HIROSHIMA MATH. J.
7 (1977), 459–472

On the Semilinear Heat Equations With Time-lag

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(Received January 14, 1977)

§1. Introduction

We are concerned with the following semilinear heat equation with time-lag:

$$\frac{\partial}{\partial t} u(t, x) = \Delta u(t, x) + f(u(t-r, x), u(t, x)), \tag{1.1}$$

where r is a positive constant. A. Inoue-T. Miyakawa-K. Yoshida [3] studied the initial boundary value problem of the above equation (1.1) in a domain $\boldsymbol{\Omega}$ of $\boldsymbol{R}^3$ for some typical $f(\lambda, \mu)$. In this paper we assume that $f(\lambda, \mu)$ is a non-negative continuous function and consider the initial value problem of (1.1) in the whole of $\boldsymbol{R}^d$; the initial condition for (1.1) is given by

$$u(t, x) = a(t, x), \qquad -r \leq t \leq 0, \tag{1.2}$$

where $a(t, x)$ is a given function on $[-r, 0] \times \boldsymbol{R}^d$. If we put

$$H(t, x, y) = (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right),$$

$$H_t a(x) = \int_{\boldsymbol{R}^d} H(t, x, y) a(y) dy,$$

then the equation (1.1) with the initial condition (1.2) is transformed into the integral equation

$$\begin{cases} u(t, x) = H_t a(0, x) + \displaystyle\int_0^t ds H_{t-s} f(u(s-r, \cdot), u(s, \cdot))(x), & t > 0, \\ u(t, x) = a(t, x), & -r \leq t \leq 0. \end{cases} \tag{1.3}$$

In this paper, when we speak of a solution of (1.1) with the initial condition (1.2), we always mean that it is a solution of (1.3). By a positive solution we mean a solution which is strictly positive for $t > 0$. We assume the following conditions:

(f.1) $f(\lambda, \mu)$ is a non-negative continuous function defined on $\boldsymbol{R}_+ \times \boldsymbol{R}_+ = [0, \infty) \times [0, \infty)$ and nondecreasing in λ for each fixed μ.

(f.2)′ For each positive number M, there exists a positive constant κ_M such that

$$|f(\lambda, \mu_1) - f(\lambda, \mu_2)| \leq \kappa_M |\mu_1 - \mu_2|, \qquad 0 \leq \lambda, \mu_1, \mu_2 \leq M.$$

(a.1) $a(t, x)$ is a non-negative bounded continuous function on $[-r, 0]\times \boldsymbol{R}^d$ and $a(0, x)$ is not identically zero.

Under these conditions the equation (1.1) with the initial condition (1.2) has a unique positive (local) solution, which is denoted by $u(t, x)$ or $u(t, x; a, f; r)$ when we want to stress the initial value a, the nonlinear term f and the time-lag r. We say that *a positive (global) solution $u(t, x)$ of* (1.1) *grows up to infinity (as $t\to\infty$) if for any positive number M and any compact set $\boldsymbol{K}$ in $\boldsymbol{R}^d$ there exists a positive number T such that $u(t, x)\geq M$ for any $x\in \boldsymbol{K}$ and $t\geq T$.*

Our problem is to find a sufficient condition for any positive global solution of (1.1) (if it exists) to grow up to infinity as $t\to\infty$. When there is no time-lag, H. Fujita [1] and K. Hayakawa [2] investigated the blowing up problem. Recently K. Kobayashi-T. Sirao-H. Tanaka [6] gave a sufficient condition for the growing up of positive solutions of (1.1) with $f(\lambda, \mu)=f(\mu)$ (without time-lag). The purpose of this paper is to extend the results of [6] to the case with time-lag.

Our main results are stated as follows. Put $f_\delta(\lambda)=\inf_{\lambda\leq\xi,\eta\leq\delta} f(\xi, \eta)$, $\delta>0$ and $f_\Delta(\lambda)=f(\lambda, \lambda)$. Assume that $f(\lambda, \mu)>0$ for $\lambda>0$ and $\mu>0$. Then, *under some additional conditions on f, the divergence of the integral $\int_0^\delta f_\delta(\lambda)\lambda^{-2-(2/d)}d\lambda$ for some $\delta>0$ implies the growing up of positive global solutions of* (1.1), *if they exist, while the convergence of $\int_0^\delta f_\Delta(\lambda)\lambda^{-2-(2/d)}d\lambda$ implies that there exists a positive solution of* (1.1) *converging to* 0 *uniformly in x as $t\to\infty$.* Similar results can be obtained in the case when $f(\lambda, 1)=0$ for $0\leq\lambda\leq 1$ and $f(\lambda, \mu)>0$ for $0<\lambda$, $\mu<1$. Finally, it will be remarked that some semilinear heat equations with time-lag can be described in terms of branching processes in a way similar to the case without time-lag.

The auther wishes to thank Professor H. Tanaka for his helpful suggestions and advice.

§2. Preliminaries

In this section we give some preliminary results, among which Theorem 2 will play an important role in the next section. First we state an elementary comparison lemma.

