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Homogenization of Certain One-dimensional Discontinuous Markov Processes

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§1. Introduction

Homogenization of diffusion processes in $\boldsymbol{R}^n$ has been investigated by Bensoussan, Lions and Papanicolaou [1] and Bensoussan [2]. The purpose of this paper is to remark that some of their methods can also be adapted to a similar problem for certain one-dimensional discontinuous Markov processes in which the limit process yields a stable process.

Let γ_+, γ_- and α be constants such that $\gamma_+, \gamma_- \geqq 0$, $\gamma_+ + \gamma_- > 0$ and $1 < \alpha < 2$. Suppose we are given $a(x, y)$ and $b(x)$, and consider the operator A_1 defined by

$$(1.1)\quad A_1 f(x) = \int_{-\infty}^{\infty} \{f(x+y) - f(x) - f'(x)y\} a(x, y) v(y)\, dy + b(x) f'(x)$$

for smooth functions f, where

$$(1.2)\qquad v(y) = \begin{cases} \gamma_+/y^{\alpha+1}, & y>0, \\ \gamma_-/|y|^{\alpha+1}, & y<0. \end{cases}$$

We make the following assumptions for the coefficients $a(x, y)$ and $b(x)$.

(1.3a) $a(x, y)$ is a non-negative bounded C^1-function on $\boldsymbol{R}^2$ with $a(x, 0) > 0$, and is a periodic function of x with period 1 for fixed y.

(1.3b) $b(x)$ is a real valued periodic continuous function on $\boldsymbol{R}$ with period 1.

Under the above assumptions there exists a Markov process $\{X_1(t)\}$ governed by A_1 and having almost all sample paths in W, the space of real valued right continuous functions on $\boldsymbol{R}_+ = [0, \infty)$ with left limits. W is equiped with the Skorohod topology. Then our problem is this: Under suitable conditions on $a(x, y)$ and $b(x)$ does the process, after the scalings $t \to t/\varepsilon^\alpha$ and $x \to \varepsilon x$, converge in the law sense to some process as $\varepsilon \downarrow 0$? So we are interested in the Markov process with generator A_ε, $\varepsilon > 0$, given by

$$(1.4)\quad A_\varepsilon f(x) = \int_{-\infty}^{\infty} \{f(x+y) - f(x) - f'(x)y\} a(\varepsilon^{-1}x,\ \varepsilon^{-1}y) v(y) dy + \varepsilon^{-\alpha+1} b(\varepsilon^{-1}x) f'(x)$$

for smooth functions. We denote by P_ε^x the probability measure on W for the Markov process starting at x and governed by A_ε. Then the problem is also stated as follows. Under what conditions on $a(x, y)$ and $b(x)$ does P_ε^x converge to some probability measure P_0^x in W as $\varepsilon \downarrow 0$?

We think of $\{X_1(t), t \geqq 0\}$ as a Markov process on the one-dimensional torus $\boldsymbol{T}=\boldsymbol{R}/\boldsymbol{Z}$, denote by $m(dx)$ the (unique) invariant probability measure of the process and introduce the following conditions.

(1.5) There exist the limits: $\bar{a}_\pm = \lim_{y\to\pm\infty} y^{-1}\int_0^y \bar{a}(z)dz$, where

$$\bar{a}(y) = \int_0^1 a(x, y)m(dx).$$

$$\int_0^1 b(x)m(dx)=0. \tag{1.6}$$

Then our main result is that, under the conditions (1.3), (1.5) and (1.6), P_ε^x converges to P_0^x as $\varepsilon \downarrow 0$ where P_0^x denotes the probability measure on W for the stable process with generator given by

$$\begin{aligned} L_0 f(x) &= \bar{a}_+\int_0^\infty \{f(x+y)-f(x)-f'(x)y\}\nu(y)dy \\ &\quad + \bar{a}_-\int_{-\infty}^0 \{f(x+y)-f(x)-f'(x)y\}\nu(y)dy. \end{aligned} \tag{1.7}$$

§2. Some properties of the Markov process governed by A_1

In this section we find an invariant measure $m(\cdot)$ of the Markov process $\{X_1(t)\}$ when considered on the torus $\boldsymbol{T}$, and then seek a periodic solution of $-A_1 u=f$ for a periodic f with $\int_0^1 f dm=0$. We introduce several spaces of functions.

- $C_0(\boldsymbol{R})$ = the space of real valued continuous functions on $\boldsymbol{R}$ vanishing at infinity.
- $C_0^2(\boldsymbol{R})$ = the space of functions $f\in C_0(\boldsymbol{R})$ such that f' and f'' exist and belong to $C_0(\boldsymbol{R})$.
- $C_u(\boldsymbol{R})$ = the space of real valued, bounded and uniformly continuous functions on $\boldsymbol{R}$.
- $B(\boldsymbol{R})$ = the space of real valued functions on $\boldsymbol{R}$, bounded and Borel measurable.

Similar notations for the spaces of functions on $\boldsymbol{R}^2$ are used, and the supremum norm of a function in any one of these spaces is denoted by $\|\cdot\|$. We also write

$\partial = \partial/\partial s$ and $D = \partial/\partial x$.

