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# *Some Remarks on Representations of p-adic Chevalley Groups*

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## Introduction

Let $F$ be a $p$-adic field, and let $\mathfrak{O}$ and $\mathfrak{P}$ be the ring of integers and the maximal ideal of $\mathfrak{O}$ respectively. F. I. Mautner [4] first constructed square-integrable irreducible unitary representations of $PGL_2(F)$ which are induced by irreducible representations of a certain maximal compact subgroup. In [5], J. A. Shalika carried it out for $SL_2(F)$ by a different method. Independently, in [6] and [7], T. Shintani extended Mautner's results to a sort of special linear group of rank $n$. Recently, in [2] and [3], P. Gérardin extended their results to reductive)$p$-adic groups whose semi-simple parts are simply connected.

In this paper, we extend the former results of [7], which are not covered by Gérardin's results, to general $p$-adic Chevalley groups. The contents of this paper are as follows. Let $G(\mathbf{z})$ be a Chevalley group over the ring of all rational integers $\mathbf{z}$. Then we have a $p$-adic Chevalley group $G(F)$ and its maximal compact subgroup $G(\mathfrak{O})$ by base changes. In § 1, we give preliminaries on the structures of $p$-adic Chevalley groups after [3]. In § 2, we prepare certain lemma about induced representations of finite groups. In § 3, we show that continuous irreducible unitary representations of $G(\mathfrak{O})$, which do not come from representations of $G(\mathfrak{O}/\mathfrak{P})$, are induced by certain irreducible representations of certain subgroups of $G(\mathfrak{O})$ (Theorem 1). In § 4, when we let $\nu$ be a continuous irreducible unitary representation of $G(\mathfrak{O})$ which does not come from a representation of $G(\mathfrak{O}/\mathfrak{P})$, we obtain a sufficient condition for $\mathrm{Ind}_{G(\mathfrak{O})}^{G(F)}\nu$ to be square-integrable.

In concluding the introduction, the author wishes to express his sincere gratitude to R. Hotta who read this paper and gave him many advices.

NOTATIONS: (i) Let $F$ be a non-archimedean local field, and let $\mathfrak{O}$, $\mathfrak{P}$ and $\pi$ be the ring of integers of $F$, the maximal ideal of $\mathfrak{O}$, and a prime element of $F$, respectively. Let $p$ be the characteristic of the finite field $\mathfrak{O}/\mathfrak{P}$.

(ii) For a ring $R$, we denote by $M(n_1, n_2, R)$ the set of $n_1$ by $n_2$ matrices with coefficients in $R$. We put $M(n, R) = M(n, n, R)$.

(iii) For each positive integer $m$, we denote by $\psi_m$ the reduction modulo $\mathfrak{P}^m$: $\mathfrak{O} \to \mathfrak{O}/\mathfrak{P}^m$. For integers $n \geqq m \geqq 1$, we denote by the same symbol $\psi_m$ the reduction modulo $\mathfrak{P}^m$: $\mathfrak{O}/\mathfrak{P}^n \to \mathfrak{O}/\mathfrak{P}^m$.

(iv) If $R$ is an arbitrary commutative ring with the identity, we denote by $R^*$ the multiplicative group of all units in $R$.

(v) We denote by $\mathbf{Z}$, $\mathbf{M}$ and $\mathbf{C}$ the ring of all rational integers, the set of all natural numbers and the field of all complex numbers, respectively.

## §1 $p$-adic Chevalley groups

The aim of this section is to describe the structures of $p$-adic Chevalley groups and their subgroups.

**1.1.** Let $\mathfrak{G}$ be a finite dimensional complex semi-simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{A}$ of $\mathfrak{G}$. Then we have the root decomposition $\mathfrak{G} = \mathfrak{A} + \sum_{\alpha\in\Phi} \mathfrak{A}^\alpha$ (direct sum), where $\Phi$ is the set of roots relative to $(\mathfrak{G}, \mathfrak{A})$. Choose a Chevalley basis $(X_\alpha)_{\alpha\in\Phi}$ in $\mathfrak{G}$ relative to $\mathfrak{A}$. Let $Q(\Phi)$ (resp. $P(\Phi)$) be the root module (resp. the weight module) of $\Phi$ in the dual space $\mathfrak{A}'$ of $\mathfrak{A}$ (cf. [1], 6, § 1). Let $\rho$ be a finite dimensional faithful representation of $\mathfrak{G}$ on a vector space $E$ over $\mathbf{C}$, and let $\boldsymbol{X}$ be the lattice generated by weights of $\rho$ ($\omega \in \mathfrak{A}'$ is called a weight of $\rho$, if there exists non-zero $v \in E$ such that $\rho(H)v = \omega(H)v$ for any $H \in \mathfrak{A}$). Then $Q(\Phi) \subset \boldsymbol{X} \subset P(\Phi)$, and we have an admissible lattice $E(\mathbf{Z})$ of $E$ for $(\rho, E)$ (cf. [8], § 2). Let $R$ be an arbitrary commutative ring with the identity. We define the automorphisms $x_\alpha(t)$ and $h(\chi)$ of $E(R) = E(\mathbf{Z}) \otimes_{\mathbf{Z}} R$ as follows: For each $\alpha \in \Phi$, $x_\alpha(t) = \sum_{n\geq 0} \rho(X_\alpha^n/n!)t^n$ $(t \in R)$. For each $\chi \in \mathrm{Hom}(\boldsymbol{X}, R^*)$, and for each $v \in E(R)$ of weight $\omega$, $h(\chi)v = \chi(\omega)v$. We denote by $A(R)$ the subgroup of $\mathrm{Aut}(E(R))$ generated by all $h(\chi)$ $(\chi \in \mathrm{Hom}(\boldsymbol{X}, R^*))$, and by $G(R)$ that generated by all subgroups $x_\alpha(R)$ $(\alpha \in \Phi)$ and $A(R)$. We call this group $G(R)$ the Chevalley group over $R$. For the above lattice $\boldsymbol{X}$, we denote by $\boldsymbol{X}'$ the set of all $H \in \mathfrak{A}$ such that $\langle H', H\rangle \in \mathbf{Z}$ for any $H' \in \boldsymbol{X}$, where $\langle\ ,\ \rangle$ is the natural pairing on $\mathfrak{A}' \times \mathfrak{A}$. Put $G(R) = \boldsymbol{X} \otimes_{\mathbf{Z}} R + \sum_{\alpha\in\Phi} R \cdot X_\alpha$ (direct sum), and we denote by $\rho_R$ the representation of the Lie algebra $G(R)$ into $\mathrm{End}(E(R))$. Then $\rho_R$ is faithful ([3], II, 2.1.6). Hence we can define the adjoint action of $G(R)$ on $\mathfrak{G}(R)$ by $\rho_R(\mathrm{Ad}\, x \cdot Y) = x\rho_R(Y)x^{-1}$ $(x \in G(R),\ Y \in \mathfrak{G}(R))$ (cf. [3], II, 2.1.6).