LEMMA 1. *Let $a_i(t, x)$, $i=1, 2$, be bounded continuous functions on $[-r, 0]\times \boldsymbol{R}^d$ and $f_i(\lambda, \mu)$, $i=1, 2$, continuous functions on $\boldsymbol{R}\times \boldsymbol{R}$. We assume that for each $M>0$ there exists a constant $\kappa=\kappa_M$ such that $|f_i(\lambda, \mu_1)-f_i(\lambda, \mu_2)|\leq \kappa|\mu_1-\mu_2|$, $i=1, 2$, for $|\lambda|$, $|\mu_1|$, $|\mu_2|\leq M$, and that at least one of $f_1(\lambda, \mu)$ and $f_2(\lambda, \mu)$ is nondecreasing in λ for each fixed μ. Moreover, we assume that $f_1\geq f_2$ and $a_1\geq a_2$. Then, we have*

$$u(t, x; a_1, f_1; r) \geq u(t, x; a_2, f_2; r)$$

for any $t \geq 0$ *belonging to a time interval in which the solutions exist.*

PROOF. We consider the case when $f_1(\lambda, \mu)$ is nondecreasing in λ. We put $u_i(t, x)=u(t, x; a_i, f_i; r)$, $i=1, 2$, and prove, for each integer $n \geq 0$, that $u_1(t, x) \geq u_2(t, x)$ for any $t \in ((n-1)r, nr]$ and $x \in \boldsymbol{R}^d$. Since the validity of the inequality for $n=0$ is a part of the assumptions of the theorem, we assume that the inequality holds for n and prove that it holds also for $n+1$. If we put $g_i(t, x, \mu)=f_i(u_i(t-r, x), \mu)$, $i=1, 2$, then $u_i(t, x)$ satisfies $\partial u_i/\partial t=\Delta u_i+g_i(t, x, u_i)$, $nr \leq t \leq (n+1)r$, and $u_1(nr, x) \geq u_2(nr, x)$, $g_1(t, x, \mu) \geq g_2(t, x, \mu)$ for $nr \leq t \leq (n+1)r$. Therefore, by a well known comparison theorem in partial differential equations we have $u_1(t, x) \geq u_2(t, x)$ for $nr \leq t \leq (n+1)r$, as was to be proved. The case when $f_2(\lambda, \mu)$ is nondecreasing in λ can be treated similarly.

In the sequel, we assume that f is a non-negative continuous function on $\boldsymbol{R}_+ \times \boldsymbol{R}_+$ satisfying the conditions (f.1) and (f.2)$'$, and that a is a non-negative bounded continuous function on $[-r, 0] \times \boldsymbol{R}^d$ satisfying (a.1). The following assertions 1°, 2° and 3° can be proved in the same way as in the corresponding lemmas of [6].

1° If any positive solution $u(t, x; a, f; r)$ of (1.1) for any time lag $r>0$ either blows up in finite time or satisfies

$$\limsup_{t \to \infty} \|u(t, \cdot\,; a, f; r)\|_\infty = \infty,$$

then any positive solution of

$$\frac{\partial u}{\partial t} = \Delta u + \varepsilon f(u(t-r, x), u(t, x))$$

has the same property for any $\varepsilon>0$ and $r>0$.

2° For any positive t there exist positive constants α and β such that $u(t, x; a, f; r) \geq \alpha \exp(-\beta|x|^2)$ (provided the solution exists up to t).

3° We consider a class of monotone radial functions:

$$\mathscr{A} = \{a \in C(\boldsymbol{R}^d): a(x) \geq 0, \not\equiv 0; a(x) \geq a(y) \quad \text{for} \quad |x| \leq |y|\}.$$

If $f(\lambda, \mu)$ is also nondecreasing in μ for each fixed λ and if $a(t, x) \in \mathscr{A}$ for any $t \in [-r, 0]$, then $u(t, x; a, f; r) \in \mathscr{A}$ for $t \geq 0$ (provided the solution exists up to t).

Making use of these preliminary results 1°, 2°, 3°, we can prove the following theorems; the proof is much the same as that of Theorems 3.3, 3.4 in [6] and so is omitted.

THEOREM 2. *Assume that f and $\tilde{f}$ satisfy* (f.1) *and* (f.2)$'$ *and also that the* *following conditions are satisfied*:

(i) $f(\lambda, \mu) > 0$ *for* $\lambda > 0, \mu > 0$.
(ii) $\tilde{f}(\lambda, \mu)$ *is nondecreasing in* μ *for each fixed* λ *and* $\tilde{f}(\lambda, 0) = \tilde{f}(0, \mu) = 0$.
(iii) $\liminf_{(\lambda \downarrow 0, \mu \downarrow 0)} \dfrac{f(\lambda, \mu)}{\tilde{f}(\lambda, \mu)} > 0$.

Further, we assume that for any time-lag $r>0$ *any positive solution* $\tilde{u}(t, x)$ *of*

$$\frac{\partial u}{\partial t} = \Delta u + \tilde{f}(u(t-r, x), u(t, x)) \tag{2.1}$$

either blows up in finite time or satisfies

$$\limsup_{t\to\infty} \|\tilde{u}(t, \cdot)\|_\infty = \infty. \tag{2.2}$$

Then any positive global solution of (1.1), *if it exists, grows up to infinity for any time-lag* $r>0$.

THEOREM 2′. *Let* f *be a non-negative continuous function defined on* $[0, 1]\times[0, 1]$ *such that* $f(\lambda, 1)=0$ *for* $0\le\lambda\le 1$ *and* $f(\lambda, \mu)>0$ *for* $0<\lambda, \mu<1$. *Assume that* $f(\lambda, \mu)$ *is nondecreasing in* λ *for each fixed* μ *and satisfies* (f.2)′ *with* $M=1$ *and that* $\tilde{f}(\lambda, \mu)$ *is a continuous function on* $\boldsymbol{R}_+\times\boldsymbol{R}_+$ *satisfying* (f.1), (f.2)′, $\tilde{f}(\lambda, 0)=\tilde{f}(0, \mu)=0$ *and also nondecreasing in* μ *for each fixed* λ. *Further, we assume that*

$$\liminf_{(\lambda\downarrow 0, \mu\downarrow 0)} \frac{f(\lambda, \mu)}{\tilde{f}(\lambda, \mu)} > 0$$

and that for any time-lag $r>0$ *any positive solution* $\tilde{u}(t, x)$ *of* (2.1) *either blows up in finite time or satisfies* (2.2). *Then any positive solution of* (1.1) *dominated by* 1 *converges to* 1 *uniformly on each compact set in* $\boldsymbol{R}^d$ *as* $t\to\infty$.

