HIROSHIMA MATH. J.
7 (1977), 855–864

Eigenfunctions of the Laplacian on a Real Hyperboloid of One Sheet

Kenji HIRAOKA, Shuichi MATSUMOTO and Kiyosato OKAMOTO

(Received May 20, 1977)

Introduction

In this paper we deal with an analogue of the Helgason conjecture [3] on the case of a real hyperboloid of one sheet. Contrary to the case of symmetric spaces any C^∞ eigenfunction of the Casimir operator on our space is a "Poisson transform" of some C^∞ function on the sphere. Our method is quite different from those of [2], [3], [5] etc. (cf. Remark 2). The authors are very grateful to Professor G. Schiffmann for helpful discussions.

§1. Notation and Preliminaries

Let X denote the real hyperboloid of one sheet in $\boldsymbol{R}^{p+1}$ $(p\geqq 2)$ defined by $x_1^2+\cdots+x_p^2-x_{p+1}^2=1$. Then the Lorentz group $G=SO_0(p, 1)$ acts canonically on X so that X is identified with the homogeneous space G/H_0, where

$$H_0=\left\{\begin{bmatrix} 1 & 0\cdots\cdots 0 \\ 0 & \\ \vdots & * \\ 0 & \end{bmatrix}\in G\right\}.$$

Put

$$H=\left\{\begin{bmatrix} \pm 1 & 0\cdots\cdots 0 \\ 0 & \\ \vdots & * \\ 0 & \end{bmatrix}\in G\right\}.$$

Then any function f in $C^\infty(G/H)$ is identified with a C^∞ function f on X such that $f(x)=f(-x)\,(x\in X)$. We regard $C^\infty(G/H)$ as a subspace consisting of all f in $C^\infty(G)$ such that $f(gh)=f(g)\,(g\in G,\ h\in H)$.

We denote by $\mathfrak{g}$ the Lie algebra of G. Then $\mathfrak{g}$ is identified with the set of all matrices (a_{ij}) $(1\leqq i, j\leqq p+1)$ such that $a_{ii}=0$ $(1\leqq i\leqq p+1)$, $a_{ij}=-a_{ji}$ $(1\leqq i\leqq j\leqq p)$ and $a_{p+1,j}=a_{j,p+1}$ $(1\leqq j\leqq p)$. We define subalgebras $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$ as follows. Let E_{ij} be the matrix such that the (i, j) component is equal to 1 and the other components are all equal to 0. We put $X_{ij}=E_{ij}-E_{ji}$ $(1\leqq i\leqq j\leqq p)$ and $Y_i=E_{i,p+1}+E_{p+1,i}$ $(1\leqq i\leqq p)$. Let $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ be the subalgebras spanned by X_{ij} $(1\leqq i, j\leqq p)$, Y_1 and $X_{1,i}+Y_i$ $(2\leqq i\leqq p)$, respectively. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. We denote by K, M, A and N the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then we have $G=KAH$. It follows that any f in $C^\infty(G/H)$ is uniquely determined by its value on KA.

Put $P=MAN$. Then P is a minimal parabolic subgroup of G. For any real number t we put $a_t=\exp tY_1$. We fix a complex number s once for all and consider the character ξ_s of P defined by $\xi_s(ma_tn)=e^{(2s+p-1)t/2}$ $(m\in M,\ a_t\in A,\ n\in N)$. Let L_s be the associated line bundle over G/P and $C^\infty(G/P, L_s)$ the space of all C^∞ sections of L_s. Then $C^\infty(G/P, L_s)$ is canonically identified with the set of all ϕ in $C^\infty(G)$ such that $\phi(gma_tn)=e^{-(2s+p-1)t/2}\phi(g)$ $(g\in G,\ m\in M,\ a_t\in A,\ n\in N)$. On the other hand $C^\infty(K/M)$ is canonically identified with the set of all ϕ in $C^\infty(K)$ such that $\phi(km)=\phi(k)$ $(k\in K,\ m\in M)$. The Iwasawa decomposition $G=KAN$ gives us the isomorphism