**1.2.** By changing base rings $R$ in 1.1, we obtain the following groups; $G = G(F)$, $K = G(\mathfrak{O})$, $G(\mathfrak{O}/\mathfrak{P}^n)$ $(n \in \mathbf{N})$, $A = A(F)$, $A(\mathfrak{O})$ and $A(\mathfrak{O}/\mathfrak{P}^n)$ $(n \in \mathbf{N})$. From now on, we shall identify $A$, $A(\mathfrak{O})$ and $A(\mathfrak{O}/\mathfrak{P}^n)$ with $\boldsymbol{X}' \otimes_{\mathbf{Z}} F^*$, $\boldsymbol{X}' \otimes_{\mathbf{Z}} \mathfrak{O}^*$ and $\boldsymbol{X}' \otimes_{\mathbf{Z}} (\mathfrak{O}/\mathfrak{P}^n)^*$ respectively by the canonical isomorphisms.

DEFINITION 1.2. *For each integer* $n \geqq 1$, *we denote by* $G(\mathfrak{P}^n)$ (*resp.* $A(\mathfrak{P}^n)$) *the subgroup of* $K$ (*resp.* $A(\mathfrak{O})$) *which is the kernel of the reduction modulo* $\mathfrak{P}^n$: $G(\mathfrak{O}) \to G(\mathfrak{O}/\mathfrak{P}^n)$ (*resp.* $A(\mathfrak{O}) \to A(\mathfrak{O}/\mathfrak{P}^n)$). *For integers* $n \geqq m \geqq 1$, *we denote by* $G(\mathfrak{P}^m/\mathfrak{P}^n)$ (*resp.* $A(\mathfrak{P}^m/\mathfrak{P}^n)$) *the subgroup of* $G(\mathfrak{O}/\mathfrak{P}^n)$ (*resp.* $A(\mathfrak{O}/\mathfrak{P}^n)$) *which is the kernel of the reduction modulo* $\mathfrak{P}^m$: $G(\mathfrak{O}/\mathfrak{P}^n) \to G(\mathfrak{O}/\mathfrak{P}^m)$ (*resp.* $A(\mathfrak{O}/\mathfrak{P}^n)$

$\to A(\mathfrak{O}/\mathfrak{P}^m))$.

From [3], 2.2.5 and 2.2.7, $G(\mathfrak{P}^n)$ is the subgroup of $K$ generated by all subgroups $x_\alpha(\mathfrak{P}^n)$ $(\alpha \in \Phi)$ and $A(\mathfrak{P}^n) = \boldsymbol{X}' \otimes_{\mathbf{Z}} (1+\mathfrak{P}^n)$, and $G(\mathfrak{P}^m/\mathfrak{P}^n)$ is the subgroup generated by all subgroups $x_\alpha(\mathfrak{P}^m/\mathfrak{P}^n)$ $(\alpha \in \Phi)$ and $A(\mathfrak{P}^m/\mathfrak{P}^n) = \boldsymbol{X}' \otimes_{\mathbf{Z}} (1+\mathfrak{P}^m/1+\mathfrak{P}^n)$.

EXAMPLE 1.2. When $G = SL_{l+1}(F)$ $(l \geqq 1)$, we have $G(\mathfrak{P}^n) = \{x \in SL_{l+1}(\mathfrak{O}) \mid x-1 \in \pi^n M(l+1, \mathfrak{O})\}$ $(n \in \mathbf{N})$ and $G(\mathfrak{P}^m/\mathfrak{P}^n) = \{x \in SL_{l+1}(\mathfrak{O}/\mathfrak{P}^n) \mid x-1 \in \pi^m M(l+1, \mathfrak{O}/\mathfrak{P}^n)\}$ $(n \geqq m \geqq 1)$.

$G = G(F)$ inherits a topology from $F$ for which $G$ is a locally compact topological group. More precisely, $G$ has a fundamental system of neighborhoods $\{G(\mathfrak{P}^n)\}_{n \geqq 0}$ which consist of open and compact subgroups of $G$. In particular, $K = G(\mathfrak{O})$ is a profinite group. For $n \geqq m \geqq 1$, we obtain the adjoint action of $G(\mathfrak{O}/\mathfrak{P}^n)$ on $G(\mathfrak{P}^m/\mathfrak{P}^n)$ from that of $G(\mathfrak{O})$ on $G(\mathfrak{P}^m)$ by the reduction modulo $\mathfrak{P}^n$.

LEMMA 1 ([3], 2.2.6, Lemma 5). *If* $2m \geqq n \geqq m \geqq 1$, *the mapping* $e$: $\mathfrak{G}(\mathfrak{P}^m/\mathfrak{P}^n) \to G(\mathfrak{P}^m/\mathfrak{P}^n)$, *defined by* $e(X) = 1 + \rho(X)$, *is an isomorphism as abelian groups commuting with the adjoint actions of* $G(\mathfrak{O}/\mathfrak{P}^n)$.

**1.3.** Let $\mathfrak{G}'$ be the dual vector space over $\mathbf{C}$ of $\mathfrak{G}$. We denote by $\mathfrak{G}'(\mathbf{Z})$ the set of all $X' \in \mathfrak{G}'$ such that $<X', X> \in \mathbf{Z}$ for all $X \in \mathfrak{G}(\mathbf{Z})$. Then we have $\mathfrak{G}'(\mathbf{Z}) = \boldsymbol{X} + \sum_{\alpha \in \Phi} \mathbf{Z} \cdot X'_\alpha$ (direct sum), where $\boldsymbol{X}$ is naturally embedded into $\mathfrak{G}'$, and where $X'_\alpha$ is a linear form defined by $<X'_\alpha, X_\alpha> = 1$, $<X'_\alpha, X_\beta> = 0$ $(\alpha \neq \beta)$, and $<X'_\alpha, \mathfrak{A}> = 0$. We define the co-adjoint action of $G(R)$ on $\mathfrak{G}'(R) = \mathfrak{G}'(\mathbf{Z}) \otimes_{\mathbf{Z}} R$;

$$<\mathrm{Ad}^{\vee} x \cdot X', \mathrm{Ad}\, x \cdot X> = <X', X> \qquad (X \in \mathfrak{G}(R),\ X' \in \mathfrak{G}'(R),\ x \in G(R)),$$

where $R$ is an arbitrary commutative ring with the identity.