Given $\nu(y)$ as in (1.2), we consider the operaor L defined by

$$Lf(x) = \int_{-\infty}^{\infty} \{f(x+y) - f(x) - f'(x)y\} \nu(y) dy$$

for smooth functions. The transition density $p(t, x, y)$ of the stable process governed by L is given by

$$p(t, x, y) = p(t, y-x) \tag{2.1a}$$

$$p(t, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi} e^{-c_0 t|\xi|^{\alpha}(1+ih \operatorname{sgn}\xi)} d\xi, \; t > 0, \; x \in \boldsymbol{R}, \tag{2.1b}$$

where $c_0 = -(\gamma_+ + \gamma_-)\Gamma(2-\alpha)\cos(\pi\alpha/2)/\alpha(\alpha-1) > 0$ and $h = (\gamma_+ - \gamma_-)\tan(\pi\alpha/2)/(\gamma_+ + \gamma_-)$. The associated semigroup and the Green operator are defined for $f \in B(\boldsymbol{R})$ by

$$S^t f(x) = \int_{-\infty}^{\infty} f(y) p(t, y-x) dy, \; G^\lambda f(x) = \int_0^\infty e^{-\lambda t} S^t f(x) dt, \; t > 0, \; \lambda > 0.$$

We also consider the space-time forms of these operators which will be denoted by the corresponding German letters:

$$\mathfrak{S}^t \mathfrak{f}(s, x) = S^t \mathfrak{f}(s+t, x), \; \mathfrak{G}^\lambda \mathfrak{f}(s, x) = \int_0^\infty e^{-\lambda t} \mathfrak{S}^t \mathfrak{f}(s. x) dt, \; \lambda > 0, \; \mathfrak{f} \in B(\boldsymbol{R}^2).$$

If we consider $\{\mathfrak{S}^t\}$ on the space $C_0(\boldsymbol{R}^2)$, then it is a strongly continuous semigroup; we denote by $\mathfrak{L}$ the generator of this semigroup. Obviously $\mathscr{D}(\mathfrak{L}) \supset C_0^2(\boldsymbol{R}^2)$ and $\mathfrak{L}\mathfrak{f} = (\partial + L)\mathfrak{f}$ for $\mathfrak{f} \in C_0^2(\boldsymbol{R}^2)$. The following lemma can be easily proved by making use of $D^n p(t, x) = D^n p(1, t^{-1/\alpha} x) t^{-(n+1)/\alpha}$, $|D^n p(1, x)| \leq const \times |x|^{-(2+n)}$, $n \geqq 0$ (similar results are also found in [8]).

LEMMA 1. (i) *For any* $\mathfrak{f} \in B(\boldsymbol{R}^2)$, $D\mathfrak{G}^\lambda \mathfrak{f}$ *exists and is given by*

$$D\mathfrak{G}^\lambda \mathfrak{f}(s, x) = -\int_0^\infty \int_{-\infty}^\infty e^{-\lambda t} \mathfrak{f}(s+t, y) Dp(t, y-x) \, dt dy,$$

from which it follows that

$$\|D\mathfrak{G}^\lambda \mathfrak{f}\| \leqq c_1 \lambda^{-(\alpha-1)/\alpha} \|f\|, \; c_1 = \Gamma\Big(1 - \frac{1}{\alpha}\Big) \int_{-\infty}^{\infty} |Dp(1, x)| dx. \tag{2.2}$$

(ii) *For any* β *satisfying* $0 < \beta < \alpha - 1$ *there exists a constant* c_2 *depending only on* α *and* β *such that*

$$|D\mathfrak{G}^\lambda \mathfrak{f}(s, x_1) - D\mathfrak{G}^\lambda \mathfrak{f}(s, x_2)| \leqq c_2 \lambda^{-(\alpha-1-\beta)/\alpha} \|\mathfrak{f}\| |x_1 - x_2|^\beta, \; \mathfrak{f} \in B(\boldsymbol{R}^2).$$

Given $a(x, y)$ and $b(x)$ satisfying (1.3), we consider the operator A_1 defined by (1.1) for smooth functions. The existence of the Markov process governed by A_1 is essentially known; in fact it can be constructed by a routine method. But we sketch the construction here because we want to have some further results such as Proposition 3.

First we construct the Markov process governed by

$$Af(x) = \int_{-\infty}^{\infty} \{f(x+y) - f(x) - f'(x)y\} a_0(x, y) v(y) dy + b_0(x) f'(x)$$

where $a_0(x, y) = a(x, y)/a(x, 0)$ and $b_0(x) = b(x)/a(x, 0)$. We put

$$\Lambda f(x) = \int_{-\infty}^{\infty} \{f(x+y) - f(x) - f'(x)y\} a_1(x, y) v(y) dy + b_0(x) f'(x),$$
$$a_1(x, y) = a_0(x, y) - 1 .$$

If $\mathfrak{u} \in \mathscr{D}(\mathfrak{L})$ and so if $\mathfrak{u} = \mathfrak{G}^\lambda \mathfrak{f}$, $\mathfrak{f} \in C_0(\boldsymbol{R}^2)$, then $D\mathfrak{u} \in C_0(\boldsymbol{R}^2)$ by Lemma 1 and hence $\Lambda \mathfrak{u} \in C_0(\boldsymbol{R}^2)$, admitting the following estimate by virtue of (2.2):

$$\|\Lambda \mathfrak{u}\| = \|\Lambda \mathfrak{G}^\lambda \mathfrak{f}\| \le (\|b_0\| + 2c_3) c_1 \lambda^{-(\alpha-1)/\alpha} \|\mathfrak{f}\| , \tag{2.3}$$
$$c_3 = \int_{-\infty}^{\infty} |y| \sup_x |a_1(x, y)| v(y) dy .$$