§3. The growing up problem

3.1. A sufficient condition for growing up

Before stating our theorem we introduce several conditions concerning f. We put $f_\delta(\lambda) = \inf_{\lambda\le\xi,\eta\le\delta} f(\xi, \eta)$ for $\lambda\le\delta$.

(f.1) $f(\lambda, \mu)$ is a non-negative continuous function defined on $\boldsymbol{R}_+\times\boldsymbol{R}_+$ and nondecreasing in λ for each fixed μ.
(f.2) $f(\lambda, \mu)$ is a locally Lipschitz continuous function on $\boldsymbol{R}_+\times\boldsymbol{R}_+$.
(f.3) $f(\lambda, \mu) > 0$ for $\lambda > 0, \mu > 0$.
(f.4) $\int_0^\delta f_\delta(\lambda)/\lambda^{2+\frac{2}{d}} d\lambda = \infty$ for some $\delta > 0$.

(f.5) There exist positive constants c and δ such that

$$f_\delta(\lambda_1\lambda_2) \geq c\lambda_2^{1+\frac{2}{d}} f_\delta(\lambda_1) \quad \text{for} \quad 0 < \lambda_1 \leq \lambda_2,\ \lambda_1 < c,\ \lambda_1\lambda_2 < c.$$

Denote by $\mathscr{F}$ the class of all functions f on $\boldsymbol{R}_+ \times \boldsymbol{R}_+$ satisfying (f.1)$\sim$(f.5).

THEOREM 3. *If $f(\lambda, \mu)$ belongs to $\mathscr{F}$, then any positive global solution of* (1.1), *if it exists, grows up to infinity as $t \to \infty$.*

To simplify the proof, we define a subclass $\tilde{\mathscr{F}}$ of $\mathscr{F}$. Namely we denote by $\tilde{\mathscr{F}}$ the class of all functions on $\boldsymbol{R}_+ \times \boldsymbol{R}_+$ satisfying (f.1), (f.2), (f.3) and the following conditions (f.4)*, (f.5)*, (f.6): Put $f_\Delta(\lambda) = f(\lambda, \lambda)$.

(f.4)* $$\int_0^\delta f_\Delta(\lambda)/\lambda^{2+\frac{2}{d}} d\lambda = \infty \quad \text{for some} \quad \delta > 0.$$

(f.5)* There exists a positive constant c such that

(a) $f_\Delta(\lambda_1\lambda_2) \geq c\lambda_2^{1+\frac{2}{d}} f_\Delta(\lambda_1)$ for $0 < \lambda_1 \leq \lambda_2,\ \lambda_1 < c$,

(b) $f_\Delta(\lambda_1\lambda_2) \geq c\lambda_2^{2+\frac{2}{d}} f_\Delta(\lambda_1)$ for $0 < \lambda_2 \leq \lambda_1 < c$.

(f.6) $f(\lambda, \mu)$ is nondecreasing in μ for each fixed λ.
We claim that
(3.1) for each $f(\lambda, \mu)$ in $\mathscr{F}$ there exists $\tilde{f}(\lambda, \mu)$ in $\tilde{\mathscr{F}}$ such that

$$\liminf_{(\lambda, \mu \downarrow 0)} f(\lambda, \mu)/\tilde{f}(\lambda, \mu) > 0.$$

In fact, applying Lemma 3.6 of [6] to $f_\delta(\lambda)$ we can find a nondecreasing locally Lipschitz continuous function $\tilde{f}_\delta(\lambda)$ satisfying (i) $\tilde{f}_\delta(0) = 0$, $\tilde{f}_\delta(\lambda) > 0$ $(\lambda > 0)$, (ii) $\int_{0+} \tilde{f}_\delta(\lambda)/\lambda^{2+2/d} d\lambda = \infty$, (iii) there exists a positive constant c such that

$$\tilde{f}_\delta(\lambda_1\lambda_2) \geq c\lambda_2^{1+2/d} \tilde{f}_\delta(\lambda_1), \qquad 0 < \lambda_1 \leq \lambda_2,\ \lambda_1 < c,$$
$$\tilde{f}_\delta(\lambda_1\lambda_2) \geq c\lambda_2^{2+2/d} \tilde{f}_\delta(\lambda_1), \qquad 0 < \lambda_2 \leq \lambda_1 < c,$$

and (iv) $\liminf_{\lambda \downarrow 0} f_\delta(\lambda)/\tilde{f}_\delta(\lambda) > 0$. Then, $\tilde{f}(\lambda, \mu) = \tilde{f}_\delta(\lambda \wedge \mu)$ has the desired properties.