EXAMPLE 1.3. When $G = SL_{l+1}(F)$ $(l \geqq 1)$, we assume that the residue characteristic $p$ of $F$ is not 2 and does not divide $l+1$. Then we have $\mathfrak{G}(\mathbf{Z}) = \{x \in M(l+1, \mathbf{Z}) \mid \mathrm{Tr}\, x = 0\}$. Define a non-degenerate bilinear form on $\mathfrak{G}(\mathbf{Z})$ by $<x, y> = \mathrm{Tr}\, xy$ $(x, y \in \mathfrak{G}(\mathbf{Z}))$. We identify $\mathfrak{G}'(\mathbf{Z})$ with $\mathfrak{G}(\mathbf{Z})$ by the isomorphism induced from the above bilinear form. Note that the above bilinear form $<\ ,\ >$ is naturally extended to a non-degenerate bilinear form on $\mathfrak{G}(\mathfrak{O}/\mathfrak{P}^n)$ $(n \geqq 1)$ by the above assumption. Thus we have $\mathrm{Ad}^{\vee} x \cdot X' = xX'x^{-1}$, $x \in SL_{l+1}(\mathfrak{O}/\mathfrak{P}^n)$, $X' \in \mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^n) = \{X' \in M(l+1, \mathfrak{O}/\mathfrak{P}^n) \mid \mathrm{Tr}\, X' = 0\}$ $(n \geqq 1)$.

**1.4.** From now on, we fix a base $B(\Phi)$ of the root system $\Phi$.

DEFINITION 1.4. *For each* $H \in \boldsymbol{X}'$, *we define the element* $\pi^H$ *of* $A = \boldsymbol{X}'$

$\otimes_Z F^*$ *by* $\omega(\pi^H)=\pi^{<\omega,H>}$ *for any* $\omega\in\mathbf{X}$. *We denote by* $\mathbf{X}$ *the set of all* $H\in\mathbf{X}'$ *such that* $<\alpha, H>\geqq 0$ *for any* $\alpha\in B(\Phi)$.

LEMMA 2 (Cartan decomposition).

$$G = KAK = \cup_{H\in\mathbf{X}'_+} K\pi^H K \quad (\textit{disjoint union}).$$

PROOF. The proof can be found in Theorem 21 of [8].

EXAMPLE 1.4. When $G=SL_{l+1}(F)$ $(l\geqq 1)$, put

$$A_+ = \left\{ \begin{pmatrix} \pi^{m_1} & & & 0 \\ & \pi^{m_2} & & \\ & & \ddots & \\ 0 & & & \pi^{m_{l+1}} \end{pmatrix} \middle| \begin{array}{l} (m_1, m_2,\dots, m_{l+1})\in\mathbf{Z}^{l+1},\ m_1+m_2+ \\ \qquad \cdots + m_{l+1}=0, \text{ and } m_1\geqq m_2\geqq\cdots\geqq m_{l+1} \end{array} \right\}.$$

Then we have

$$SL_{l+1}(F) = \cup_{a\in A_+} SL_{l+1}(\mathfrak{O})\cdot a\cdot SL_{l+1}(\mathfrak{O}) \quad (\text{disjoint union}).$$

## §2. Preliminaries for induced representations of finite groups

Let $H$ be a subgroup of a finite group $G$, and let $\nu$: $H\to GL(V)$ be a linear representation of $H$ where $V$ is a finite dimensional vector space over $\mathbf{C}$. We denote by $\mathrm{Ind}_H^G\nu$ the representation of $G$ induced from $\nu$. We assume that $H$ is abelian and normal. We denote by $\hat{H}$ the set of all characters of $H$. Then $G$ operates on $\hat{H}$ in an obvious way i.e., for $\chi\in\hat{H}$, $g\in G$ and $h\in H$, ${}^g\chi(h)=\chi(g^{-1}hg)$. For each $\chi\in\hat{H}$, we denote by $I_\chi$ the subgroup of $G$ fixing $\chi$. Let $\mu$: $G\to GL(W)$ be an irreducible representation of $G$, where $W$ is a finite dimensional vector space over $\mathbf{C}$. For each $\chi\in\hat{H}$, put $W_\chi=\{w\in W|\mu(h)w=\chi(h)w$ for any $h\in H\}$. Then we see immediately that $W_\chi$ is a $I_\chi$-invariant subspace of $W$, and that $\mu$ induces naturally a representation $\mu_\chi$ of $I_\chi$ on $W_\chi$. With these notations, we have the following Lemma.

LEMMA 3 ([7], §1). *Let* $\chi_0$ *be a character of* $H$ *such that* $W_{\chi_0}\neq\{0\}$, *and let* $O$ *be the* $G$*-orbit in* $\hat{H}$ *containing* $\chi_0$. *Then* $W=\sum_{\chi\in O}W_\chi$, *and* $\mu_\chi$ *is an irreducible representation of* $I_\chi$ *and* $\mathrm{Ind}_{I_\chi}^G\mu_\chi$ *is equivalent to* $\mu$. *Conversely, for* $\chi\in\hat{H}$, *let* $\nu_\chi$ *be an irreducible representation of* $I_\chi$ *such that* $\nu_\chi(h)=\chi(h)\cdot 1$ *for any* $h\in H$. *Then* $\mathrm{Ind}_{I_\chi}^G\nu_\chi$ *is an irreducible representation of* $G$. *Moreover, for* $\chi$, $\tau\in\hat{H}$, $\mathrm{Ind}_{I_\chi}^G\nu_\chi$ *is equivalent to* $\mathrm{Ind}_{I_\chi}^G\nu_\chi$ *if and only if there exists* $k\in G$ *such that* $\chi={}^k\tau$, *and* $\nu_\chi$ *is equivalent to* $\nu_\tau^k$ *as representations of* $I_\chi$, *where* $\nu_\chi^k$ *is defined by* $\nu_\tau^k(g)=\nu_\tau(k^{-1}gk)$ $(g\in I_\chi)$.

## §3. Irreducible representations of the maximal compact subgroup $K$

**3.1.** Let $\chi$ be an additive character of $F$. We say that $\chi$ is of *order* 0 if $\chi$

is trivial on $\mathfrak{O}$ and non-trivial on $\mathfrak{P}^{-1}$. From now on, fix a character $\chi$ of $F$, of order 0. Let $f, f'$ and $f''$ be integers such that $f \geqq 2, f=f'+f''$, and $2f' \leqq f \leqq 2f'+1$.

