.7 and K2-Theorem 2.13.

(iii) Finally, we prove the proposition. Assume that

$$l+w=0 \qquad (l\in T\mathfrak{L}_2,\ w\in T\mathfrak{W}_2).$$

Since $T[D^*(8), S^{4n+3}]=[T^*, S^{4n+3}]$ in $\mathfrak{N}_{4n+3}(T^*)$ by Lemma 2.12, we see that $rl=0=rw$ by the T-images of the equalities of (5.4) and Theorem 2.7. Thus $w=0$ by (ii). Therefore we have the proposition by (i) and (ii). q.e.d.

By the split exact sequence (5.1) and Lemma 5.5 (i), we have immediately the following

PROPOSITION 5.8. *The extension homomorphism*

$$T'\colon \tilde{\Omega}_*(Z_3) \longrightarrow \tilde{\Omega}_*(T^*),$$

induced by the inclusion $T'\colon Z_3\subset T^*$ *in* (5.1), *is monomorphic, and*

$$\operatorname{Im} T' = \tilde{\Omega}_*(T^*)_{\mathrm{odd}}.$$

By the above two propositions and Lemma 5.5, we have immediately the following

THEOREM 5.9. *The principal oriented bordism module* $\tilde{\Omega}_*(T^*)$ *of the binary tetrahedral group* T^* *is the direct sum*

$$\tilde{\Omega}_*(T^*) = T(\mathfrak{L}_2\oplus\mathfrak{W}_2)\oplus T'\tilde{\Omega}_*(Z_3),$$

where the summands are the 2-*primary and odd components, and are given in Propositions* 5.6 *and* 5.8, *respectively.*

§6. $\Omega_*(O^*)$

In this section, we study the binary octahedral group O^*.

Consider the direct summands

(6.1) $\qquad \mathfrak{W}_2$ and $\mathfrak{Q}_2$ of $\tilde{\Omega}_*(D^*(8))$,

of (5.3) and K2-7.1, respectively, (cf. K2-Theorem 8.12).

LEMMA 6.2. (i) $\#\tilde{\Omega}_n(O^*)_{\mathrm{odd}} = \#\tilde{\Omega}_n(Z_3)$ *if* $n \equiv 3\ (4)$, $= 1$ *otherwise.*

(ii) $\#\tilde{\Omega}_n(O^*)_{\mathrm{even}} = \#\sum_j H_{4j+3}(O^*;\ \Omega_{n-4j-3})_{\mathrm{even}} \times \#((\mathfrak{W}_2 \oplus \mathfrak{Q}_2) \cap \tilde{\Omega}_n(D^*(8)))$.

PROOF. In the same way as the proof of Lemma 5.5, we see that

$$\tilde{H}_p(O^*;\ \Omega_q)_{\mathrm{odd}} = \tilde{H}_p(Z_3;\ \Omega_q) \quad \text{if} \quad p \equiv 3\ (4), \quad = 0 \quad \text{otherwise},$$

$$\#\tilde{H}_p(O^*;\ \Omega_q)_{\mathrm{even}} = \begin{cases} (\#\tilde{H}_p(D^*(8);\ \Omega_q))^{1/2} & \text{if} \quad p \equiv 1,\ 2\ (4), \\ \#\tilde{H}_p(D^*(8);\ \Omega_q) & \text{if} \quad p \equiv 0\ (4), \end{cases}$$

by Proposition 1.6. Therefore we have the lemma by K2-Proposition 4.9 (ii) and K2-Lemma 7.9 (i). q.e.d.

Consider the inclusion

$$(6.3) \qquad O'\colon Z_3[\exp(2\pi i/3)] \longrightarrow O^*[x,\ y,\ z,\ t], \qquad \operatorname{Im} O' = Z_3[x],$$

(cf. (1.2)), the transfer homomorphism

$$(6.4) \qquad t_{O'}\colon \Omega_*(O^*) \longrightarrow \Omega_*(Z_3),$$

defined by restricting O^*-actions to Z_3 by the inclusion O', and the classes

$$(6.5) \qquad [O^*,\ S^{4n+3}] \in \tilde{\Omega}_{4n+3}(O^*) \qquad (\text{in } (2.5)).$$

LEMMA 6.6. $t_{O'}[O^*,\ S^{4n+3}] = \alpha^{(3)}_{4n+3}$,
where $\alpha^{(3)}_{4n+3} = [Z_3,\ S^{4n+3}]$ *is the class in Theorem* 3.8 (i).