By virtue of (3.1) and Theorem 2, it is enough to prove Theorem 3 replacing $\mathscr{F}$ by $\tilde{\mathscr{F}}$. By 2° and Lemma 1 in §2, it is also enough to treat the case when $a(\cdot, \cdot)$ satisfies $a(0, x) = \alpha \exp(-\beta|x|^2)$, $0 < \alpha < c$, $\beta > 0$, where c is the constant appearing in (f.5)*. So we assume that $f \in \tilde{\mathscr{F}}$, $a(0, x) = \alpha \exp(-\beta|x|^2)$ and define $u_n(t, x)$, $n \geq 0$, as follows:

$$u_0(t, x) = \begin{cases} H_t a(0, x) = \alpha(1 + 4\beta t)^{-d/2} \exp\{-\beta|x|^2/(1 + 4\beta t)\}, & t > 0, \\ a(t, x), & -r \leq t \leq 0. \end{cases}$$

$$u_n(t,\ x) = \begin{cases} H_t a(0,\ x) + \int_0^t dsH_{t-s}f(u_{n-1}(s-r,\ \cdot),\ u_{n-1}(s,\cdot)), & t>0, \\ a(t,\ x), \quad -r \le t \le 0, \quad (n \ge 1). \end{cases}$$

Let $u(t,\ x)$ be the solution of (1.1). Then by (f.1) and (f.6) we have

$$u(t,\ x) \ge u_n(t,\ x), \qquad n \ge 0,$$

provided that $u(\cdot,\ \cdot)$ exists up to t. To simplify the notation we put $\gamma = 1 + \frac{2}{d}$ and

$$\theta(t) = \alpha(1+4\beta t)^{-d/2}$$

$$\varphi(t) = \int_0^t \frac{f_\Delta(\theta(s))}{\theta(s)} ds = \frac{\alpha^{2/d}}{2\beta d} \int_{\theta(t)}^{\alpha} \frac{f_\Delta(\lambda)}{\lambda^{2+2/d}} d\lambda.$$

We note that the assumption (f.4)* implies $\varphi(t) \to \infty$ as $t \to \infty$. The following lemma is a modification of Lemma 2.2 of [6] adapted to the present situation, and fundamentally the proof is also similar. But, since the proof is somewhat complicated, we give it in full.

LEMMA 4. *Let $f(\lambda,\ \mu)$ belong to $\tilde{\mathscr{F}}$ and $a(t,\ x)$ be a bounded continuous function such that $a(0,\ x) = \alpha\exp(-\beta|x|^2)$, $0<\alpha<c$, $\beta>0$. Then we have for any positive integer n and $t \ge nr$*

$$u_n(t,\ x) \ge (1+4\beta nr)^{-d/2}\{1 + B_n(t,\ x)\}u_0(t-nr,\ x), \tag{3.2}$$

where

$$B_n(t,\ x) = C_n \varphi(t-nr)^{1+\gamma+\cdots+\gamma^{n-1}} \exp\left\{-\frac{\beta(\gamma+\cdots+\gamma^n)}{1+4\beta(t-nr)}|x|^2\right\}, \quad n \ge 1,$$

$$C_n = \frac{\{c(1+4\beta r)^{-(1+\gamma)d/2}\}^{1+\gamma+\cdots+\gamma^{n-1}}}{(1+4\beta nr)^{-d/2}(1+\gamma+\cdots+\gamma^n)^{d/2}} \times \prod_{k=0}^{n-1} \left\{\frac{(1+4\beta kr)^{-d/2}}{(1+\gamma+\cdots+\gamma^k)^{1+\gamma d/2}}\right\}^{\gamma^{n-k-1}}, \quad n \ge 1.$$

PROOF. We prove this lemma by induction.
Step 1. We consider the case $n=1$. First we note that for $s \ge r$

$$\begin{cases} u_0(s,\ x) = \alpha(1+4\beta s)^{-d/2}\exp\{-\beta|x|^2/(1+4\beta s)\} \\ \qquad \ge (1+4\beta r)^{-d/2}u_0(s-r,\ x), \\ u_0(s-r,\ x) \ge (1+4\beta r)^{-d/2}\, u_0(s-r,\ x). \end{cases} \tag{3.3}$$

Since $f(\lambda,\ \mu)$ is nondecreasing in λ and μ, we have

$$u_1(t,\ x) \ge u_0(t,\ x) + \int_r^t dsH_{t-s}f_\Delta((1+4\beta r)^{-d/2}u_0(s-r,\ \cdot))(x). \tag{3.4}$$

Applying (f.5)* with

$$\lambda_1 = \theta(s-r) < c,$$
$$\lambda_2 = (1+4\beta r)^{-d/2}\exp\{-\beta|x|^2/(1+4\beta(s-r))\} \leq 1,$$

we have

$$\begin{aligned}(3.5)\quad f_\Delta((1+4\beta r)^{-d/2}u_0(s-r,x)) &= f_\Delta(\lambda_1\lambda_2)\\ &\geq \min\{c\lambda_2^\gamma f_\Delta(\lambda_1),\ c\lambda_2^{1+\gamma}f_\Delta(\lambda_1)\} = c\lambda_2^{1+\gamma}f_\Delta(\lambda_1)\\ &= c(1+4\beta r)^{-(1+\gamma)d/2}\exp\{-(1+\gamma)\beta|x|^2/(1+4\beta(s-r))\}f_\Delta(\theta(s-r)).\end{aligned}$$

In order to estimate the integrand in the right hand side of (3.4), we write

$$\begin{aligned}&H_{t-s}\exp\{-(1+\gamma)\beta|\cdot|^2/(1+4\beta(s-r))\}\\ &\quad = \{1+4\beta'(t-s)\}^{-d/2}\exp\{-\beta'|x|^2/(1+4\beta'(t-s))\},\\ &\beta' = (1+\gamma)\beta/\{1+4\beta(s-r)\}.\end{aligned}$$