DEFINITION 3.1. *For each $X' \in \mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^f)$, we define a function $\chi^f_{X'}$ on $G(\mathfrak{P}^{f''})$ by $\chi^f_{X'}(g)=\chi(\pi^{-f}<X', e^{-1}(\psi_f(g))>)$ for any $g \in G(\mathfrak{P}^{f''})$ where $e$ is the isomorphism of $\mathfrak{G}(\mathfrak{P}^{f''}/\mathfrak{P}^f)$ onto $G(\mathfrak{P}^{f''}/\mathfrak{P}^f)$ defined in Lemma 1 of* § 1.

LEMMA 4. (i) *The function $\chi^f_{X'}$ is an one-dimensional representation of $G(\mathfrak{P}^{f''})$ which is trivial on $G(\mathfrak{P}^f)$.*

(ii) *For any $k \in K$, we have ${}^k\chi^f_{X'}=\chi^f_{\mathrm{Ad}^\vee(\psi_f(k))\cdot X'}$.*

(iii) *The mapping $X' \mapsto \chi^f_{X'}$ is an isomorphism of the additive group $\mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^f)$ onto the multiplicative group of all one-dimensional representations of $G(\mathfrak{P}^{f''})$ which are trivial on $G(\mathfrak{P}^f)$.*

PROOF. (i) is clear by the definition of $\chi^f_{X'}$.

(ii) By Lemma 1, we have ${}^k\chi^f_{X'}(g)=\chi^f_{X'}(k^{-1}gk)=\chi(\pi^{-f}<X', e^{-1}(\psi_f(k^{-1}gk))>)=\chi(\pi^{-f}<X', \mathrm{Ad}(\psi_f(k))^{-1}\cdot(e^{-1}(\psi_f(g)))>)=(\pi^{-f}<\mathrm{Ad}^\vee\psi_f(k)\cdot X', e^{-1}(\psi_f(g))>)=\chi^f_{\mathrm{Ad}^\vee\psi_f(k)\cdot X'}(g)$ for any $k \in K$ and any $g \in G(\mathfrak{P}^{f''})$.

(iii) By the well-known commutator relations in the Chevalley group $G(R)$ over a commutative ring $R$ (see [8]), and by the fact that $2f'' \geqq f$, we see that $G(\mathfrak{P}^{f''}/\mathfrak{P}^f) \cong G(\mathfrak{P}^{f''})/G(\mathfrak{P}^f)$ is abelian. Hence every one-dimensional representation of $G(\mathfrak{P}^{f''})$ which is trivial on $G(\mathfrak{P}^f)$ can be regarded as a character of $G(\mathfrak{P}^{f''}/\mathfrak{P}^f)$. Hence in order to prove (iii), it is enough to show that the mapping $X' \mapsto \chi^f_{X'}$ is an isomorphism of $\mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^{f'})$ onto $G(\widehat{\mathfrak{P}^{f''}/\mathfrak{P}^f})$, where we denote by $G(\widehat{\mathfrak{P}^{f''}/\mathfrak{P}^f})$ the multiplicative group of all characters of $G(\mathfrak{P}^{f''}/\mathfrak{P}^f)$. Since $\chi$ is of order 0, we have a non-degenerate bilinear form $(X', X) \mapsto \chi(\pi^{-f}<X', X>)$ on $\mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^f) \times \mathfrak{G}(\mathfrak{P}^{f''}/\mathfrak{P}^f)$. Hence an assigning each $X' \in \mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^{f'})$ to a character $X \mapsto \chi(\pi^{-f}<X', X>)$ of $\mathfrak{G}(\mathfrak{P}^{f''}/\mathfrak{P}^f)$ gives the isomorphism $\mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^{f'}) \cong G(\mathfrak{P}^{f''}/\mathfrak{P}^f)$. Combining this isomorphism with an isomorphism $\mathfrak{G}(\widehat{\mathfrak{P}^{f''}/\mathfrak{P}^f}) \cong G(\widehat{\mathfrak{P}^{f''}/\mathfrak{P}^f})$ induced from the isomorphism $e$: $\mathfrak{G}(\mathfrak{P}^{f''}/\mathfrak{P}^f) \cong G(\mathfrak{P}^{f''}/\mathfrak{P}^f)$, we have the desired isomorphism $X' \mapsto \chi^f_{X'}$: $\mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^f) \cong G(\widehat{\mathfrak{P}^{f''}/\mathfrak{P}^f})$. q.e.d.

**3.2.** Let $\nu$ be a non-trivial continuous irreducible unitary representation of the maximal compact subgroup $K$ of $G$ on a Hilbert space.

DEFINITION 3.2. *We call an integer $f$ the conductor of $\nu$, if $\nu$ is trivial on $G(\mathfrak{P}^f)$ and non-trivial on $G(\mathfrak{P}^{f-1})$. We denote by $f=f(\nu)$ (Note that $K$ has a fundamental system of neighborhoods $\{G(\mathfrak{P}^n)\}_{n \geqq 1}$).*

We assume that $f=f(\nu) \geqq 2$, and let $f', f''$ be integers such that $f=f'+f''$, $2f' \leqq f \leqq 2f'+1$. For each $X' \in \mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^{f'})$, put $V_{X'}=\{v \in V | \nu(g)v=\chi^f_{X'}(g)v$

for any $g\in G(\mathfrak{P}^{f''})\}$. By (ii) of Lemma 4, for each $k\in K$, ${}^k\chi^f_{X'}=\chi^f_{X'}$ if and only if $\mathrm{Ad}^\vee(\psi_f(k))X'=X'$. So we denote by $Z_K(X')$ the set of all $k\in K$ fixing $\chi^f_{X'}$. Then $V_{X'}$ is a $Z_K(X')$-invariant subspace of $V$. Put $O_\nu=\{X'\in\mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^{f'})|V_{X'}\neq 0\}$. Then, since $G(\mathfrak{P}^{f''})/G(\mathfrak{P}^f)$ is a finite normal abelian subgroup of $G(\mathfrak{O})/G(\mathfrak{P}^f)$, $O_\nu$ is not empty. For each $X'\in O_\nu$, we denote by $\nu_{X'}$ the representation of $Z_K(X')$ on $V_{X'}$ defined by $\nu_{X'}(g)=\nu(g)|_{V_{X'}}$ for any $g\in Z_K(X')$.

With these notations, we have the following generalization of Theorem 1 of [7], §2.