PROOF. Since $Z_3[x] = q^{-1}Z_3 q$ for some $q \in \boldsymbol{H}$, where $Z_3 \subset S^1$, the equivariant diffeomorphism of $(Z_3[x],\ S^{4n+3})$ onto $(Z_3,\ S^{4n+3})$ is defined by multiplying q. Hence we have the lemma. q.e.d.

PROPOSITION 6.7. (i) *Let* $\mathfrak{O}_1$ *be the* Ω_**-submodule of* $\tilde{\Omega}_*(O^*)$ *generated by* $16[O^*,\ S^{4n+3}]$, $n \geqq 0$. *Then* $\mathfrak{O}_1 = \tilde{\Omega}_*(O^*)_{\mathrm{odd}}$ *and*

$$\mathfrak{O}_1 \approx \mathfrak{Z}_{3,1} \quad by \quad t_{O'} \quad of \quad (6.4).$$

Here $\mathfrak{Z}_{3,1}$ *is the submodule of* $\tilde{\Omega}_*(Z_3)$ *in Theorem* 3.8, *and so* $\mathfrak{O}_1$ *is the quotient module of the free* Ω_**-module* $\Omega_*\{\{16[O^*,\ S^{4n+3}]\colon n \geqq 0\}\}$ *by the submodule generated by the elements*

$$16\sum_{j=0}^{n} V^{(3)}_{2(n-j)}[O^*,\ S^{4j+3}], \qquad n \geqq 0.$$

(ii) $[O^*,\ S^{4n+3}] \in \tilde{\Omega}_*(O^*)$ *is of order* $2^4 3^{n+1}$.

PROOF. We see (i) in the same way as the proof of Proposition 4.7, by

Lemmas 6.2 (i) and 6.6. We see (ii), since $16[O^*, S^{4n+3}]$ is of order $a_{4n+3}^{(3)} = 3^{n+1}$ by (i), Lemma 6.6 and Theorem 3.8 (ii). q.e.d.

PROPOSITION 6.8. (i) *Let $\mathfrak{O}$ and $\mathfrak{O}_2$ be the Ω_*-submodules of $\tilde{\Omega}_*(O^*)$ generated by $[O^*, S^{4n+3}]$, $n \geqq 0$, and $3^{n+1}[O^*, S^{4n+3}]$, $n \geqq 0$, respectively. Then*

$$\mathfrak{O} = \mathfrak{O}_1 \oplus \mathfrak{O}_2, \quad \mathfrak{O}_1 = \mathfrak{O}_{\mathrm{odd}}, \quad \mathfrak{O}_2 = \mathfrak{O}_{\mathrm{even}},$$

where $\mathfrak{O}_1$ is the one of the above proposition, and $\mathfrak{O}_2$ is the quotient module of the free Ω_*-module $\Omega_*\{\{3^{n+1}[O^*, S^{4n+3}]: n \geqq 0\}\}$ by the submodule generated by the elements $2^4 3^{n+1}[O^*, S^{4n+3}]$, $n \geqq 0$.

(ii) $\#(\mathfrak{O}_2 \cap \tilde{\Omega}_n(O^*)) = \#\sum_j H_{4j+3}(O^*; \Omega_{n-4j-3})_{\mathrm{even}}$.

PROOF. (i) By (ii) of the above proposition, $o_{4n+3} = 3^{n+1}[O^*, S^{4n+3}]$ is of order 16. Since $[O^*, S^{4n+3}]$ is a generator of $H_{4n+3}(O^*; Z) = Z_{48}$, we see that

$$(*) \qquad x o_{4n+3} = 0 \quad \textit{if and only if} \quad x \in 16\Omega_*,$$

in the same way as the proof of K2-Lemma 4.3 (ii). By using $(*)$ and the Smith homomorphism Δ in (5.7), we see that

$$\textstyle\sum_j x_j o_{4j+3} = 0 \quad \text{if and only if} \quad x_j \in 16\Omega_*,$$

in the same way as the proof (i) of Proposition 5.6. Hence we see the structure of $\mathfrak{O}_2$ and (i).