Thus we can define $\mathfrak{A}$ on $\mathscr{D}(\mathfrak{L})$ by $\mathfrak{A}\mathfrak{u} = \mathfrak{L}\mathfrak{u} + \Lambda \mathfrak{u}$ for $\mathfrak{u} \in \mathscr{D}(\mathfrak{L})$. If λ is sufficiently large, then $\|\Lambda \mathfrak{G}^\lambda\| < 1$ by (2.3) and hence $\mathfrak{u} = \mathfrak{G}^\lambda (I - \Lambda \mathfrak{G}^\lambda)^{-1} \mathfrak{f}$ gives us the unique solution in $\mathscr{D}(\mathfrak{A}) = \mathscr{D}(\mathfrak{L})$ of the equation $(\lambda - \mathfrak{A})\mathfrak{u} = \mathfrak{f}$, $\mathfrak{f} \in C_0(\boldsymbol{R}^2)$. Again by (2.3) we see that $\mathfrak{A}$ is the smallest closed extension of the operator $\partial + A$ restricted on $C_0^2(\boldsymbol{R}^2)$. Then it follows that $\mathfrak{A}$ has the strong negative property, that is, $\mathfrak{f} \in \mathscr{D}(\mathfrak{A})$ and $\mathfrak{f}(s_0, x_0) = \max_{(s,x) \in \boldsymbol{R}^2} \mathfrak{f}(s, x) > 0$ imply that $\mathfrak{A}\mathfrak{f}(s_0, x_0) \le 0$, because $\partial + A$ has the same property in $C_0^2(\boldsymbol{R}^2)$ (for example, see [6]). Therefore, there exists a unique strongly continuous sub-Markov semigroup $\{\mathfrak{T}^t\}$ on $C_0(\boldsymbol{R}^2)$ with generator $\mathfrak{A}$. Clearly, $\{\mathfrak{T}^t\}$ is associated with the unique strongly continuous sub-Markov semigroup $\{T^t\}$ on $C_0(\boldsymbol{R})$ in such a way that $\mathfrak{T}^t(s,x) = T^t \mathfrak{f}(x)$ where $f(x) = \mathfrak{f}(s+t, x)$. The associated Markov process $\boldsymbol{X} = \{W, w(t), P^x, x \in \boldsymbol{R}\}$ is nothing but the one governed by A.

As in [7] we can prove that $\boldsymbol{X}$ has the strong Feller property. Denote by $P(t, x, \cdot)$ the transition function of the process $\boldsymbol{X}$ and put

$$\mathfrak{R}^\lambda \mathfrak{f}(s, x) = \int_0^\infty e^{-\lambda t} dt \int_{-\infty}^{\infty} \mathfrak{f}(s+t, y) P(t, x, dy), \quad \mathfrak{f} \in \mathrm{B}(\boldsymbol{R}^2) .$$

Then for all sufficiently large λ,

$$\mathfrak{R}^\lambda \mathfrak{f} = \mathfrak{G}^\lambda (I - \Lambda \mathfrak{G}^\lambda)^{-1} \mathfrak{f}, \quad \mathfrak{f} \in B(\boldsymbol{R}^2) . \tag{2.4}$$

In fact this follows from the following two remarks: (a) (2.4) holds for $\mathfrak{f}\in C_0(\boldsymbol{R}^2)$, and (b) if $\{\mathfrak{f}_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of functions in $B(\boldsymbol{R}^2)$ converging pointwisely to $\mathfrak{f}$, then using Lemma 1 we can prove that $\mathfrak{R}^\lambda\mathfrak{f}_n$ and $\mathfrak{G}^\lambda(I-\Lambda\mathfrak{G}^\lambda)^{-1}\mathfrak{f}_n$ converge pointwisely to $\mathfrak{R}^\lambda\mathfrak{f}$ and $\mathfrak{G}^\lambda(I-\Lambda\mathfrak{G}^\lambda)^{-1}\mathfrak{f}$, respectively, as $n\to\infty$. Putting $\mathfrak{f}\equiv 1$ in (2.4), we obtain $\mathfrak{R}^\lambda 1=1/\lambda$ which means that $\boldsymbol{X}$ is conservative. If we put

$$\mathfrak{g}(s, x) = \int_{-\infty}^{\infty} \frac{1}{t}\chi_{[0,t]}(s)e^{\lambda s}f(y)P(t-s, x, dy),\ f\in B(\boldsymbol{R})$$

then

$$\int_{-\infty}^{\infty} f(y)P(t, x, dy) = \mathfrak{R}^\lambda\mathfrak{g}(0, x) = \mathfrak{G}^\lambda(I-\Lambda\mathfrak{G}^\lambda)^{-1}\mathfrak{g}(0, x),$$

provided that λ is sufficiently large. This formula implies the strong Feller property of $\boldsymbol{X}$ since the last member of the above is continuous in x by (ii) of Lemma 1. $\{T^t\}$ is also a strongly continuous semigroup on the space $C_u(\boldsymbol{R})$; we denote by $\boldsymbol{A}$ the generator of this semigroup. Then by Lemma 1 the domain $\mathscr{D}_u$ of $\boldsymbol{A}$ is contained in $C_u^1(\boldsymbol{R})$, the space of functions belonging to $C_u(\boldsymbol{R})$ together with their first derivatives.

Now the Markov process $\boldsymbol{X}_1$ governed by A_1 is obtained from $\boldsymbol{X}=\{W, w(t), P^x, x\in\boldsymbol{R}\}$ by making time substitution $t\to\tau(t)$, where τ is the inverse function of $\int_0^t ds/a(w(s), 0)$. Since $a(x, y)$ and $b(x)$ are periodic functions of x with period 1 for each fixed y, both $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{X}}_1$ induce the Markov processes $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{X}}_1$ on the torus $\boldsymbol{T}$ respectively in the natural way. For example, $\tilde{\boldsymbol{X}}$ has the transition function $\tilde{P}(t, \tilde{x}, \cdot)$ given by $\tilde{P}(t, \tilde{x}, \tilde{U})=P(t, x, U)$, $U=\{y\in\boldsymbol{R};\ y+n\in\tilde{U}$ for some n in $\boldsymbol{Z}\}$.