THEOREM 1. *Let $\nu$ be a continuous irreducible unitary representation of $K$ such that $f=f(\nu)\geqq 2$, and let $f', f''$ be integers such that $f=f'+f''$, $2f'\leqq f\leqq 2f'+1$. Then*

(i) *$G(\mathfrak{O}/\mathfrak{P}^f)$ operates transitively on $O_\nu$ by the adjoint action, and for $X'\in O_\nu$, we have $X'\not\equiv 0 \pmod{\mathfrak{P}}$.*

(ii) *$\nu_{X'}$ is the representation of $Z_K(X')$ which coincides with $\chi^f_{X'}\cdot 1$ on $G(\mathfrak{P}^{f''})$, and $\mathrm{Ind}_{Z_K(X')}^K\nu_{X'}$ is equivalent to $\nu$. Conversely, for $X'\in\mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^f)$ such that $X'\not\equiv 0 \pmod{\mathfrak{P}}$, let $\mu$ be an irreducible unitary representation of $Z_K(X')$ which coincides with $\chi^f_{X'}\cdot 1$ on $G(\mathfrak{P}^{f''})$. Then $\nu=\mathrm{Ind}_{Z_K(X')}^K\mu$ is a continuous irreducible unitary representation of $K$ such that $f(\nu)=f$ and $X'\in O_\nu$.*

PROOF. Fix an element $X'_0$ of $O_\nu$. For each $k\in K$, we have $\nu(k)V_{X_0'}=V^f_{\mathrm{Ad}^\vee(\psi_f(k))\cdot X'_0}$. Indeed, for any $g\in G(\mathfrak{P}^{f''})$ and for any $\nu(g)\nu(k)v=\nu(k)\nu(k^{-1}gk)v=\nu(k)\chi^f_{X'_0}(k^{-1}gk)v=\nu(k)\chi^f_{\mathrm{Ad}^\vee(\psi_f(k))\cdot X'_0}(g)v=\chi^f_{\mathrm{Ad}^\vee(\psi_f(k))\cdot X'_0}(g)\nu(k)v$. Then we have $\mathrm{Ad}^\vee(\psi_f(k))\cdot X'_0\in O_\nu$ for any $k\in K$. Since the representation $\nu$ of $K$ can be regarded as an irreducible unitary representation of the finite group $G(\mathfrak{O})/G(\mathfrak{P})$, we have $V=\sum_{k\in K}V_{\mathrm{Ad}^\vee(\psi)_f(k))\cdot X'_0}$. Therefore we have $V=\sum_{k\in K}V_{\mathrm{Ad}^\vee(\psi)_f(k))\cdot X'_0}\subset\sum_{X'\in O_\nu}V_{X'}\subset V$, whe eas we have $O_\nu=\{\mathrm{Ad}^\vee(\psi_f(k))\cdot X'_0|k\in K\}$. This shows that $G(\mathfrak{O}/\mathfrak{P}^f)\cong G(\mathfrak{O})/G(\mathfrak{P}^f)$ operates transitively on $O_\nu$ by the co-adjoint action. For $X'\in O_\nu$, we assume that $X'\equiv 0 \pmod{\mathfrak{P}}$. Then $\chi^f_{X'}$ is trivial on $G(\mathfrak{P}^{f''})$. Hence the representation $\nu$ is trivial on $G(\mathfrak{P}^{f''})$. This contradicts $f(\nu)=f$.

(ii) In the proof of Lemma 4, we have seen that $G(\mathfrak{P}^{f''})/G(\mathfrak{P})$ is a normal abelian subgroup of the finite group $G(\mathfrak{O})/G(\mathfrak{P})$. Hence (ii) is an immediate consequence of Lemma 3 in §2. q.e.d.

In the above Theorem 1, the condition, $f(\nu)\geqq 2$, means that the representation $\nu$ of $K$ does not come from an representation of $G(\mathfrak{O}/\mathfrak{P})$. Hence Theorem 1 says that continuous irreducible unitary representations of the maximal compact subgroup $K$ of $G$ which do not come from representations of $G(\mathfrak{O}/\mathfrak{P})$ are induced from certain irreducible representations of certain subgroups of $K$.

## §4. Unitary representations of $G$ induced from irreducible representations of tie maximal compact subgroup $K$ of $G$

**4.1.** Let $dg$ be the Haar measure on $G$ such that $\int_K dg=1$. Let $U$ be a continuous unitary representation of $G$ on a Hilbert space $\mathfrak{H}$.

DEFINITION 4.1. *$U$ is said to be square-integrable if there exists $v\in \mathfrak{H}-\{0\}$ such that*

$$\int_G (U(g)v, v)\overline{(U(g)v, v)}dg < +\infty,$$

*where $(\ ,\ )$ is an inner product of $\mathfrak{H}$ and $\overline{(U(g)v, v)}$ is the complex conjugate of $(U(g)v, v)$.*

If $U$ is square-integrable, then there exists a number $d>0$, called the *formal degree* of $U$ depending only the equivalence class of $U$ and on the normalization of the Haar measure $dg$ on $G$ such that

$$\int_G (U(g)u_1, v_1)\overline{(U(g)u_2, v_2)}dg = d^{-1}(u_1, u_2)\overline{(v_1, v_2)}$$

for all $u_i, v_i\in\mathfrak{H}$ $(i=1, 2)$ (Schur's orthogonality relation). Let $\nu$ be a continuous irreducible unitary representation of $K$ on a finite dimensional Hilbert space $V$. We denote by $\mathfrak{H}_\nu$ the set of all $V$-valued functions $f$ satisfying the following conditions:

(i) $f(kg)=\nu(k)f(g)$ for any $k\in K$ and for any $g\in G$,

(ii) $\int_G (f(g), f(g))dg < +\infty$,

where $(\ ,\ )$ is an inner product of $V$. We define an inner product $<\ ,\ >$ on $\mathfrak{H}_\nu$ by

$$<f, h> = \int_G (f(g), h(g))dg \qquad (f, h\in\mathfrak{H}_\nu).$$

Then $\mathfrak{H}_\nu$ becomes a Hilbert space. We define a representation $U_\nu$ of $G$ on $\mathfrak{H}_\nu$ as follows:

$$(U_\nu(g)f)(g') = f(g'g) \qquad (g, g'\in G, f\in\mathfrak{H}_\nu).$$

We denote by $\mathrm{Ind}_K^G\,\nu$ the above unitary representation $U_\nu$ and by $U_\nu|K$ the representation of $K$ on $\mathfrak{H}_\nu$ obtained by restricting $U_\nu$ to $K$. Put $I(U_\nu|K, \nu)=\dim \mathrm{Hom}_K(V, \mathfrak{H}_\nu)$. This is called the *multiplicity* of $\nu$ in $U_\nu|K$.