$$C^\infty(G/P, L_s)\ni\phi\longrightarrow\phi|_K\in C^\infty(K/M).$$

For any g in G, f in $C^\infty(G/H)$ and ϕ in $C^\infty(G/P, L_s)$, we define $(\pi(g)f)(x)=f(g^{-1}x)$ and $(\pi_s(g)\phi)(x)=\phi(g^{-1}x)$ $(x\in G)$. Then π and π_s are representations of G on $C^\infty(G/H)$ and $C^\infty(G/P, L_s)$. Let $d\pi$ and $d\pi_s$ be the infinitesimal representations of $\mathfrak{g}$ defined by π and π_s, respectively. We denote by the same notation the representations of the universal enveloping algebra of $\mathfrak{g}$ which are uniquely determined by $d\pi$ and $d\pi_s$, respectively.

§2. The Casimir Operator

Let Ω be the Casimir operator of $\mathfrak{g}$. Then

$$\Omega=\frac{1}{2(p-1)}\Big(-\sum_{1\leqq i<j\leqq p}X_{ij}^2+\sum_{1\leqq i\leqq p}Y_i^2\Big),$$

where X_{ij}^2 and Y_i^2 denote the squares in the universal enveloping algebra of $\mathfrak{g}$. We denote by Ω_K the Casimir operator of $\mathfrak{k}$ defined by the bilinear form $(2(p-1))^{-1}B(X, Y)$ $(X, Y\in\mathfrak{k})$, where B is the Killing form of $\mathfrak{g}$. Then we have the following

LEMMA 1. 1) *For any f in $C^\infty(G/H)$,*

$$(d\pi(\Omega)f)(ka_t)$$
$$=\frac{1}{2(p-1)}\Big\{\frac{\partial^2}{\partial t^2}+(p-1)\operatorname{th}t\frac{\partial}{\partial t}+\frac{d\pi(\Omega_K)}{\operatorname{ch}^2 t}\Big\}f(ka_t)$$
$$(k\in K,\ a_t\in A).$$

2) *For any ϕ in $C^\infty(G/P, L_s)$,*

$$d\pi_S(\Omega)\phi = \frac{1}{2(p-1)}\left(s - \frac{p-1}{2}\right)\left(s + \frac{p-1}{2}\right)\phi.$$

The proof is the same as in [5] so that we omit the proof.

§3. The Intertwining Operator $\mathscr{P}_S$

For any $x={}^t(x_1,\ldots, x_{p+1})$ and $y={}^t(y_1,\ldots, y_{p+1})$ in $\boldsymbol{R}^{p+1}$, we put $<x, y> = x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1}$. For any ϕ in $C^\infty(G/P, L_s)$ we define

$$(\mathrm{I}_s\phi)(g) = \int_K |<g^{-1}ke_1, e_2>|^{(2s-p+1)/2}\phi(k)dk \qquad (g\in G),$$

where dk is the normalized Haar measure on K, $e_1={}^t(1, 0,\ldots, 0, 1)$ and $e_2={}^t(1, 0,\ldots, 0)$. The integral converges and defines a holomorphic function of s when $\mathrm{Re}\, s>(p-3)/2$. It can be extended meromorphically to the whole complex plane which has poles of order one at $s-(p-1)/2\in\{-1, -3, -5,\ldots\}$. We put $\mathscr{P}_s=\dfrac{1}{\Gamma((2s-p+3)/4)}\mathrm{I}_s$. Then $\mathscr{P}_s$ is defined for all complex number s and it is easy to see that $\mathscr{P}_s$ is an intertwining operator of $C^\infty(G/P, L_s)$ into $C^\infty(G/H)$. Moreover one has the following lemma.