Since for $r \leq s \leq t$

$$\begin{aligned}\{1+4\beta'(t-s)\}^{-d/2} &= \left\{\frac{1+4\beta(1+\gamma)(t-r)-4\beta\gamma(s-r)}{1+4\beta(t-r)}\right\}^{-d/2}\\ &\qquad\times\left\{\frac{1+4\beta(s-r)}{1+4\beta(t-r)}\right\}^{d/2}\\ &\geq (1+\gamma)^{-d/2}\{(1+4\beta(s-r))(1+4\beta(t-r))^{-1}\}^{d/2},\end{aligned}$$

$$\frac{\beta'}{1+4\beta'(t-s)} \leq \frac{\beta(1+\gamma)}{1+4\beta(t-r)},$$

we have

$$\begin{aligned}(3.6)\quad &H_{t-s}\exp\{-(1+\gamma)\beta|\cdot|^2/(1+4\beta(s-r))\}\\ &\geq (1+\gamma)^{-d/2}\{(1+4\beta(s-r))(1+4\beta(t-r))^{-1}\}^{d/2}\\ &\qquad\times\exp\{-\beta(1+\gamma)|x|^2/(1+4\beta(t-r))\}\\ &= (1+\gamma)^{-d/2}u_0(t-r,x)\exp\{-\beta\gamma|x|^2/(1+4\beta(t-r))\}/\theta(s-r).\end{aligned}$$

Therefore, noting $u_0(t,x) \geq (1+4\beta r)^{-d/2}u_0(t-r,x)$ and the definition of φ, we have from (3.4), (3.5) and (3.6)

$$
\begin{aligned}
&u_1(t, x) \\
&\geq (1+4\beta r)^{-d/2} u_0(t-r, x)\Big[1 + c(1+4\beta r)^{-\gamma d/2}(1+\gamma)^{-d/2} \\
&\qquad\qquad \times \exp\{-\beta\gamma|x|^2/(1+4\beta(t-r))\} \int_r^t \frac{f_\Delta(\theta(s-r))}{\theta(s-r)}\, ds\Big] \\
&= (1+4\beta r)^{-d/2} u_0(t-r, x)\{1 + B_1(t, x)\}, \qquad t \geq r.
\end{aligned}
$$

Step 2. Next, assuming that (3.2) holds for n we prove that (3.2) holds also for $n+1$. Write

$$
(3.7) \qquad u_{n+1}(t, x) = u_0(t, x) + \int_0^t ds H_{t-s} f(u_n(s-r, \cdot), u_n(s, \cdot))(x).
$$

From (3.3) we have, for $s \geq (n+1)r$,

$$
(3.8) \qquad \begin{cases} u_0(s-nr, x) \geq (1+4\beta r)^{-d/2} u_0(s-(n+1)r, x), \\ u_0(s-(n+1)r, x) \geq (1+4\beta r)^{-d/2} u_0(s-(n+1)r, x). \end{cases}
$$

First we shall estimate $f(u_n(s-r, x), u_n(s, x))$ from below. Since $B_n(t, x)$ is nondecreasing in t, the use of induction hypothesis and (3.8) implies that for $s \geq (n+1)r$

$$
\begin{aligned}
(3.9) \quad &\min\{u_n(s-r, x), u_n(s, x)\} \\
&\geq (1+4\beta nr)^{-d/2}\{1 + B_n(s-r, x)\}(1+4\beta r)^{-d/2} u_0(s-(n+1)r, x) \\
&= \lambda_1\lambda_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_1 &= \theta(s-(n+1)r), \qquad (<c), \\
\lambda_2 &= (1+4\beta nr)^{-d/2}\{1 + B_n(s-r, x)\}(1+4\beta r)^{-d/2} \\
&\qquad\qquad \times \exp[-\beta|x|^2/\{1+4\beta(s-(n+1)r)\}].
\end{aligned}
$$

Since $f(\lambda, \mu)$ is nondecreasing in λ and μ, we have from (3.8) for $s \geq (n+1)r$

$$
f(u_n(s-r, x), u_n(s, x)) \geq f(\lambda_1\lambda_2, \lambda_1\lambda_2) = f_\Delta(\lambda_1\lambda_2).
$$

We now apply (f.5)* to $f_\Delta(\lambda_1\lambda_2)$. In case $\lambda_1 < \lambda_2$ we have from (a) of (f.5)*

$$
\begin{aligned}
&f_\Delta(\lambda_1\lambda_2) \geq c\lambda_2^\gamma f_\Delta(\lambda_1) \\
&= c(1+4\beta nr)^{-\gamma d/2}\{1 + B_n(s-r, x)\}^\gamma (1+4\beta r)^{-\gamma d/2} \\
&\qquad \times \exp[-\beta\gamma|x|^2/\{1+4\beta(s-(n+1)r)\}] f_\Delta(\theta(s-(n+1)r),
\end{aligned}
$$

while in case $\lambda_1 \geq \lambda_2$

$$f_\Delta(\lambda_1\lambda_2) \geq c\lambda_2^{1+\gamma} f_\Delta(\lambda_1)$$

$$= c(1+4\beta nr)^{-(1+\gamma)d/2}\{1+B_n(s-r,x)\}^{(1+\gamma)}(1+4\beta r)^{-(1+\gamma)d/2}$$

$$\times \exp\left[-\beta(1+\gamma)|x|^2/\{1+4\beta(s-(n+1)r)\}\right] f_\Delta(\theta(s-(n+1)r)).$$