LEMMA 5 ([7], §3). (i) *If $I(U_\nu|K, \nu)<+\infty$, then $U_\nu$ decomposes into a* *direct sum of at most* $I(U_\nu|K, \nu)$ *many irreducible representations. In particular, if* $I(U_\nu|K, \nu)=1$, *then* $U_\nu$ *is irreducible.*

(ii) *If* $U_\nu$ *is irreducible, then* $U_\nu$ *is square-integrable and its formal degree equals* dim $V$.

**4.2.** Let $G=\bigcup_{H\in X'_+} K\pi^H K$ be the Cartan decomposition of $G$ in Lemma 2. Let $\nu$ be a continuous irreducible unitary representation of $K$ on a Hilbert space $V$. For each $H\in X'_+$, put $K^H=K\cap\pi^{-H}K\pi^H$ and we denote by $\nu^H$ the representation of $K^H$ on $V$ defined by $\nu^H(k)=\nu^H(\pi^H k\pi^{-H})$ for any $k\in K^H$. Let $\mathfrak{H}_\nu$ be the representation space of $\mathrm{Ind}_K^G \nu$ defined in 4.1. For each $H\in X'_+$, we denote by $\mathfrak{H}_\nu^H$ the set of all $f\in\mathfrak{H}_\nu$ whose supports are contained in $K\pi^H K$. Then $\mathfrak{H}_\nu^H$ is a closed subset of $\mathfrak{H}_\nu$ and invariant under the representation $U_\nu|K$, and moreover we have $\mathfrak{H}_\nu=\sum_{H\in X'_+}\mathfrak{H}_\nu^H$ (direct sum as a Hilbert space). We denote by $U_\nu^H|K$ the representation $k\mapsto U(k)$ of $K$ on $\mathfrak{H}_\nu^H$ and by $\nu|K^H$ the representation of $K^H$ on $V$ obtained by restricting $\nu$ to $K^H$. Put $I(\nu|K^H, \nu)=\dim\mathrm{Hom}_{K^H}(V, V)$.

LEMMA 6 ([7], § 3). (i) *For each* $H\in X'_+$, $U_\nu^H|K$ *is equivalent to* $\mathrm{Ind}_{K^H}^K\nu^H$.

(ii) *For each* $H\in X'_+$, $I(U_\nu^H|K, \nu)=I(\nu|K^H, \nu^H)$.

(iii) $I(U_\nu|K, \nu)=\sum_{H\in X'_+} I(\nu|K^H, \nu^H)$ (*Remark, The equality in* (iii) *admits the infinity, i.e.,* $+\infty=+\infty$).

**4.3** Let $\nu$ be a continuous irreducible unitary representation of $K$ with the conductor $f(\nu)=f\geqq 2$, and let $f', f''$ be integers such that $f=f'+f''$, $2f'\leqq f\leqq 2f'+1$. Let $O_\nu$ be the set of all $X'\in\mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^{f'})$ such that $V_{X'}\neq\{0\}$. Thus every element of $O_\nu$ is uniquely written as the form $H'+\sum_{\alpha\in\Phi}u_\alpha X'_\alpha$ where $H'\in X\otimes_{\mathbf{Z}}\mathfrak{O}/\mathfrak{P}^{f'}$ and $u_\alpha\in\mathfrak{O}/\mathfrak{P}^{f'}$ ($\alpha\in\Phi$). For each $X'=H'+\sum_{\alpha\in\Phi}u_\alpha X'_\alpha\in O_\nu$ and each integer $m$ such that $1\leqq m\leqq f'$, we denote by $\mathrm{Supp}_m(X')$ the set of all roots $\alpha$ such that $\psi_m(u_\alpha)\neq 0$ (We recall that $\psi_m$ is the reduction modulo $\mathfrak{P}^m$: $\mathfrak{O}/\mathfrak{P}^{f'}\to\mathfrak{O}/\mathfrak{P}^m$). For each $H\in X'_+$, we denote by $P_m(H)$ the set of all positive roots $\alpha$ such that $<\alpha, H>\geqq m$. Put $B(\Phi)=\{\alpha_1,\dots,\alpha_l\}$ (the fixed base of the root system $\Phi$). Then every root is uniquely written as the form $\sum_{i=1}^l n_i\alpha_i$ where all $n_i$ are non-negative integers and have the same sign. For each $\alpha_j\in B(\Phi)$, we denote by $(\alpha_j)$ the set of all roots $\alpha=\sum_{i=1}^l n_i\alpha_i$ such that $n_j\geqq 1$.

PROPOSITION 1. *Let* $\nu$ *be a continuous irreducible unitary representation of* $K$ *with the conductor* $f(\nu)=f\geqq 2$, *and let* $f', f''$ *be integers such* $f=f'+f''$, $2f'\leqq f\leqq 2f'+1$. *Let* $H$ *be an element of* $X'_+$. *Assume that there exists an integer* $m$ ($1\leqq m\leqq f'$) *such that for any* $X'\in O_\nu$ *and for any* $\alpha_i\in B(\Phi)$, $(\alpha_i)\cap\mathrm{Supp}_m(X')\neq\phi$ *and* $B(\Phi)\cap P_m(H)\neq\phi$. *Then* $I(\nu|K^H, \nu^H)=0$.

PROOF. We shall prove by absurdity. Assume that $I(\nu|K^H, \nu^H)>0$. Let $V$ be a representation space of $\nu$. Then there exists a non-trivial linear transformation $T$ of $V$ satisfying the following condition; for any $k\in K^H$,

$$(1) \qquad v(k)T = Tv(\pi^H k \pi^{-H}).$$

Now by the assumption $B(\Phi)\cap P_m(H)\neq\phi$, there exists $\alpha_{i_0}\in B(\Phi)\cap P_m(H)$ $(1\leqq i_0\leqq \mathfrak{l})$. Then