LEMMA 2.

$$\mathscr{P}_s\circ d\pi_s(\Omega) = d\pi(\Omega)\circ\mathscr{P}_s.$$

We put $\mathscr{H}_s=\{f\in C^\infty(G/H);\ d\pi_s(\Omega)f=(2(p-1))^{-1}(s+(p-1)/2)(s-(p-1)/2)f\}$. Then we obtain

COROLLARY. *$\mathscr{P}_s$ maps $C^\infty(G/P, L_s)$ into $\mathscr{H}_s$.*

Notice that K/M is canonically isomorphic to S^{p-1}. We denote by Λ the set of all integers or all non negative integers in case $p=2$ or $p\geqq 3$, respectively. Then the zonal spherical function ω_m with height $m(m\in\Lambda)$ is given by

$$\begin{aligned}&\omega_m(\exp(\theta_{p-1}X_{p-1,p})\exp(\theta_{p-2}X_{p-2,p-1})\cdots\exp(\theta_1X_{12})),\\ &=\begin{cases}\mathrm{F}(m+p-2, -m, (p-1)/2, (1-\cos\theta_1)/2) & (p\geqq 3),\\ e^{im\theta_1} & (p=2).\end{cases}\end{aligned}$$

In the rest of this paper we assume that $p\geqq 3$. In the case $p=2$ the proof is much easier. We denote by Λ_+ or Λ_- the set of all m in Λ which are even or odd, respectively. Let τ be the left regular representation of K on $C^\infty(K/M)$. For any m in Λ we denote by Γ_m the subspace of $C^\infty(K/M)$ which is spanned by the elements $\tau(k)\omega_m$ $(k\in K)$. Let τ_m be the restriction of τ to Γ_m. Then, as is well-known, $\{(\tau_m, \Gamma_m)\}_{m\in\Lambda}$ exhausts up to equivalence the set of all irreducible representations (of K) of class one with respect to M.

For any ϕ in $C^\infty(K/M)$ we put

$$\phi_m(k) = d_m\int_K \overline{\chi_m(k_1)}\phi(k_1^{-1}k)dk_1 \qquad (k\in K),$$

where χ_m and d_m denote the character and the degree of τ_m, respectively. Let $\mathscr{H}_m$ be the space of vectors in $C^\infty(G/H)$ which transform according to τ_m. We denote $\mathscr{H}_{s,m}=\mathscr{H}_s\cap\mathscr{H}_m$. For any f in $\mathscr{H}_s$ we put

$$f_m(g) = d_m\int_K \overline{\chi_m(k)}f(k^{-1}g)dk \qquad (g\in G).$$

Then it is obvious that $f_m\in\mathscr{H}_{s,m}$ and that $f=0$ if and only if $f_m=0$ for all m in Λ.

LEMMA 3. 1) *For any* ϕ *in* $C^\infty(K/M)$ *the expansion* $\phi(k)=\sum_{m\in\Lambda}\phi_m(k)$ *converges absolutely and uniformly on* K.

2) *For any* f *in* $\mathscr{H}_s$ *the expansion* $f(ka_t)=\sum_{m\in\Lambda}f_m(ka_t)$ *converges absolutely and uniformly on* K.

This lemma is proved by the usual routine.

When $\mathrm{Re}\, s$ is sufficiently large, the following lemma is an immediate consequence of the fact that $\mathscr{P}_s$ is an intertwining operator. By the analytic continuation we obtain

LEMMA 4. *For any* ϕ *in* $C^\infty(K/M)$,

$$(\mathscr{P}_s\phi)_m = \mathscr{P}_s\phi_m \qquad (m\in\Lambda).$$

COROLLARY. *For any* ϕ *in* $C^\infty(K/M)$,

$$(\mathscr{P}_s\phi)(ka_t) = \sum_{m\in\Lambda}(\mathscr{P}_s\phi_m)(ka_t)$$

converges absolutely and uniformly on K.