(ii) There is a group homomorphism

$$\begin{aligned} \varphi\colon & \textstyle\sum_j H_{4j+3}(O^*; \Omega_{n-4j-3})_{\mathrm{even}} \\ & = \textstyle\sum_j H_{4j+3}(O^*; Z)_{\mathrm{even}} \otimes \Omega_{n-4j-3} \longrightarrow \mathfrak{O}_2 \cap \tilde{\Omega}_n(O^*) \end{aligned}$$

defined by $\varphi(z_j \otimes x) = x o_{4j+3}$, where $z_j = \mu o_{4j+3}$. Then, it is clear that φ is isomorphic by (i). q.e.d.

PROPOSITION 6.9. (i) *The extension*

$$O\colon \tilde{\Omega}_*(D^*(8)) \longrightarrow \tilde{\Omega}_*(O^*),$$

induced by the inclusion O in (1.9), *is monomorphic ont he submodule $\mathfrak{W}_2 \oplus \mathfrak{Q}_2$ of* (6.2).

(ii) $\tilde{\Omega}_*(O^*)_{\mathrm{even}} = \mathfrak{O}_2 \oplus O(\mathfrak{W}_2 \oplus \mathfrak{Q}_2)$.

PROOF. (i) Consider the orientation ignoring homomorphism $r\colon \Omega_*(\ ) \to \mathfrak{N}_*(\ )$. Then $r\colon \tilde{\Omega}_*(D^*(8)) \to \mathfrak{N}_*(D^*(8))$ is monomorphic on $\mathfrak{W}_2 \oplus \mathfrak{Q}_2$ by K2-Theorem 8.12 and Rohlin's Theorem [2, Th. 16.2]. Hence we see that

$$r \circ O = O \circ r\colon \tilde{\Omega}_*(D^*(8)) \longrightarrow \mathfrak{N}_*(O^*)$$

is monomorphic on $\mathfrak{W}_2 \oplus \mathfrak{Q}_2$, by Theorem 2.11, (5.4) and K2-Lemma 7.3 (ii). Thus we have (i).

(ii) Suppose

$$l+w=0 \quad \text{for} \quad l \in \mathfrak{O}_2 \quad \text{and} \quad w \in O(\mathfrak{W}_2 \oplus \mathfrak{Q}_2).$$

Then $rl=0=rw$ by Theorem 2.11, (6.4) and K2-Lemma 7.3 (ii). Hence $w=0$ by the proof of (i). Therefore we have (ii) by Lemma 7.2. q.e.d.

By the above three propositions, we have the following

THEOREM 6.10. *The principal oriented bordism module $\tilde{\Omega}_*(O^*)$ of the binary octahedral group O^* is the direct sum*

$$\tilde{\Omega}_*(O^*) = \mathfrak{O}_1 \oplus \mathfrak{O}_2 \oplus O(\mathfrak{W}_2 \oplus \mathfrak{Q}_2)$$

of the submodules given in the above propositions.

§7. $\Omega_*(I^*)$

Finally, we consider the binary icosahedral group I^*.

LEMMA 7.1. (i) $\#\tilde{\Omega}_n(I^*)_{\mathrm{odd}} = \#\tilde{\Omega}_n(Z_{15})$ *if* $n \equiv 3\ (4)$, $=1$ *otherwise.*

(ii) $\#\tilde{\Omega}_n(I^*)_{\mathrm{even}} = \#\sum_j H_{4j+3}(I^*;\ \Omega_{n-4j-3})_{\mathrm{even}} \times \#(\mathfrak{W}_2 \cap \tilde{\Omega}_n(D^*(8)))$,

where $\mathfrak{W}_2$ is the one of (5.3).