Hence we have for $s \geq (n+1)r$

$$(3.10)\quad f(u_n(s-r,x), u_n(s,x))$$

$$\geq c(1+4\beta nr)^{-(1+\gamma)d/2} B_n(s-r,x)^\gamma (1+4\beta r)^{-(1+\gamma)d/2}$$

$$\times \exp\left[-\beta(1+\gamma)|x|^2/\{1+4\beta(s-(n+1)r)\}\right] f_\Delta(\theta(s-(n+1)r))$$

$$= c(1+4\beta nr)^{-(1+\gamma)d/2}\,(1+4\beta r)^{-(1+\gamma)d/2} C_n^\gamma$$

$$\times \exp(-\beta''|x|^2)\varphi(s-(n+1)r)^{\gamma+\cdots+\gamma^n} f_\Delta(\theta(s-(n+1)r)),$$

where $\beta''=\beta(1+\gamma+\cdots+\gamma^{n+1})/\{1+4\beta(s-(n+1)r)\}$. Next, in order to estimate the integrand in the right hand side of (3.7), we notice that for $(n+1)r \leq s \leq t$

$$\{1+4\beta''(t-s)\}^{-d/2}$$

$$\geq \left\{\frac{1+4\beta(1+\gamma+\cdots+\gamma^{n+1})\,(t-(n+1)r)}{1+4\beta(t-(n+1)r)}\right\}^{-d/2}$$

$$\times \left\{\frac{1+4\beta(s-(n+1)r)}{1+4\beta(t-(n+1)r)}\right\}^{d/2}$$

$$\geq (1+\gamma+\cdots+\gamma^{n+1})^{-d/2}\left\{\frac{1+4\beta(s-(n+1)r)}{1+4\beta(t-(n+1)r)}\right\}^{d/2},$$

$$\frac{\beta''}{1+4\beta''(t-s)} \leq \frac{\beta(1+\gamma+\cdots+\gamma^{n+1})}{1+4\beta(t-(n+1)r)}.$$

Then we have for $(n+1)r \leq s \leq t$

$$(3.11)\quad H_{t-s}\exp(-\beta''|\cdot|^2)$$

$$= (1+4\beta''(t-s))^{-d/2}\exp\{-\beta''|x|^2/(1+4\beta''(t-s))\}$$

$$\geq \frac{(1+\gamma+\cdots+\gamma^{n+1})^{-d/2}}{\theta(s-(n+1)r)}$$

$$\times \exp\left\{-\frac{\beta(\gamma+\cdots+\gamma^{n+1})}{1+4\beta(t-(n+1)r)}|x|^2\right\} u_0(t-(n+1)r, x).$$

Therefore, from (3.7), (3.10) and (3.11) we have for $t \geq (n+1)r$

$$\begin{aligned}(3.12)\qquad & u_{n+1}(t, x) - u_0(t, x) \\ & \geq c(1 + 4\beta nr)^{-(1+\gamma)d/2}(1 + 4\beta r)^{-(1+\gamma)d/2} \\ & \quad \times C_n^\gamma(1 + \gamma + \cdots + \gamma^{n+1})^{-d/2} \\ & \quad \times \exp\left\{-\frac{\beta(\gamma + \cdots + \gamma^{n+1})}{1 + 4\beta(t - (n+1)r)}|x|^2\right\} u_0(t - (n+1)r, x) \\ & \quad \times \int_{(n+1)r}^t \varphi(s - (n+1)r)^{\gamma+\cdots+\gamma^n} \frac{f_\Delta(\theta(s - (n+1)r))}{\theta(s - (n+1)r)}\, ds .\end{aligned}$$

Since

$$\begin{aligned}&\int_{(n+1)r}^t \varphi(s - (n+1)r)^{\gamma+\cdots+\gamma^n} \frac{f_\Delta(\theta(s - (n+1)r))}{\theta(s - (n+1)r)}\, ds \\ &= (1 + \gamma + \cdots + \gamma^n)^{-1} \varphi(t - (n+1)r)^{1+\gamma+\cdots+\gamma^n}, \\ &u_0(t, x) \geq \{1 + 4\beta(n+1)r\}^{-d/2} u_0(t - (n+1)r, x), \\ &\qquad\qquad\qquad\qquad t \geq (n+1)r ,\end{aligned}$$

inserting the explicit representation of C_n into the right hand side of (3.12), we finally obtain

$$u_{n+1}(t, x) \geq (1 + 4\beta(n+1)r)^{-d/2}\{1 + B_{n+1}(t, x)\} u_0(t - (n+1)r, x), \qquad t \geq (n+1)r,$$

and so the lemma is proved.

Now we proceed to the proof of Theorem 3. We may assume that f belongs to $\tilde{\mathscr{F}}$ and $a(0, x) = \alpha \exp(-\beta|x|^2)$, $0 < \alpha < c$, $\beta > 0$. By Lemma 4 we have for $n \geq 1$ and $t \geq nr$,

$$\begin{aligned}(3.13)\qquad u(t, x) &\geq (1 + 4\beta nr)^{-d/2} B_n(t, x) u_0(t - nr, x) \\ &= \alpha(1 + 4\beta nr)^{-d/2}(1 + 4\beta(t - nr))^{-d/2} B_n(t, x) \\ &\qquad \times \exp\{-\beta|x|^2/(1 + 4\beta(t - nr))\} \\ &= D_1 D_2 D_3 D_4,\end{aligned}$$

where

$$\begin{aligned}D_1 &= \alpha(1 + 4\beta(t - nr))^{-d/2}(1 + \gamma + \cdots + \gamma^n)^{-d/2}, \\ D_2 &= \{c(1 + 4\beta r)^{-(1+\gamma)d/2} \varphi(t - nr)\}^{1+\gamma+\cdots+\gamma^{n-1}} \\ &\qquad \times \exp\{-\beta(1 + \gamma + \cdots + \gamma^n)|x|^2/(1 + 4\beta(t - nr))\},\end{aligned}$$

$$D_3 = \prod_{k=0}^{n-1} (1 + 4\beta kr)^{-(d/2)\gamma^{n-k-1}},$$

$$D_4 = \prod_{k=0}^{n-1} (1 + \gamma + \cdots + \gamma^k)^{(1+\gamma d/2)\gamma^{n-k-1}}.$$