PROPOSITION 1. *For any* ϕ *in* Γ_m $(m\in\Lambda)$,

$$(\mathscr{P}_s\phi)(ka_t) = (\mathscr{P}_s\omega_m)(a_t)\phi(k) \qquad (k\in K,\ a_t\in A).$$

PROOF. For any ϕ in Γ_m we put

$$\phi_M(k) = \int_M \phi(mk)dm \qquad (k\in K),$$

where dm is the normalized Haar measure on M. Then clearly we have $\phi_M=\phi(e)\omega_m$.

$$
\begin{aligned}
(\mathscr{P}_s\phi)(a_t) &= \frac{1}{\Gamma((2s-p+3)/4)}\int_K |<ke_1, a_te_2>|^{(2s-p+1)/2}\phi(k)\,dk \\
&= \frac{1}{\Gamma((2s-p+3)/4)}\int_K |<mke_1, a_te_2>|^{(2s-p+1)/2}\phi(mk)\,dk \\
&= \frac{1}{\Gamma((2s-p+3)/4)}\int_K |<ke_1, a_te_2>|^{(2s-p+1)/2}\phi(mk)\,dk \\
&= \frac{1}{\Gamma((2s-p+3)/4)}\int_K |<ke_1, a_te_2>|^{(2s-p+1)/2}\phi_M(k)\,dk \\
&= \frac{1}{\Gamma((2s-p+3)/4)}\int_K |<ke_1,a_te_2>|^{(2s-p+1)/2}\phi(e)\omega_m(k)\,dk \\
&= (\mathscr{P}_s\omega_m)(a_t)\phi(e).
\end{aligned}
$$

Since $\mathscr{P}_s$ is an intertwining operator, we have

$$(\mathscr{P}_s\phi)(ka_t) = (\mathscr{P}_s\omega_m)(a_t)\phi(k).$$

§4. K-finite Eigenfunctions

In this section we study the space $\mathscr{H}_{s,m}$ $(m \in \Lambda)$ by means of the separation variables. Fix any f in $\mathscr{H}_{s,m}$. Then by definition

$$d\pi(\Omega)f = \frac{1}{2(p-1)}\left(s - \frac{p-1}{2}\right)\left(s + \frac{p-1}{2}\right)f.$$

On the other hand, from Lemma 1 we have

$$(d\pi(\Omega)f)(ka_t) = \frac{1}{2(p-1)}\left\{\frac{\partial^2}{\partial t^2} + (p-1)\,\mathrm{th}\,t\frac{\partial}{\partial t} + \frac{d\tau(\Omega_K)}{\mathrm{ch}^2 t}\right\}f(ka_t).$$

Since $d\tau_m(\Omega_K) = m(m+p-2)I$, from the above formulas we get

$$\left\{\frac{\partial^2}{\partial t^2} + (p-1)\,\mathrm{th}\,t\frac{\partial}{\partial t} + \frac{m(m+p-2)}{\mathrm{ch}^2 t} - \left(s - \frac{p-1}{2}\right)\left(s + \frac{p-1}{2}\right)\right\}f(ka_t) = 0.$$

Now we define $F_f(k, t) = f(ka_t)$ and $F_f^{\pm}(k, t) = 1/2\{f(kk_0a_t) \pm f(ka_t)\}$, where $k_0 = \mathrm{Diag}(-1, -1, 1, \dots, 1)$. For any t in $\boldsymbol{R}$ let $V_t^{\pm}$ denote the subspace of $\mathscr{H}_m$ which is spanned by $F_f^{\pm}(\cdot, t)(f \in \mathscr{H}_{s,m})$. Then it is easy to see that $V_t^{\pm}$ is an invariant subspace of $\mathscr{H}_m$. Let M' be the normalizer of A in K. Then $M' = M \cup k_0M$. We denote by $\sigma_{\pm}$ the representation of M' which is trivial on M such that $\sigma_{\pm}(k_0) = \pm I$. On the other hand it is clear that $V_t^{\pm}$ is contained in the induced representation from M' to K generated by $\sigma_{\pm}$. It follows from the Frobenius reciprocity law that $V_t^{\pm} \neq \{0\}$ if and only if the restriction of τ_m to M' contains $\sigma_{\pm}$ (which is equivalent to saying that $\tau_m(k_0)\omega_m = \pm\omega_m$). Since $\tau_m(k_0)\omega_m = (-1)^m\omega_m$, $V_t^{\pm} \neq \{0\}$ if and only if $(-1)^m = \pm 1$. For any f in $\mathscr{H}_{s,m}$ and k in K, we define $\mathrm{F}_f^k(t) = f(ka_t)$.