LEMMA 2. *$\tilde{P}(t, \tilde{x}\ \tilde{U})$ is strictly positive for any $t>0$, $\tilde{x}\in\boldsymbol{T}$ and non-empty open set $\tilde{U}$ in $\boldsymbol{T}$.*

PROOF. We may assume that $\gamma_+>0$. For open sets U and V in $\boldsymbol{R}$, we write $U\xrightarrow{t}V$ when $P(s, x, V)>0$ for any $s\in(0, t]$ and $x\in U$. Then, $U_1\xrightarrow{t}U_2\xrightarrow{s}U_3$ clearly implies $U_1\xrightarrow{t+s}U_3$. For $U=(a, b)$ and $V=(c, d)$, we write $U<V$ when $a<b<c<d$. We also write $U\lhd V$ when $U<V$ and $d-a<\delta$, where δ is a fixed positive constant such that $a_0(x, y)>1/2$ holds for any $y\in(-\delta,\delta)$. First we claim that $U\lhd V$ implies that $U\xrightarrow{t}V$ for some positive t. In fact, if f is a non-negative C^∞-function on $\boldsymbol{R}$ such that supp $(f)\subset$ the closure of V and $0<f\leqq 1$ on V, then

$$\begin{aligned} P(t, x, V)/t \geqq T^tf(x)/t &\longrightarrow \int_{-\infty}^{\infty} f(x+y)a_0(x, y)\nu(y)dy,\ t\downarrow 0,\ x\in U,\\ &\geqq 2^{-1}\delta^{-\alpha-1}\gamma_+\int_V f(z)dz > 0,\ x\in U, \end{aligned}$$

which implies that $U \xrightarrow{t} V$ for some $t>0$ since the above convergence is uniform in x. Next, assume that $U<V$ and the both lengths of U and V are less than δ. Then we can find open intervals $U_1,\dots, U_n$ such that $U \lhd U_1 \lhd \cdots \lhd U_n \lhd V$, and hence $U \xrightarrow{t_0} U_1 \xrightarrow{t_1} \cdots \xrightarrow{t_n} V$ for some positive $t_0,\cdots, t_n$. Therefore, if $U<V$ and if both lengths of U and V are less than δ, then $U \xrightarrow{t} V$ for some $t>0$. In terms of the process on $\boldsymbol{T}$ this result can now be stated as follows: with self-evident notation, if $\tilde{U}$ and $\tilde{V}$ are open intervals in $\boldsymbol{T}$ of lengths less than δ, then $\tilde{U} \xrightarrow{t} \tilde{V}$ for some $t>0$. From $\tilde{U} \xrightarrow{t} \tilde{V} \xrightarrow{s} \tilde{U} \xrightarrow{t} \cdots \xrightarrow{t} \tilde{V}$, we have $\tilde{U} \xrightarrow{nt+(n-1)s} \tilde{V}$ for some $t, s>0$ and for any integer $n \geqq 1$. This clearly proves the lemma.

Since $\tilde{\boldsymbol{X}}$ has the strong Feller property, it has the same property in the strict sense by a theorem of Mokobodzki ([5]). This combined with Lemma 2 implies that there exist a unique invariant probability measure $m_0(\cdot)$ of $\tilde{\boldsymbol{X}}$ and positive constants c_4 and c_5 such that

$$\|\tilde{T}^t\tilde{f} - \mathrm{m}_0(\tilde{f})\| \leqq c_4 e^{-c_5 t}\|\tilde{f}\|,\ t > 0,\ \tilde{f} \in B(\boldsymbol{T}), \tag{2.5}$$

where $m_0(\tilde{f}) = \int_{\boldsymbol{T}} \tilde{f} dm_0$ (see [9]). It then follows that $m(dx) = cm_0(dx)/a(x, 0)$ with $c = \{\int_{\boldsymbol{T}} m_0(dx)/a(x, 0)\}^{-1}$ is the unique invariant probability measure for $\tilde{\boldsymbol{X}}_1$, and we finally obtain the following proposition.

Proposition 3. (i) *There exists a unique invariant probability measure* $m(\cdot)$ *of* $\tilde{\boldsymbol{X}}_1$.
(ii) *If f is a periodic continuous function on $\boldsymbol{R}$ with period* 1 *and satisfying* $\int_0^1 f dm = 0$, *then* $u = \int_0^\infty T^t(f/a(\cdot, 0))dt$ *exists, belongs to* $\mathcal{D}_u(\subset C_u^1(\boldsymbol{R}))$ *and is a periodic solution of* $-A_1 u = f$, *or more precisely, of* $-a(\cdot, 0)\boldsymbol{A}u = f$.

§3. The main theorem and the proof

Given $a(x, y)$ and $b(x)$ satisfying (1.3), we consider the operator A_ε defined by (1.4) for each $\varepsilon>0$. By the preceeding section there exists a Markov process $\boldsymbol{X}_\varepsilon$ governed by A_ε. Denote by P_ε^x the probability measure on the path space W induced by this process starting at x.

Theorem. *In addition to* (1.3), *we assume that* (1.5) *and* (1.6) *are satisfied. Then,* P_ε^x *converges to* P_0^x *as* $\varepsilon \downarrow 0$, *where* P_0^x *is the probability measure on* W *of the stable process governed by* L_0 *of* (1.7) *and starting at* x.