$$(2) \qquad (\alpha_{i_0}) \subset P_m(H).$$

In fact, let $\beta=\sum_{i=1}^{\mathfrak{l}} n_i\alpha_i$ be any root of $(\alpha_{i_0})$. Then, since $n_i\geqq 0$ and $<\alpha_i, H>\geqq 0$ for all $i$ $(1\leqq i\leqq \mathfrak{l})$, we have $<\beta, H> = \sum_{i=1}^{\mathfrak{l}} n_i<\alpha_i, H> \geqq <\alpha_{i_0}, H>\geqq m$. Hence $\beta$ belongs to $P_m(H)$. We denote by $U$ the subgroup of $K$ generated by the set $\{x_\alpha(t)|\alpha\in(\alpha_{i_0}), t\in\mathfrak{P}^{f-m}\}$. Then, under some order in $(\alpha_{i_0})$, every element of $U$ is uniquely written as the product $\prod_{\alpha\in(\alpha_{i_0})} x_\alpha(t_\alpha)$ $(t_\alpha\in\mathfrak{P}^{f-m})$ ([8], § 3, Lemma 17). For any $u\in U$, say $u=\prod_{\alpha\in(\alpha_{i_0})} x_\alpha(t_\alpha)$ $(t_\alpha\in\mathfrak{P}^{f-m})$, we have $\pi^H u\pi^{-H}=\prod_{\alpha\in(\alpha_{i_0})}\cdot \pi^H x_\alpha(t_\alpha)\pi^{-H}=\prod_{\alpha\in(\alpha_{i_0})} x_\alpha(\pi^{<\alpha,H>}t_\alpha)$. By (2), $\pi^{<\alpha,H>}t_\alpha\in\mathfrak{P}^f$ for all $\alpha\in(\alpha_{i_0})$. Thus we have $\pi^H u\pi^{-H}\in G(\mathfrak{P}^f)$. Hence by (1), we have $v(u)T=T$ for any $u\in U$. Since $T$ is not trivial, there exists $v\in V-\{0\}$ such that $v(u)v=v$ for any $u\in U$. Therefore, since $V=\sum_{X'\in O_v} V_{X'}$, there exists a non-zero $V_{X'}$-component $v_{X'}$ of $v$ such that $v(u)v_{X'}=v_{X'}$ for any $u\in U$. Hence we have $\chi_{X'}^f(u)=1$ for any $u\in U$. On the other hand, by the assumption $(\alpha_{i_0})\cap\mathrm{Supp}_m(X')\neq\phi$, there exists a root $\gamma$ belonging to $(\alpha_{i_0})\cap\mathrm{Supp}_m(X')$. Therefore, let $X'$ be the form $H'+\sum_{\alpha\in\Phi} u_\alpha X'_\alpha$ where $H'\in \boldsymbol{X}\otimes_{\mathbf{Z}}\mathfrak{O}/\mathfrak{P}^f$, $u_\alpha\in\mathfrak{O}/\mathfrak{P}^f$ $(\alpha\in\Phi)$, then we have $\psi_m(u_\gamma)\neq 0$ in $\mathfrak{O}/\mathfrak{P}^m$. Here we take $t_\alpha\in\mathfrak{P}^{f-m}$ $(\alpha\in(\alpha_{i_0}))$ such that for $\alpha=\gamma$, $t_\gamma\notin\mathfrak{P}^{f-m+1}$ and $\chi(\pi^{-f}u_\gamma t_\gamma)\neq 1$, and for $\alpha\neq\gamma$, $t_\alpha\in\mathfrak{P}^{f-m+1}$. This is possible, because $\chi$ is of order 0. Put $u_0=\prod_{\alpha\in(\alpha_{i_0})} x_\alpha(t_\alpha)$, then we have $u_0\in U$. By the definition of $\chi_{X'}^f$, we have $\chi_{X'}^f(u_0)=\chi(\pi^{-f}<X', e^{-1}(\psi_f(u_0))>)=\chi(\pi^{-f}<X', \sum_{\alpha\in(\alpha_{i_0})}\psi_f(u_\alpha)\cdot X_\alpha>)=\chi(\pi^{-f}t_\gamma u_\gamma)\neq 1$. This is contradiction. q. e. d.

COROLLARY 1. *Let $v, f, f'$ and $f''$ be as in Proposition 1. Assume that there exists an integer $m$ $(1\leqq m\leqq f')$ such that for any $X'\in O_v$ and for any integer $i$ $(1\leqq i\leqq \mathfrak{l})$, $(\alpha_i)\cap\mathrm{Supp}_m(X')\neq\phi$. Then there exist only finitely many elements $H$ of $\boldsymbol{X}'_+$ such that $I(v|K^H, v^H)>0$. If $m=1$, then $I(v|K^H, v^H)=0$ for any $H\neq 0$ in $\boldsymbol{X}'_+$.*

PROOF. If $I(v|K^H, v^H)>0$, then we have $B(\Phi)\cap P_m(H)=\phi$ by the above Proposition. Therefore, for all $\alpha_i\in B(\Phi)$ $(1\leqq i\leqq \mathfrak{l})$, we have $0\leqq <\alpha_i, H> <m\cdots$ $(*)$. Since the root module $Q(\Phi)$ is of finite index in the lattice $\boldsymbol{X}$, and the bilinear form $<\ ,\ >$ is non-degenerated on $\boldsymbol{X}\times\boldsymbol{X}'$, there must exist only finitely many $H\in\boldsymbol{X}'_+$ satisfying $(*)$. In the case that $m=1$, there does not exist such $H\neq 0$. q. e. d.

By Corollary 1 and Lemma 6, we have the following Corollary.

COROLLARY 2. *Let $v, f, f'$ and $f''$ be as in Proposition 1. Assume that* *there exists an integer* $m$ $(1 \leqq m \leqq f')$ *such that for any* $X' \in O_\nu$ *and for any* $\alpha_i \in B(\Phi)$ $(1 \leqq i \leqq l)$, $(\alpha_i) \cap \mathrm{Supp}_m(X') \neq \phi$. *Then* $I(U_\nu|K, \nu) < +\infty$. *If* $m=1$, *then* $I(U_\nu|K, \nu)=1$.

By Lemma 6 and the above Propositions, we have the following Theorem.

THEOREM 2. *Let* $\nu$ *be a continuous irreducible unitary representation with conductor* $f(\nu)=f \geqq 2$, *and let* $f, f'$ *and* $f''$ *be integers such that* $f=f'+f''$, $2f' \leqq f \leqq 2f'+1$. *Assume that there exists an integer* $m$ $(1 \leqq m \leqq f')$ *such that for any* $X' \in O_\nu$ *and for any* $\alpha_i \in B(\Phi)$ $(1 \leqq i \leqq l)$, $(\alpha_i) \cap \mathrm{Supp}_m(X') \neq \phi$. *Then* $U_\nu = \mathrm{Ind}_K^G \nu$ *decomposes into a direct sum of at most* $I(U_\nu|K, \nu)$ *many irreducible representations of* $G$. *In particular, if* $m=1$, *then* $U_\nu = \mathrm{Ind}_K^G \nu$ *is a square-integrable irreducible unitary representation of* $G$ *whose formal degree is the degree of the representation* $\nu$.