Let us consider an ordinary differential equation

$$(1) \qquad \left\{\frac{d^2}{dt^2} + (p-1)\,\mathrm{th}\,t\frac{d}{dt} + \frac{m(m+p-2)}{\mathrm{ch}^2 t} - \left(s - \frac{p-1}{2}\right)\left(s + \frac{p-1}{2}\right)\right\}\mathrm{F}(t) = 0$$

under the condition

$$(2) \qquad \mathrm{F}(-t) = (-1)^m \mathrm{F}(t).$$

Then in the above we have proved that $\mathrm{F}_f^k(t)$ satisfies the equation (1) under the condition (2). We put $x = \mathrm{th}^2 t$. Fix any solution $\mathrm{F}(t)$ of the differential equation (1) and we put

$$u(x) = (1-x)^{-(2s+p-1)/4}\mathrm{F}(x).$$

Then u satisfies the hypergeometric equation;

$$x(1-x)\frac{d^2u}{dx^2} + \{c - (a+b+1)x\}\frac{du}{dx} - abu = 0,$$

where $a = s/2 - m/2 - p/4 + 3/4$, $b = s/2 + m/2 + p/4 - 1/4$ and $c = 1/2$. Thus we conclude that $\mathrm{F}_f^k(t)$ coincides, up to constant, with $\mathrm{F}_{s,m}(t)$, where

$$\mathrm{F}_{s,m}(t) = \begin{cases} \mathrm{ch}\,t^{-(2s+p-1)/2}\mathrm{F}\left(\frac{s}{2} - \frac{m}{2} - \frac{p}{4} + \frac{3}{4}, \frac{s}{2} + \frac{m}{2} + \frac{p}{4} - \frac{1}{4}, \frac{1}{2}, \mathrm{th}^2 t\right) & (m \in \Lambda_+), \\ \mathrm{ch}\,t^{-(2s+p-1)/2}\,\mathrm{th}\,t\,\mathrm{F}\left(\frac{s}{2} - \frac{m}{2} - \frac{p}{4} + \frac{5}{4}, \frac{s}{2} + \frac{m}{2} + \frac{p}{4} + \frac{1}{4}, \frac{2}{3}, \mathrm{th}^2 t\right) & (m \in \Lambda_-). \end{cases}$$

It follows that $\mathrm{F}_f(k, t) = \phi(k)\mathrm{F}_{s,m}(t)$ for some ϕ in $\mathscr{H}_m$. Thus we proved the following

PROPOSITION 2. *$\mathscr{H}_{s,m}$ is an irreducible K-module which is equivalent to τ_m.*

Fix any m in Λ. Then by the corollary to Lemma 2, $\mathscr{P}_s\omega_m \in \mathscr{H}_{s,m}$. Hence there exists a constant $C_{s,m}$ such that $(\mathscr{P}_s\omega_m)(a_t)=C_{s,m}\mathrm{F}_{s,m}(t)$, where the constant $C_{s,m}$ is given as follows:

$$C_{s,m} = (\mathscr{P}_s\omega_m)(a_0)$$
$$= \frac{2^{p-3}\Gamma(p/2)\Gamma((p-1)/2)\Gamma((p-2)/2)\Gamma((2s-2p+5)/4)}{\pi(p-3)!\Gamma((2s-2m-p+5)/4)\Gamma((2s+2m+p+1)/4)} \qquad (m \in \Lambda_+).$$

$$C_{s,m} = \frac{d}{dt}(\mathscr{P}_s\omega_m)(a_t)\Big|_{t=0}$$
$$= \frac{-2^{p-2}\Gamma(p/2)\Gamma((p-1)/2)\Gamma((p-2)/2)\Gamma((2s-p+5)/4)}{\pi(p-3)!\Gamma((2s-2m-p+3)/4)\Gamma((2s+2m+p-1)/4)} \qquad (m \in \Lambda_-).$$