We prove this theorem by making use of the calculus of stochastic integrals as in [1] and [2]. For each $\varepsilon>0$ and $x \in \boldsymbol{R}$ the path functions of the process $\boldsymbol{X}_\varepsilon$ starting at x can be constructed on a suitable probability space as a solution

$X_\varepsilon(t)$ of the stochastic integral equation (3.1) below. More precisely, on a suitable probability space $(\Omega, \mathscr{F}, P)$ with an increasing family $\{\mathscr{F}_t\}_{t\geqq 0}$ of sub-σ-fields we can find (i) an $\{\mathscr{F}_t\}$-adapted Poisson random measure $N(dtdy)$ on $\boldsymbol{R}_+\times\boldsymbol{R}$ with characteristic measure $dt\nu(y)dy$, and (ii) an $\{\mathscr{F}_t\}$-adapted right continuous process $\{X_\varepsilon(t)\}$ on $\boldsymbol{R}$ with left limits, in such a way that

$$(3.1)\qquad X_\varepsilon(t) = x + \varepsilon\int_0^t\int_{-\infty}^{\infty}\sigma(\varepsilon^{-1}X_\varepsilon(s-),\ \varepsilon^{-1}y)M(dsdy) + \varepsilon^{-\alpha+1}\int_0^t b(\varepsilon^{-1}X_\varepsilon(s))ds$$

holds with probability one, where $M(dsdy)=N(dsdy)-ds\nu(y)dy$ and $\sigma(x, y)$ is defined by

$$(3.2)\quad \sigma(x, y) = \begin{cases} \inf\{y' > 0: \int_y^{\infty}\nu(z)dz > \int_{y'}^{\infty}a(x, z)\nu(z)dz\} & \text{for } y > 0, \\ \sup\{y' < 0: \int_{-\infty}^{y}\nu(z)dz > \int_{-\infty}^{y'}a(x, z)\nu(z)dz\} & \text{for } y < 0. \end{cases}$$

The condition (1.3a) implies that $\sigma(x,y)$ is a periodic function in x with period 1 and $|\sigma(x, y)|\leqq const.|y|$. It is remarked that $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ and $N(dtdy)$ may depend upon ε.

LEMMA 4. *For each x the family $\{P_\varepsilon^x,\ 0<\varepsilon\leqq 1\}$ of probability measures on the space W is tight.*

PROOF. For $\delta>0$ and $n\geqq 1$ we put

$$\mathscr{V}_\delta^n(X_\varepsilon) = \sup\max_{1\leqq i\leqq r}\ \sup_{t_{i-1}\leqq s,t<t_i}|X_\varepsilon(s) - X_\varepsilon(t)|,$$

$$\mathscr{W}_\delta^n(X_\varepsilon) = \inf\max_{1\leqq i\leqq r}\ \sup_{t_{i-1}\leqq s,t<t_i}|X_\varepsilon(s) - X_\varepsilon(t)|,$$

where both the supremum and the infimum are taken over all partitions Δ of $[0, n]: 0=t_0<t_1<\cdots<t_r=n$ such that $\delta<t_i-t_{i-1}\leqq 2\delta$, $1\leqq i\leqq r$. Then, by Theorem 15.2 in [3] it is enough to prove that the following tightness criterion is satisfied

$$(3.3)\quad \begin{cases} \text{(a)}\ \lim\limits_{l\to\infty}\sup\limits_{0<\varepsilon\leqq 1}P\{\sup\limits_{0\leqq t\leqq n}|X_\varepsilon(t)|>l\} = 0 & \text{for any } n\geqq 1, \\ \text{(b)}\ \lim\limits_{\delta\downarrow 0}\sup\limits_{0<\varepsilon\leqq 1}P\{\mathscr{W}_\delta^n(X_\varepsilon)>\eta\} = 0 & \text{for any } n\geqq 1 \text{ and } \eta>0. \end{cases}$$

By (1.5) and Proposition 3 there exists a periodic solution $\varphi(x)\in\mathscr{D}_u$ of $-A_1\varphi=b$. We put $Y_\varepsilon(t)=X_\varepsilon(t)+\varepsilon\varphi(\varepsilon^{-1}X_\varepsilon(t))$ and then apply the transformation formula of stochastic integrals*) to obtain

$$(3.4)\qquad \begin{aligned} Y_\varepsilon(t) = {} & X_\varepsilon(t) + \varepsilon\varphi(\varepsilon^{-1}x) + \int_0^t \varphi'(\varepsilon^{-1}X_\varepsilon(s-))dX_\varepsilon(s) \\ & + \sum_{s\leqq t}\{\varepsilon\varphi(\varepsilon^{-1}X_\varepsilon(s)) - \varepsilon\varphi(\varepsilon^{-1}X_\varepsilon(s-)) \\ & \qquad - \varphi'(\varepsilon^{-1}X_\varepsilon(s-))(X_\varepsilon(s) - X_\varepsilon(s-))\} \\ = {} & x + \varepsilon\varphi(\varepsilon^{-1}x) + \varepsilon\int_0^t\int_{-\infty}^{\infty}\sigma_\varepsilon M(dsdy) \\ & + \varepsilon\int_0^t\int_{-\infty}^{\infty}\varphi'(\varepsilon^{-1}X_\varepsilon(s-))\sigma_\varepsilon M(dsdy) \\ & + \varepsilon\int_0^t\int_{-\infty}^{\infty}\{\varphi(\varepsilon^{-1}X_\varepsilon(s-) + \sigma_\varepsilon) - \varphi(\varepsilon^{-1}X_\varepsilon(s-)) \\ & - \varphi'(\varepsilon^{-1}X_\varepsilon(s-))\sigma_\varepsilon\}M(dsdy) + \varepsilon^{-\alpha+1}\int_0^t b(\varepsilon^{-1}X_\varepsilon(s))ds \\ & + \varepsilon^{-\alpha+1}\int_0^t \varphi'(\varepsilon^{-1}X_\varepsilon(s))b(\varepsilon^{-1}X_\varepsilon(s))ds \\ & + \varepsilon\int_0^t\int_{-\infty}^{\infty}\{\varphi(\varepsilon^{-1}X_\varepsilon(s) + \sigma_\varepsilon) \\ & - \varphi(\varepsilon^{-1}X_\varepsilon(s)) - \varphi'(\varepsilon^{-1}X(s))\sigma_\varepsilon\}ds\nu(y)dy \\ = {} & x + \varepsilon\varphi(\varepsilon^{-1}x) + \int_0^t\int_{-\infty}^{\infty}\rho(\varepsilon, s, y)M(dsdy)\,, \end{aligned}$$

