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Openness of loci, P-excellent rings and modules

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Introduction

The study of the problem of openness of certain loci over noetherian rings was firstly undertaken systematically by A. Grothendieck [7]. There he has initiated the theory of *excellent rings*, where the important notion is that of the fibers of homomorphisms (e.g., formal fibers). In recent years, R. Y. Sharp has developed the theory of *acceptable rings* replacing regularity in the definition of excellent rings by Gorensteinness (cf. [15], [16]). On the other hand, some authors recently have recognized the importance of the so-called *Nagata's criterion for openness of loci* and they have proved it for many important properties (cf. [6], [8]).

The aim of this paper is to generalize the above-mentioned theories for rings to modules and to study the problem of openness of certain loci of finitely generated modules over noetherian rings. In particular, we shall present a theory of P-*excellent rings* and P-*excellent modules* (where P is a given property for finitely generated modules over noetherian local rings) which includes the theories of excellent rings and acceptable rings as special cases. For this purpose, we have to extend the definitions and axioms which have been considered only for rings by the above authors to modules and we shall generalize their results to modules.

The contents are divided into two parts. In Part I, we shall study the problem of openness of loci and the theory of P-*excellent modules* axiomatically. Concerning general properties on P-*morphisms* and *well-fibered modules* (cf. Def. 6, Def. 12), we shall mainly follow the method due to R. Y. Sharp [15]. Firstly we shall prove a fundamental theorem on P-excellent modules: P-excellent modules are stable under homomorphisms essentially of finite type. Next, we shall show the following theorems under certain conditions on P (which are satisfied for P = Cohen-Macaulay, Gorenstein and complete intersection):

(1) All modules over a P-excellent ring are P-excellent.

(2) If an A-module M is P, then M is P-excellent.

(3) Suppose that there is an A-module M which is P and that Supp(M) coincides with Spec(A). Then A is P-excellent.

Lastly, we shall treat the finite descent of P-excellent rings which generalizes a theorem of S. Greco [5] on excellent rings.

In order that these results in Part I have real meaning, we must show that the properties in which we are interested satisfy the axioms in Part I. Thus, in Part II of this paper, we shall study Nagata's criterion and prove it for certain properties. Here we are much influenced by an interesting paper by S. Greco and M. G. Marinari [6]. (In particular, we shall prove Nagata's criterion for n-Gorenstein property which is remained unsettled in [6].)

Part I. P-excellent rings and modules

All rings are assumed to be commutative noetherian rings and modules are finitely generated unless otherwise stated.

Let P be a property concerning modules over local rings. When P is a property which is defined only for rings (e.g., P = regular, normal etc.), we can extend it to modules by considering free modules over rings with the property P. For example, we say that a module M over a local ring A is regular if $M=0$ or A is a regular local ring and M is free over A. Zero modules are always assumed to be P.

For a module M over a (not necessarily local) ring A, put $\mathrm{P}(A; M)=\{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \text{ is P}\}$. We call it the P-*locus* of an A-module M. We also write $\mathrm{P}(M)$ instead of $\mathrm{P}(A; M)$ when there is no fear of confusion for A. Then, a fundamental problem is: When is $\mathrm{P}(M)$ open in $\mathrm{Spec}(A)$?

EXAMPLE 1. For any ring A, the sets $\mathrm{Dom}(A)=\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_{\mathfrak{p}} \text{ is an integral domain}\}$ and $\mathrm{Red}(A)=\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_{\mathfrak{p}} \text{ is reduced}\}$ are open. In fact, $\mathrm{Dom}(A)=\bigcup(\mathrm{Spec}(A)-\mathrm{Supp}(\mathfrak{p}))$, where $\mathfrak{p}$ runs over the set $\mathrm{Min}(A)$ of minimal prime ideals of A, and $\mathrm{Red}(A)=\mathrm{Spec}(A)-\mathrm{Supp}(\mathfrak{N}(A))$, where $\mathfrak{N}(A)$ is the nilradical of A.

EXAMPLE 2. Henceforth, we shall consider mainly the following properties as P:

(1) P = Reg = a free module over a regular local ring.

(2) P = CI = a free module over a complete intersection local ring. (A local ring A is said to be a *complete intersection* if the completion of A is of the form B/I, where B is a regular local ring and I is an ideal of B generated by a B-regular sequence.)