REMARK. By Lemma 6 and Corollary 1 to Proposition 1, we have $I(U_\nu|K, \nu) = \sum I(\nu|K^H, \nu^H)$ where the summation is taken over all $H \in \boldsymbol{X}'_+$ such that $<\alpha_i, H> < m$ for all $\alpha_i \in B(\Phi)$.

**4.4.** We shall compare our results with those of [7] obtained by T. Shintani. From now on, we put $G=SL_{l+1}(F)$ and $K=SL_{l+1}(\mathfrak{O})$ $(l \geqq 1)$. Let $\nu$ be a continuous irreducible unitary representation of $K$ on a Hilbert space $V$ with conductor $f(\nu)=f \geqq 2$, and $f', f''$ be $f=f'+f''$, $2f' \leqq f \leqq 2f'+1$. By Example 1.3 and 3.2, for this representation $\nu$, $O_\nu$ is the set of all $x \in \mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^f)$ such that $V_x \neq \{0\}$, where $\mathfrak{G}'(\mathfrak{O}/\mathfrak{P}^f) = \{x \in M(l+1, \mathfrak{O}/\mathfrak{P}^f) | \mathrm{Tr}\, x = 0\}$. For some $x \in O_\nu$, we define a monic polynomial of the degree $l+1$ over the finite local ring $\mathfrak{O}/\mathfrak{P}^f$ by $C_\nu(t) = \det(t \cdot 1 - x)$ where $t$ is an indeterminate. Then $C_\nu(t)$ does not depend upon the choice of an element $x$ of $O_\nu$. Hence the polynomial $C_\nu(t)$ corresponds the above representation $\nu$ of $K$. T. Shintani proved the following facts in [7]; If $C_\nu(t)$ is an irreducible polynomial, then the unitary representation $\mathrm{Ind}_K^G \nu$ of $G$ has finitely many irreducible components. In particular, put $C_\nu(t) = t^{l+1} - a_1 t^l + \cdots + a_{l+1}$ $(a_i \in \mathfrak{O}/\mathfrak{P}^{f'})$. If $\psi_1(C_\nu(t)) = t^{l+1} + \psi_1(a_1) t_l + \cdots + \psi_1(a_{l+1})$ is an irreducible polynomial over the finite field $\mathfrak{O}/\mathfrak{P}$, then $\mathrm{Ind}_K^G \nu$ is a square-integrable irreducible representation and its formal degree equals $\dim V$. Moreover, he constructed all continuous irreducible unitary representation $\nu$ whose corresponding polynomials $\psi_1(C_\nu(t))$ are irreducible.

PROPOSITION 3. *Let* $\nu$ *be a continuous irreducible unitary representation of* $K=SL_{l+1}(\mathfrak{O})$ *with conductor* $f(\nu)=f \geqq 2$, *and let* $f', f''$ *be integers* $f=f'+f''$, $2f' \leqq f \leqq 2f'+1$. *For each integer* $m$ $(1 \leqq m \leqq f')$, *if* $\psi_m(C_\nu(t))$ *is an irreducible polynomial over* $\mathfrak{O}/\mathfrak{P}^m$, *then we have* $(\alpha_i) \cap \mathrm{Supp}_m(x) \neq \phi$ *for any* $x \in O_\nu$ *and for any* $\alpha_i \in B(\Phi)$ $(1 \leqq i \leqq l)$.

PROOF. Put $A=\left\{\begin{pmatrix} a_1 & & 0 \\ & a_2 \ddots & \\ 0 & & a_{l+1} \end{pmatrix} \in SL_{l+1}(F)\right\}$. For each $i$, let $e_i$ be a character of $A$ defined by $e_i\left(\begin{pmatrix} a_1 & & 0 \\ & a_i \ddots & \\ 0 & & a_{l+1} \end{pmatrix}\right)=a_i$. Then as a root system of $\mathfrak{G}$ and its base, we can take $\Phi=\{e_i-e_j|i\neq j,\ 1\leqq i\leqq l+1,\ 1\leqq j\leqq l+1\}$ and $B(\Phi)=\{e_1-e_2, e_2-e_3,\dots, e_l-e_{l+1}\}$. Now if there exist $x\in O_v$ and $\alpha_{i_0}=e_{i_0}-e_{i_0+1}\in B(\Phi)$ $(1\leqq i_0\leqq l)$ such that $(\alpha_{i_0})\cap\mathrm{Supp}_m(x)=\phi$, then by the definition of $\mathrm{Supp}_m(x)$ and by Example 1.3, $\psi_m(x)\in M(l+1, \mathfrak{O}/\mathfrak{P}^m)$ is the form $\begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}$ where $x_1 \in M(i_0, \mathfrak{O}/\mathfrak{P}^m)$, $x_2\in M(i_0, l+1-i_0, \mathfrak{O}/\mathfrak{P}^m)$ and $x_3\in M(l+1-i_0, \mathfrak{O}/\mathfrak{P}^m)$. Hence $\psi_m(C_v(t))=\det(t\cdot 1-\psi_m(x))$ is clearly reducible. q.e.d.

Thus in the case of $G=SL_{l+1}(F)$, there exist continuous irreducible unitary representations of $K=SL_{l+1}(\mathfrak{O})$ which satisfy the condition that $m=1$ in Theorem 2.

## References


[1] N. Bourbaki, Groupes et algèbres de lie, Chap. IV, V et VI, Hermann, Paris, 1968.

[2] P. Gérardin, On the discrete series for Chevalley groups, Proc. Amer. Math. Soc., Summer institute on harmonic analysis and homogeneous spaces, 1972.

[3] ———, Construction de séries discrètes $p$-adiques, Lecture Notes in Math. **462** (1975), Springer-Verlag, Berlin. Heidelberg. New York.

[4] F. I. Mautner, Spherical functions over $p$-adic fields II, Amer. J. Math. **86** (1964), 171–200.

[5] J. A. Shalika, Representations of the two by two unimodular groups over local fields, Part II, Seminar on representations of Lie groups, Princeton, 1966.

[6] T. Shintani, On certain series of square-integrable irreducible unitary representations of special linear groups over $p$-adic fields, Sûgaku, **19** (1968), 231–239.

[7] ———, On certain square-integrable irreducible unitary representations of some $p$-adic linear groups, J. Math. Soc. Japan, **20** (1968), 522–565.

[8] R. Steinberg, Lectures on Chevalley groups, Yale Univ. Lecture Notes, 1967.



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