where $\sigma_\varepsilon = \sigma(\varepsilon^{-1}X_\varepsilon(s-), \varepsilon^{-1}y)$ and $\rho(\varepsilon, t, y)$ is an $\{\mathscr{F}_t\}$-predictable process defined by

$$\begin{aligned} \rho(\varepsilon, t, y) &= \sigma_1(\varepsilon, t, y) + \rho_2(\varepsilon, t, y), \\ \rho_1(\varepsilon, t, y) &= \varepsilon\sigma(\varepsilon^{-1}X_\varepsilon(t-), \varepsilon^{-1}y), \\ \rho_2(\varepsilon, t, y) &= \varepsilon\varphi(\varepsilon^{-1}X_\varepsilon(t-) + \sigma(\varepsilon^{-1}X_\varepsilon(t-), \varepsilon^{-1}y)) \\ &\qquad - \varepsilon\varphi(\varepsilon^{-1}X_\varepsilon(t-))\,. \end{aligned}$$

The condition (1.3a) implies that $|\sigma(x, y)| \leqq const.|y|$ and hence

$$(3.5)\qquad |\rho(\varepsilon, t, y)|,\ |\rho_1(\varepsilon, t, y)|,\ |\rho_2(\varepsilon, t, y)| \leqq c_4|y|.$$

Therefore we have

*) Usually the transformation formula is valid for a C^2-function φ. The present case in which $\varphi\in\mathscr{D}_u$ ($\subset C_u^1(\boldsymbol{R})$) can be treated by approximating φ by smooth functions if necessary.

$$P\{\sup_{0\leqq t\leqq n}|X_\varepsilon(t)|>l\}$$

$$\leqq P\{\sup_{0\leqq t\leqq n}|\int_0^t\int_{-\infty}^{\infty}\rho(\varepsilon,s,y)M(dsdy)|>l-|x|-2\|\varphi\|\}$$

$$\leqq E\{|\int_0^n\int_{-\infty}^{\infty}\rho(\varepsilon,s,y)M(dsdy)|\}/(l-|x|-2\|\varphi\|)$$

$$\leqq\Big[E\{\int_0^n\int_{|y|\leqq 1}|\rho(\varepsilon,s,y)|^2 ds\nu(y)dy\}^{1/2}$$

$$+2E\{\int_0^n\int_{|y|>1}|\rho(\varepsilon,s,y)|ds\nu(y)dy\}\Big]/(l-|x|-2\|\varphi\|)$$

$$\leqq const.\ n/(l-|x|-2\|\varphi\|),$$

which proves (a) of (3.3). Next for $\theta>0$ we define

$$Z_{1,\varepsilon}(t)=Z_{1,\varepsilon}^{(1)}(t)-Z_{1,\varepsilon}^{(2)}(t)+Z_{1,\varepsilon}^{(3)}(t),$$

$$Z_{1,\varepsilon}^{(1)}(t)=\int_0^t\int_{|y|>\theta}\rho_1(\varepsilon,s,y)N(dsdy),$$

$$Z_{1,\varepsilon}^{(2)}(t)=\int_0^t\int_{|y|>\theta}\rho_1(\varepsilon,s,y)ds\nu(y)dy,$$

$$Z_{1,\varepsilon}^{(3)}(t)=\int_0^t\int_{|y|\leqq\theta}\rho_1(\varepsilon,s,y)M(dsdy),$$

and also $Z_\varepsilon(t)$, $Z_\varepsilon^{(1)}(t)$, $Z_\varepsilon^{(2)}(t)$, $Z_\varepsilon^{(3)}(t)$ similarly using ρ instead of ρ_1. Take $\varepsilon_0>0$ and fix it for a while. For $\varepsilon\geqq\varepsilon_0$ we use (3.1) to obtain

$$\mathscr{W}_\delta^n(X_\varepsilon)\leqq\mathscr{W}_\delta^n(Z_{1,\varepsilon}^{(1)})+\mathscr{V}_\delta^n(Z_{1,\varepsilon}^{(2)})+\mathscr{V}_\delta^n(Z_{1,\varepsilon}^{(3)})$$
$$+\mathscr{V}_\delta^n(\varepsilon^{-\alpha+1}\int_0^t b(\varepsilon^{-1}Z_\varepsilon(s))ds),$$

and hence

$$\text{(3.6)}\qquad P\{\mathscr{W}_\delta^n(X_\varepsilon)>\eta\}\leqq P\{\mathscr{W}_\delta^n(Z_{1,\varepsilon}^{(1)})>\eta'\}+P\{\mathscr{V}_\delta^n(Z_{1,\varepsilon}^{(3)})>\eta/2\}$$
$$\leqq P\{\mathscr{W}_\delta^n(Z^{(1)})>\eta'\}+4\eta^{-1}E\{|Z_{1,\varepsilon}^{(3)}(n)|\}$$
$$\leqq P\{\mathscr{W}_\delta^n(Z^{(1)})>\eta'\}+c_5\eta^{-1}(n\theta^{2-\alpha})^{1/2},$$