We notice that

$$D_1 \geq \alpha(1 + 4\beta(t - nr))^{-d/2}\gamma^{-(n+1)d/2}(\gamma - 1)^{d/2}. \tag{3.14}$$

Since

$$c(1 + 4\beta r)^{-(1+\gamma)d/2}\varphi(t - nr) \exp\{-\beta(1 + \gamma)|x|^2/(1 + 4\beta(t - nr))\}$$
$$\equiv \Phi(t, x, n) \geq 1$$

for x belonging to a compact set provided $t-nr$ is large enough,

$$D_2 \geq \Phi(t, x, n)^{1+\gamma+\cdots+\gamma^{n-1}} \geq \Phi(t, x, n)^{\gamma^{n-1}}. \tag{3.15}$$

Since $\sum_{k=0}^{\infty}\gamma^{-k}\log(1+4\beta kr)<\infty$, we have

$$D_3 = \exp\left\{-\gamma^{n-1}\frac{d}{2}\sum_{k=0}^{n-1}\gamma^{-k}\log(1 + 4\beta kr)\right\} \tag{3.16}$$
$$> \exp(-A_1\gamma^{n-1}),$$

where $A_1=(d/2)\sum_{k=0}^{\infty}\gamma^{-k}\log(1+4\beta kr)$. Further, since

$$\prod_{k=0}^{n-1}(1 + \gamma + \cdots + \gamma^k)^{-(1+\gamma d/2)\gamma^{-k}}$$
$$\geq (\gamma - 1)^{(1+\gamma d/2)\sum_{k=0}^{n-1}\gamma^{-k}}\gamma^{-(1+\gamma d/2)\sum_{k=0}^{n-1}(k+1)\gamma^{-k}},$$

we have

$$D_4 \geq (\gamma_0^{A_2}\gamma^{-A_3})^{\gamma^{n-1}}, \tag{3.17}$$

where $\gamma_0=(\gamma-1)\wedge 1$, $A_2=(1+\gamma d/2)\sum_{k=0}^{\infty}\gamma^{-k}<\infty$ and $A_3=(1+\gamma d/2)\sum_{k=0}^{\infty}(k+1)\gamma^{-k}<\infty$. If we put

$$A = c(1 + 4\beta r)^{-(1+\gamma)d/2}e^{-A_1}\gamma_0^{A_2}\gamma^{-A_3},$$

then from (3.13)$\sim$(3.17) we have

$$u(t, x) \geq \alpha(\gamma - 1)^{d/2}\{1 + 4\beta(t - nr)\}^{-d/2}[\gamma^{-(n+1)d\gamma^{1-n}/2}$$
$$\times A\varphi(t - nr)\exp\{-\beta(1 + \gamma)|x|^2/(1 + 4\beta(t - nr))\}]^{\gamma^{n-1}}$$

and hence

$$u(t, x) \geq \alpha(\gamma - 1)^{d/2}\{1 + 4\beta(t - nr)\}^{-d/2}\Big[\frac{1}{2} A\varphi(t - nr)$$
$$\times \exp\{-\beta(1+\gamma)|x|^2/(1 + 4\beta(t - nr))\}\Big]^{\gamma^{n-1}}$$

for $t \geq nr$ provided n is sufficiently large so that $\gamma^{-(n+1)d\gamma^{1-n}/2} \geq \frac{1}{2}$ holds. For any compact set $\boldsymbol{K}$ in $\boldsymbol{R}^d$ we can find a positive t_0 such that

$$\frac{A}{2}\varphi(t - nr)\exp[-\beta(1+\gamma)|x|^2/\{1 + 4\beta(t - nr)\}] \geq 2$$

for any $x \in \boldsymbol{K}$ and $t \geq t_0 + nr$. Since for any positive M there exists a positive integer N such that for any $n \geq N$ and $t_0 \leq t_1 \leq t_0 + r$

$$\alpha(\gamma - 1)^{d/2} 2^{\gamma^{n-1}}(1 + 4\beta t_1)^{-d/2} \geq M,$$

we have

$$u(t, x) \geq M$$

for any $x \in \boldsymbol{K}$, $n \geq N$ and $t_0 + nr \leq t \leq t_0 + (n+1)r$. This completes the proof of Theorem 3.

Theorem 3′. *Let f be a Lipschitz continuous function on* $[0, 1] \times [0, 1]$ *such that $f(\lambda, 1) = 0$ for $0 \leq \lambda \leq 1$ and $f(\lambda, \mu) > 0$ for $0 < \lambda, \mu < 1$. If $f(\lambda, \mu)$ is nondecreasing in λ for each fixed μ and satisfies the conditions* (f.4) *and* (f.5), *then any positive solution $u(t, x)$, dominated by* 1, *of the equation* (1.1) *converges to* 1 *uniformly on each compact set in $\boldsymbol{R}^d$ as $t \to \infty$.*

This is an immediate consequence of Theorem 3 and Theorem 2′.

3.2. A sufficient condition for non-growing up

Theorem 5. *Assume that $f(\lambda, \mu)$ satisfies the conditions* (f.1), (f.2), (f.6) *and the following conditions*:

(f.7) $$\int_0^\delta f_\Delta(\lambda)/\lambda^{2+\frac{2}{d}}\, d\lambda < \infty \quad \textit{for some} \quad \delta > 0.$$

(f.8) $f_\Delta(\lambda)/\lambda$ *is nondecreasing in* $\lambda > 0$.