PROOF. In the same way as the proof of Lemma 5.5, we see the lemma by Proposition 1.6. q.e.d.

Consider the inclusions

$$I_k\colon Z_k \longrightarrow I^*[x, y], \qquad k = 3, 4, 5, \tag{7.2}$$

such that

$$\operatorname{Im} I_3 = Z_3[(xy)^2], \quad \operatorname{Im} I_4 = Z_4[x], \quad \operatorname{Im} I_5 = Z_5[y^2], \quad (\text{cf. } (1.2)).$$

PROPOSITION 7.3. *Consider the extension*

$$I_k\colon \tilde{\Omega}_*(Z_k) \longrightarrow \tilde{\Omega}_*(I^*), \qquad k = 3, 5,$$

induced by the inclusion I_k of (7.2).

(i) *I_k is monomorphic on the submodule $\mathfrak{Z}_{k,1} \subset \tilde{\Omega}_*(Z_k)$ of Theorem* 3.8.

(ii) $\tilde{\Omega}_*(I^*)_{\mathrm{odd}} = I_3\mathfrak{Z}_{3,1} \oplus I_5\mathfrak{Z}_{5,1}$.

PROOF. The bordism spectral sequences of $\tilde{\Omega}_*(BG)$, for $G=I^*$, Z_k $(k=3, 5)$, are trivial by Corollary 3.5, and the inclusion I_k induces a monomorphism of E^2-terms in total degree $4n+3$ by Lemma 1.7 (i), (1.4) and the structure of Ω_* (cf. K2-Theorem 3.6). Hence I_k: $\tilde{\Omega}_{4n+3}(Z_k) \to \tilde{\Omega}_{4n+3}(I^*)$ is monomorphic, and we have the desired result by Theorem 3.8.

(ii) We have immediately the desired result by (i) and Lemma 7.1 (i). q.e.d.

LEMMA 7.4. *The order of the class* $[I^*, S^{4n+4}] \in \tilde{\Omega}_*(I^*)$ *in* (2.5) *is equal to* $2^3 3^{n+1} 5^{[n/2]+1}$.

PROOF. It is clear that $\mu[I^*, S^{4n+3}]$ is a generator of $H_{4n+3}(I^*; Z) = Z_{120}$. Hence the order of $[I^*, S^{4n+3}]$ is $2^{a(2)}3^{a(3)}5^{a(5)}$ for some $a(2) \geqq 3$, $a(3) \geqq 1$, $a(5) \geqq 1$ by Lemma 7.1. We see that $a(2)=3$ by Theorem 3.7 and Proposition 1.6.

Consider the transfer homomorphism

$$t_{I_k}\colon \tilde{\Omega}_{4n+3}(I^*) \longrightarrow \tilde{\Omega}_{4n+3}(Z_k) \qquad (k = 3 \text{ or } 5)$$

induced by I_k of (7.2). Then we see that

$$t_{I_k}[I^*, S^{4j+3}] = [Z_k, S^{4j+3}] = \alpha^{(k)}_{4j+3}$$

in the same way as Lemma 6.6. Hence t_{I_k} maps the k-primary component of $\tilde{\Omega}_{4n+3}(I^*)$ isomorphically onto $\tilde{\Omega}_{4n+3}(Z_k)$ by Lemma 8.1 and Theorem 3.8. Therefore, $a(k)$ is equal to the order $a^{(k)}_{4n+3}$ of $\alpha^{(k)}_{4n+3}$, which is 3^{n+1} if $k=3$ and $5^{[n/2]+1}$ if $k=5$ by Theorem 3.8 (ii). q.e.d.

PROPOSITION 7.5. *Let* $\mathfrak{I}$ *be the submodule of* $\tilde{\Omega}_*(I^*)$ *generated by the classes*

$$\iota_n = 3^{n+1}5^{[n/2]+1}[I^*, S^{4n+3}], \qquad n \geqq 0.$$

(i) *Then* $\mathfrak{I}$ *is the quotient module of the free* Ω_**-module* $\Omega_*\{\{\iota_n : n \geqq 0\}\}$ *by the submodule generated by the elements* $8\iota_n$, $n \geqq 0$.