where $c_5=4c_4(2-\alpha)^{-1/2}(\gamma_++\gamma_-)^{1/2}$ and

$$\eta'=2^{-1}\eta-2\delta\{c_4\int_{|y|>\theta}|y|\nu(y)dy+\varepsilon_0^{-\alpha+1}\|b\|\},$$

$$Z^{(1)}(t)=c_4\int_0^t\int_{|y|>\theta}|y|N(dsdy).$$

For $0<\varepsilon<\varepsilon_0$ we use (5.4) to obtain

$$\mathscr{W}_\delta^\eta(X_\varepsilon) \leqq \mathscr{W}_\delta^\eta(Z_\varepsilon)+\mathscr{V}_\delta^\eta(\varepsilon\varphi(\varepsilon^{-1}X_\varepsilon(t))$$
$$\leqq \mathscr{W}_\delta^\eta(Z_\varepsilon^{(1)}) + \mathscr{V}_\delta^\eta(Z_\varepsilon^{(2)}) + \mathscr{V}_\delta^\eta(Z_\varepsilon^{(3)}) + 2\varepsilon_0\|\varphi\|,$$

and hence as in (3.6) we have

$$P\{\mathscr{W}_\delta^\eta(X_\varepsilon) > \eta\} \leqq P\{\mathscr{W}_\delta^\eta(Z^{(1)}) > \eta''\} + c_5\eta^{-1}(n\theta^{2-\alpha})^{1/2}, \tag{3.7}$$

where $\eta''=2^{-1}\eta-2\{c_4\delta\int_{|y|>\theta}|y|\nu(y)dy +\varepsilon_0\|\varphi\|\}$. Now, (*b*) of (3.3) follows from (3.6) and (3.7). The proof is finished.

LEMMA 5. *For any C^∞-function f with compact support and $0\leqq s<t$ we have*

$$E\{f(X_\varepsilon(t))\,|\,\mathscr{F}_s\} - f(X_\varepsilon(s)) = E\left\{\int_s^t L_0f(X_\varepsilon(\tau))d\tau\,|\,\mathscr{F}_s\right\} + o(1),$$

where $o(1)$ means that the expectation of its absolute value tends to 0 with ε uniformly in s and t on each finite interval.

PROOF. We use the same notations as in the proof of Lemma 4. For a C^∞-function f with compact support and for $0\leqq s<t$ we have from (3.4)

$$E\{f(Y_\varepsilon(t))\,|\,\mathscr{F}_s\} - f(Y_\varepsilon(s))$$
$$= E\left[\int_s^t\int_{-\infty}^\infty\{f(Y_\varepsilon(\tau) + \rho)-f(Y_\varepsilon(\tau)) - f'(Y_\varepsilon(\tau))\rho\}d\tau\nu(y)dy\,|\,\mathscr{F}_s\right],$$
$$\rho = \rho(\varepsilon, \tau, y),$$

and hence

$$E\{f(X_\varepsilon(t))\,|\,\mathscr{F}_s\} -f(X_\varepsilon(s))$$
$$= E\left[\int_s^t\int_{-\infty}^\infty\{f(Y_\varepsilon(\tau) + \rho) - f(Y_\varepsilon(\tau)) - f'(Y_\varepsilon(\tau))\rho\}d\tau\nu(y)dy\,|\,\mathscr{F}_s\right] + o(1)$$
$$= E\left[\int_s^t\int_{-\infty}^\infty\{f(X_\varepsilon(\tau) + \rho_1) - f(X_\varepsilon(\tau)) - f'(X_\varepsilon(\tau))\rho_1\}d\tau\nu(y)dy\,|\,\mathscr{F}_s\right]$$
$$+ o(1) + \text{the remainder term}, \qquad \rho_1 = \rho_1(\varepsilon, \tau, y).$$

Writing the remainder term explicitly and noting that f is smooth enough, we can see that the remainder term is also $o(1)$ as $\varepsilon \downarrow 0$. Thus we can write

$$\begin{aligned}(3.8)\quad & E\{f(X_\varepsilon(t))\,|\,\mathscr{F}_s\} - f(X_\varepsilon(s)) \\ &= E\left[\int_s^t\int_{-\infty}^{\infty}\{f(X_\varepsilon(\tau)+y) - f(X_\varepsilon(\tau)) \right. \\ &\qquad\qquad \left. - f'(X_\varepsilon(\tau))y\}\bar{a}(\varepsilon^{-1}y)d\tau\nu(y)dy\,|\,\mathscr{F}_s\right] \\ &\quad + E\left\{\int_s^t g(\varepsilon^{-1}X_\varepsilon(\tau), X_\varepsilon(\tau))\,d\tau\,|\,\mathscr{F}_s\right\} + o(1),\end{aligned}$$

where

$$\begin{aligned} g(\xi, \eta) = \int_{-\infty}^{\infty} \{f(\eta+y) - f(\eta) - f'(\eta)y\}\{a(\xi, \varepsilon^{-1}y) \\ - \bar{a}(\varepsilon^{-1}y)\}\nu(y)dy.\end{aligned}$$