Then, for any time-lag r there exists a positive solution $u(t, x)$ of (1.1) *converging to* 0 *uniformly in x as $t \to \infty$.*

Assume that the initial value $a(t, x)$ is equal to $a(x) = \alpha\exp(-\beta|x|^2)$ for any $-r \leq t \leq 0$. We consider the following equation

$$\begin{cases} \dfrac{dw}{dt} = \dfrac{f_\Delta(b\theta(t)w(t))}{\theta(t)}, \\ w(0) = 1, \end{cases}$$

where $\theta(t) = \sup_{x \in \boldsymbol{R}^d} H_t a(x) = \alpha(1+4\beta t)^{-d/2}$ and

$$\begin{aligned} b &= \max\{ \sup_{t \geq r, x \in \boldsymbol{R}^d} H_{t-r}a(x)/H_t a(x), \sup_{0 \leq t \leq r, x \in \boldsymbol{R}^d} a(x)/H_t a(x)\} \\ &= (1 + 4\beta r)^{d/2} > 1. \end{aligned}$$

Then, as in Lemma 5.2 of [6] we can prove that $u(t, x; a, f; r) \leq w(t) H_t a(x)$. The rest of the proof is much the same as that of Theorem 5.1 of [6], and so is omitted.

§4. Remarks to associated branching models

Some semilinear heat equations with time-lag can be described by branching processes in the frame of N. Ikeda-M. Nagasawa-S. Watanabe [4]. For simplicity we consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + u^m(t-r, x) u^n(t, x) - u(t, x), \tag{4.1}$$

where m and n are non-negative integers such that $m+n \geq 2$. Let S be the direct sum $\boldsymbol{R}^d + [-r, 0) \times \boldsymbol{R}^d$ which is to be the basic state space of the branching process described below. At time $t=0$, a single particle commences a Brownian motion $\{X(t)\}$ on $\boldsymbol{R}^d$, starting from the origin and continuing for an exponential holding time ζ (branching time) independent of $\{X(t)\}$ with $P(\zeta > t) = e^{-t}$. At time ζ, the particle splits in $m+n$ new particles, n particles among which continue along independent Brownian paths on $\boldsymbol{R}^d$ starting from $X(\zeta)$ until new branching time; the other m particles are swept out to the place $(-r, X(\zeta)) \in [-r, 0) \times \boldsymbol{R}^d$ at time ζ and, after obeying to the deterministic process $\{(-r+t-\zeta, X(\zeta))\}$ for $\zeta \leq t < \zeta + r$, at time $\zeta + r$ they land on $\boldsymbol{R}^d$ at the place $X(\zeta)$ from which they again commence independent Brownian motions on $\boldsymbol{R}^d$ until new branching times. Each of these particles, in turn, is subject to the same branching rule as above. Let $a(s, x)$ be a continuous function on $[-r, 0] \times \boldsymbol{R}^d$ such that $0 \leq a(s, x) \leq 1$. If, at time t, $k(t)$ particles $X_1(t), \ldots, X_{k(t)}(t)$ are in $\boldsymbol{R}^d$ and $\ell(t)$ particles $(\rho_1(t), Y_1(t)), \ldots, (\rho_{\ell(t)}(t), Y_{\ell(t)}(t))$ are in $[-r, 0) \times \boldsymbol{R}^d$, then

$$u(t, x) = E\{\prod_{i=1}^{k(t)} a(0, x + X_i(t)) \prod_{j=1}^{\ell(t)} a(\rho_j(t), x + Y_j(t))\}$$

satisfies the equation

(4.2) $$u(t, x) = e^{-t}H_t a(0, x) + \int_0^t e^{-s}H_s\{u^m(t-s-r, \cdot)u^n(t-s, \cdot)\}ds$$

Next, if we put $v(t, x)=1-u(t, x)$, then $v(t, x)$ satisfies (1.1) with $f(\lambda, \mu)= -(1-\lambda)^m(1-\mu)^n+1-\mu$, for which the assumption of Theorem 2′ are satisfied with $\tilde{f}(\lambda, \mu)=\min(\lambda^{1+2/d}, \mu^{1+2/d})$. It is easy to see that $\tilde{f}$ satisfies the assumption of Theorem 3.

The branching model associated with the equation

(4.3) $$\frac{\partial u}{\partial t} = \Delta u + u^m(t-r, x)u^n(t, x)$$

can also be obtained by introducing "age" as in M. Nagasawa [7], T. Sirao [8] and K. Kobayashi [5].

References

[1] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t=\Delta u+u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, Sect. I, **13** (1966), 109–124.

[2] K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic differential equations, Proc. Japan Acad., **49** (1973), 503–505.

[3] A. Inoue, T. Miyakawa and K. Yoshida, Some properties of solutions for semilinear heat equations with time-lag, to appear in J. Differential Equations.

[4] N. Ikeda, M. Nagasawa and S. Watanabe, Branching Markov processes, I, II, III, J. Math. Kyoto Univ., **8** (1968), 233–278, 365–410, **9** (1969), 95–160.

[5] K. Kobayashi, Some remarks to the construction of branching Markov processes with age and sign, Hiroshima Math. J., **3** (1973), 439–449.

[6] K. Kobayashi, T. Sirao and H. Tanaka, On the growing up problem for semilinear heat equations, to appear in J. Math. Soc. Japan.

[7] M. Nagasawa, Construction of branching Markov processes with age and sign, Kōdai Math. Semi. Rep., **20** (1968), 469–508.

[8] T. Sirao, On signed branching Markov processes with age, Nagoya Math. J., **32** (1968), 155–225.

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