(ii) $\#(\mathfrak{I} \cap \tilde{\Omega}_n(I^*)) = \#\sum_j H_{4j+3}(I^*; \Omega_{n-4j-3})_{\mathrm{even}}$.

PROOF. The proposition can be proved in the same way as the proof of Proposition 6.8, by using the above lemma. q.e.d.

Now, we use the following

THEOREM 7.6 (K1-Theorem 2.18). $\tilde{\Omega}_*(Z_4)$ *is the direct sum*

$$\tilde{\Omega}_*(Z_4) = \mathfrak{H}_2 \oplus \mathfrak{G}_2, \qquad \mathfrak{G}_2 = \mathfrak{G}_{2,0} \oplus \mathfrak{G}_{2,1}.$$

Here $\mathfrak{H}_2$ *and* $\mathfrak{G}_2$ *are the submodules in* K2-*Theorem* 2.18, *and* $\mathfrak{G}_{2,\varepsilon}$ ($\varepsilon=0$ *or* 1) *is the submodule of* $\tilde{\Omega}_*(Z_4)$ *generated by the elements*

$$iE^{4n+2\varepsilon+1}W(\omega), \quad n \geqq 0, \quad \omega \in \pi, \quad (i\colon Z_2 \subset Z_4),$$

and is the quotient module of the free Ω_*-*module* $\Omega_*\{\{iE^{4n+2\varepsilon+1}W(\omega)\colon n\geqq 0, \omega \in \pi\}\}$ *by the submodule generated by the elements*

$$2iE^{4n+2\varepsilon+1}W(\omega), \quad A_{2n+\varepsilon,2}(\omega) \quad (|\omega| \geqq 2), \quad B_{2n+\varepsilon,2}(\omega, \omega'), \quad \omega, \quad \omega' \in \pi, \quad n \geqq 0.$$

PROPOSITION 7.7. *Consider the extension*

$$I_4\colon \tilde{\Omega}_*(Z_4) \longrightarrow \tilde{\Omega}_*(I^*)$$

induced by the inclusion I_4 *of* (7.2).

(i) *Then* I_4 *is monomorphic on the submodule* $\mathfrak{G}_{2,1} \subset \tilde{\Omega}_*(Z_4)$ *of the above theorem, which is isomorphic to* $\mathfrak{W}_2 \subset \tilde{\Omega}_*(D^*(8))$ *of* (5.3) *by the extension induced by the inclusion* $Z_4 \subset D^*(8)$.

(ii) $\tilde{\Omega}_*(I^*)_{\text{even}} = \mathfrak{J} \oplus I_4\mathfrak{G}_{2,1}$,
where $\mathfrak{J}$ *is the one of Proposition* 7.5.

PROOF. (i) We can prove that $r \circ I_4\colon \tilde{\Omega}_*(Z_4) \to \mathfrak{N}_*(I^*)$ is monomorphic on $\mathfrak{G}_{2,1}$ in the same way as the proof (ii) of Proposition 5.6, by using Theorem 2.7 and K2-Proposition 3.10. The fact that $\mathfrak{G}_{2,1} \approx \mathfrak{W}_2$ is seen by Theorem 7.6 and K2-Theorem 4.8 (iii).

(ii) In the same way as the proof of Proposition 6.9 (ii), we see that $\mathfrak{J} \cap I_4\mathfrak{G}_{2,1}=0$ by the above proof and Theorem 2.7. Therefore, we have (ii) by (i), Proposition 7.4 (ii) and Lemma 7.1 (ii). q.e.d.

Now, we have the following theorem by Propositions 7.3 and 7.7.

THEOREM 7.8. *The principal oriented bordism module* $\tilde{\Omega}_*(I^*)$ *of the binary icosahedral group* I^* *is the direct sum*

$$\tilde{\Omega}_*(I^*) = \mathfrak{J} \oplus I_4\mathfrak{G}_{2,1} \oplus I_3\mathfrak{Z}_{3,1} \oplus I_5\mathfrak{Z}_{5,1}$$

of the submodules given in Propositions 7.3, 7.5 *and* 7.7.

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