Since $\int_0^1 g(\xi, \eta)m(d\xi) = 0$, the function ψ defined by

$$\psi(\xi, \eta) = \int_0^{\infty} T^t(g/a(\cdot, 0))dt,\ g(\cdot) = g(\cdot, \eta),$$

satisfies $-A_1\psi(\cdot, \eta) = g(\cdot, \eta)$ for each η by Proposition 3. Since f is a C^∞-function with compact support, the function $\psi(\xi, \eta)$ has enough smoothness property so that we can apply the transformation formula of stochastic integrals to $\psi(\varepsilon^{-1}X_\varepsilon(t), X_\varepsilon(t))$, and we have

$$\begin{aligned}(3.9)\qquad & \varepsilon^\alpha E\{\psi(\varepsilon^{-1}X_\varepsilon(t), X_\varepsilon(t))\,|\,\mathscr{F}_s\} - \varepsilon^\alpha\psi(\varepsilon^{-1}X_\varepsilon(s), X_\varepsilon(s)) \\ &= \varepsilon E\left\{\int_s^t \psi_\eta(\varepsilon^{-1}X_\varepsilon(\tau), \varepsilon^{-1}y)b(\varepsilon^{-1}X_\varepsilon(\tau))d\tau\,|\,\mathscr{F}_s\right\} \\ &\quad - E\left\{\int_s^t g(\varepsilon^{-1}X_\varepsilon(\tau), X_\varepsilon(\tau))d\tau\,|\,\mathscr{F}_s\right\} \\ &\quad + \varepsilon^\alpha E\left[\int_s^t\int_{-\infty}^{\infty}\{\psi(\varepsilon^{-1}X_\varepsilon(\tau) + \varepsilon^{-1}\rho_1, X_\varepsilon(\tau) + \rho_1)\right. \\ &\quad - \psi(\varepsilon^{-1}X_\varepsilon(\tau) + \varepsilon^{-1}\rho_1, X_\varepsilon(\tau)) \\ &\quad \left. - \psi_\eta(\varepsilon^{-1}X_\varepsilon(\tau), X_\varepsilon(\tau))\rho_1\}\,d\tau\nu(y)dy\,|\,\mathscr{F}_s\right].\end{aligned}$$

The last term of the above is equal to

$$\begin{aligned} & E\left[\int_s^t\int_{-\infty}^{\infty}\{\psi(\varepsilon^{-1}X_\varepsilon(\tau) + y, X_\varepsilon(\tau) + \varepsilon y) - \psi(\varepsilon^{-1}X_\varepsilon(\tau) + y, X_\varepsilon(\tau))\right. \\ & \left. - \psi_\eta(\varepsilon^{-1}X_\varepsilon(\tau), X_\varepsilon(\tau))\varepsilon y\}a(\varepsilon^{-1}X_\varepsilon(\tau), y)\,d\tau\nu(y)\,dy\,|\,\mathscr{F}_s\right],\end{aligned}$$

which yields $o(1)$ as $\varepsilon \downarrow 0$ after elementary calculations. Therefore, from (3.9) we have

$$E\left\{\int_s^t g(\varepsilon^{-1}X_\varepsilon(\tau),\ X_\varepsilon(\tau))d\tau \,|\, \mathscr{F}_s\right\} = o(1),\ \varepsilon \downarrow 0,$$

which combined with (3.8) yields

$$\begin{aligned} &E\{f(X_\varepsilon(t)) \,|\, \mathscr{F}_s\} - f(X_\varepsilon(s)) \\ &= E\left[\int_s^t \int_{-\infty}^{\infty} \{f(X_\varepsilon(\tau) + y) - f(X_\varepsilon(\tau)) \right. \\ &\qquad \left. - f'(X_\varepsilon(\tau))y\}\bar{a}(\varepsilon^{-1}y)d\tau\nu(y)dy \,|\, \mathscr{F}_s\right] + o(1) \\ &= E\left\{\int_s^t L_0 f(X_\varepsilon(\tau))d\tau \,|\, \mathscr{F}_s\right\} + o(1). \end{aligned}$$

Here we have used the assumption (1.5). The proof is finished.

The proof of the theorem is now completed as follows. If $\mathscr{B}_s$ denotes the coordinate σ-fields $\sigma\{w(\tau): \tau \leqq s\}$ in W, then Lemma 5 states that for any C^∞-function f with compact support and for $0 \leqq s < t$

$$E_\varepsilon^x\{f(w(t)) \,|\, \mathscr{B}_s\} - f(w(s)) = E_\varepsilon^x\left\{\int_s^t L_0 f(w(\tau))d\tau \,|\, \mathscr{B}_s\right\} + o(1).$$

Then, for $0 \leqq s < s_1 < t < t_1$ and a $\mathscr{B}_s$-measurable bounded continuous function $\Phi(w)$ on W we have

$$\begin{aligned} (3.10)\quad & E_x^\varepsilon\left\{(t_1 - t)^{-1}\int_t^{t_1} f(w(\tau))d\tau\Phi(w) - (s_1 - s)^{-1}\int_s^{s_1} f(w(\tau))d\tau\Phi(w)\right\} \\ &= E_\varepsilon^x\left\{(t_1 - t)^{-1}\int_t^{t_1} d\tau_1 (s_1 - s)^{-1}\int_s^{s_1} d\tau_2 \int_{\tau_2}^{\tau_1} L_0 f(w(\tau))d\tau\Phi(w)\right\} \\ &\qquad + o(1). \end{aligned}$$

Suppose that P_ε^x converges to some limit P_0^x as $\varepsilon \downarrow 0$ via a subsequence $\varepsilon_1 > \varepsilon_2 > \cdots \downarrow 0$. In (3.10) we make first $\varepsilon \downarrow 0$ via this subsequence and then $s_1 \downarrow s$, $t_1 \downarrow t$. Then the result yields

$$E_0^x\{f(w(t)) \,|\, \mathscr{B}_s\} - f(w(s)) = E_0^x\left\{\int_s^t L_0 f(w(\tau))d\tau \,|\, \mathscr{B}_s\right\},\ \text{a. s.},$$

and hence P_0^x must be the measure of the stable process governed by L_0 as was to be proved.

Here is an example in which (1.6) is satisfied. Assume that $\gamma_+=\gamma_->0$, $a(x, y)=a(1-x, -y)$ and $b(x)=-b(1-x)$. Then, $(T_1^t b)(x)=-(T_1^t b)(1-x)$ and so (1.6) is satisfied.

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