Now we assume the following

$$(A) \qquad s+\frac{p}{2}+\frac{1}{2} \notin 2\boldsymbol{Z} \quad \text{and} \quad s-\frac{p}{2}+\frac{1}{2} \notin 2\boldsymbol{Z}.$$

PROPOSITION 3. *Under the assumption* (A), $\mathscr{P}_s$ *gives a K-isomorphism of* Γ_m *onto* $\mathscr{H}_{s,m}$.

PROOF. In view of Proposition 2 we have only to prove the injectiveness. For any ϕ in Γ_m Proposition 1 implies that

$$(\mathscr{P}_s\phi)(ka_t) = (\mathscr{P}_s\omega_m)(a_t)\phi(k) = C_{s,m}\mathrm{F}_{s,m}(t)\phi(k).$$

Since $\mathrm{F}_{s,m}(t)\neq 0$, $\mathscr{P}_s$ is injective if and only if $C_{s,m}\neq 0$. Using the above formulas for $C_{s,m}$, it is easy to check that $C_{s,m}\neq 0$ under the assumption (A).

PROPOSITION 4. *Let* s *satisfy the assumption* (A). *Then there exists a polynomial* P_s *such that* $|C_{s,m}|^{-1}\leq \mathrm{P}_s(m)$ *for all* m *in* Λ.

PROOF. For any m in Λ_+ we know that

$$C_{s,m} = C_p\frac{\Gamma((2s-p+5)/4)}{\Gamma((2s-2m-p+5)/4)\Gamma((2s+2m+p+1)/4)},$$

where

$$C_p = \frac{2^{p-3}\Gamma(p/2)\Gamma((p-1)/2)\Gamma((p-2)/2)}{\pi(p-3)!}.$$

On the other hand

$$\Gamma\left(\frac{s}{2}-\frac{m}{2}-\frac{p}{4}+\frac{5}{4}\right)\Gamma\left(\frac{s}{2}+\frac{m}{2}+\frac{p}{4}+\frac{1}{4}\right)$$
$$=\frac{(-1)^{m/2}\pi\Gamma((2s+2m+p+1)/4)}{\sin\pi(2s-p+5)/4\Gamma((-2s+2m+p-1)/4)}.$$

We put $Q_s(x)=\prod_{1\leq j\leq q}(s/2+x/2+p/4+1/4-j)$, where q is the smallest positive integer such that $q>\mathrm{Re}\,s+1/2$. Then it is easy to prove that there exists a positive constant γ_s such that $|C_{s,m}|^{-1}|Q_s(m)|^{-1}\leq\gamma_s$ for all m in Λ_+. For Λ_- we get a similar polynomial Q'_s and a constant γ'_s. The proposition is now obvious.

§5. Proof of the main theorem

First we need one more lemma.

LEMMA 5. *Fix any f in $\mathscr{H}_s$. Then for any polynomial* P, $\Sigma P(m)f_m(ka_t)$ *and* $\Sigma P(m)\{(d/dt)f_m(ka_t)\}$ *converge absolutely and uniformly on* K.

PROOF. Let f be in $\mathscr{H}_s$. Then, since

$$\Omega_K+\left(\frac{p-2}{2}\right)^2=\left\{m(m+p-2)+\left(\frac{p-2}{2}\right)^2\right\}I$$
$$=\left(m+\frac{p-2}{2}\right)^2 I$$

on Γ_m, for any positive integer n we have

$$d_m\int_K\overline{\chi_m(k)}\left[\left\{\Omega_K+\left(\frac{p-2}{2}\right)^2\right\}^n f\right](k^{-1}g)dk$$
$$=\left(m+\frac{p-2}{2}\right)^{2n}f_m(g).$$

On the other hand for any polynomial P there exists a positive number n such that $|P(m)|\leq(m+(p-2)/2)^{2n}$. Applying $[\Omega_K+((p-2)/2)^2]^n$ to Lemma 3, we conclude that $\Sigma P(m)f_m(ka_t)$ converges absolutely and uniformly on K. If we replace $f(ka_t)$ by $(d/dt)f(ka_t)$, the proof is complete.

THEOREM. *Under the assumption (A), the map $\mathscr{P}_s$ is a linear isomorphism of $C^\infty(G/P, L_s)$ onto $\mathscr{H}_s$.*

PROOF. In view of the corollary to Lemma 2 it is sufficient to prove that $\mathscr{P}_s$ is bijective. For any f in $\mathscr{H}_s$, from Lemma 3 we have $f(ka_t)=\sum f_m(ka_t)$. The right hand side converges absolutely and uniformly on K. By Propositions

1 and 3, for any m in Λ there exists a unique ϕ_m in Γ_m such that

$$\begin{aligned} f_m(ka_t) &= (\mathscr{P}_s\phi_m)(ka_t) \\ &= (\mathscr{P}_s\omega_m)(a_t)\phi_m(k). \end{aligned}$$

For any polynomial we have

$$\begin{aligned} &\sum_{m\in\Lambda} |\mathrm{P}(m)\phi_m(k)| \\ &= \sum_{m\in\Lambda_+} |\mathrm{P}(m)\phi_m(k)| + \sum_{m\in\Lambda_-} |\mathrm{P}(m)\phi_m(k)| \\ &\leqq \sum_{m\in\Lambda_+} |\mathrm{P}(m)|\,|(\mathscr{P}_s\omega_m)(a_0)|^{-1}|f_m(k)| \\ &+ \sum_{m\in\Lambda_-} |\mathrm{P}(m)|\left|\left[\frac{d}{dt}(\mathscr{P}_s\omega_m)(a_t)\Big|_{t=0}\right]\right|^{-1}\left|\frac{d}{dt}f_m(k)\right|. \end{aligned}$$

It follows from Proposition 4 and Lemma 5 that there exists a polynomial Q such that

$$\sum_{m\in\Lambda} |\mathrm{P}(m)\phi_m(k)| \leqq \sum_{m\in\Lambda_+} |\mathrm{Q}(m)|\,|f_m(k)| + \sum_{m\in\Lambda_-} |\mathrm{Q}(m)|\left|\frac{d}{dt}f_m(k)\right|.$$

Hence from Lemma 5 $\sum_{m\in\Lambda} \mathrm{P}(m)\phi_m(k)$ converges absolutely and uniformly on K. Using [6], we see that $\sum\phi_m(k)$ defines a C^∞ function on K/M which we denote by $\phi(k)$. It is now obvious that $\mathscr{P}_s\phi=f$, which shows that $\mathscr{P}_s$ is surjective. Suppose $\mathscr{P}_s\phi=0$ for some ϕ in $C^\infty(G/P, L_s)$. According to Lemma 3, we expand $\phi(k)=\sum\phi_m(k)$. Then by the corollary to Lemma 4 $0=\mathscr{P}_s\phi=\sum\mathscr{P}_s\phi_m$. Hence by Lemma 3 $\mathscr{P}_s\phi_m=0$ for all m in Λ. It follows from Proposition 3 that $\phi_m=0$ for all m in Λ. Thus $\phi=0$, which completes the proof of the theorem.

REMARK 1. The real hyperboloid of one sheet is an affine symmetric space [1]. For the general affine symmetric spaces one can easily formulate an analogue of the Helgason conjecture [3]. However our case is, essentially, the only case that any C^∞ eigenfunction can be obtained as an image of a C^∞ section of L_s.

REMARK 2. Our result can be proved by the method similar to that in [5] (see [4]).

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Department of Mathematics,
Faculty of Science,
Hiroshima University