(3) P = Gor = a Gorenstein module.

(4) P = CM = a Cohen-Macaulay module.

(5) $\mathrm{P}=\mathrm{S}_n$ (Serre's condition, [7], Chap. 4, (5.7.2)).

(6) $\mathrm{P}=\mathrm{R}_n$ (Serre's condition, [7], Chap. 4, (5.8.2)).

(7) $\mathrm{P}=\mathrm{Q}_n$ (where Q is a given property and n is a nonnegative integer) $=\mathrm{Q}$ *in codimension* $\leqq n$, that is, M is $\mathrm{Q}_n \Leftrightarrow M_{\mathfrak{p}}$ is Q for all $\mathfrak{p} \in \mathrm{Spec}(A)$ such that

$\dim(M_{\mathfrak{p}}) \leqq n$. For example, $\mathrm{Reg}_n = \mathrm{R}_n$ in (6).

(8) $\mathrm{P} = n$-Nor $= n$-*normal* $\underset{\text{def.}}{=\!=}$ S_{n+1} and R_n (cf. [3]). For example, 1-normal = normal (denoted by Nor) and 0-normal = reduced (denoted by Red).

(9) $\mathrm{P} = n$-QN $= n$-*quasi normal* (or $(n+1)$-Gorenstein, cf. [4]) $\underset{\text{def.}}{=\!=}$ S_{n+1} and Gor_n. For example, 1-quasi normal = quasi normal (denoted by QN) (cf. [19], [20]). We shall use '*quasi reduced*' (denoted by QR) instead of '0-quasi normal.'

In what follows, we shall consider some conditions for P (cf. [6], [10], [18]). In this paper, we shall always assume that our property P satisfies the next conditions (I) and (II):

(I) If a module M over a local ring A is P, then the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is P for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

(II) Free modules over regular local rings are P.

(III) Let $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat local homomorphism of local rings and M be a non-zero A-module. Put $k = A/\mathfrak{m}$.

(a) If the B-module $M \otimes_A B$ is P, then M is P, and

(b) If M and the $B \otimes_A k$-module $B \otimes_A k$ are P, then the B-module $M \otimes_A B$ is P.

(III′) Let $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat local homomorphism of local rings and M be a non-zero A-module. Put $k = A/\mathfrak{m}$. The B-module $M \otimes_A B$ is P if and only if M and $B \otimes_A k$ are P.

(NC) (Nagata's criterion) Let A be a ring and M be an A-module. If $\mathrm{P}(A/\mathfrak{p}) = \mathrm{P}(A/\mathfrak{p}; A/\mathfrak{p})$ is thick in $\mathrm{Spec}(A/\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{P}(M) \cap \mathrm{Supp}(M)$, then $\mathrm{P}(M)$ is open in $\mathrm{Spec}(A)$. (A subset Y of a topological space X is said to be *thick* in X if Y contains a non-empty open set of X, that is, the interior Y° of Y is non-empty.)

(QC) (quotient condition) Let M be an A-module with $\mathrm{P}(M) = \mathrm{Spec}(A)$. Then $\mathrm{P}(A/\mathfrak{p})$ is thick in $\mathrm{Spec}(A/\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$.

REMARK 3. The following facts hold:

(1) (III) for Reg (cf. [9], (21.D), Th. 51),
(III′) for CI (cf. [2], Th. 2),
(III′) for Gor (cf. [14], Th. (2.8)), and
(III′) for CM (cf. [7], (6.3.3)).

Note that (III′) for Reg does not hold.

(2) (NC) for Reg (cf. [7], (6.12.2); [9], (32.A)),
(NC) for CI, Gor and Gor_n (for rings) (cf. [6]),
(NC) for CM (cf. [7], (6.11.8)),
(NC) for R_n and S_n (for rings) (cf. [8]),
(QC) for CI and Gor (for rings) (cf. [6]),

(QC) for CM (cf. [7], (6.11.9)).

(In Part II of this paper, we shall extend these results for (NC) to modules and prove (NC) for some other properties.)

In the rest of Part I, we always assume that P satisfies the condition (III) unless otherwise stated.

Definition 4 (cf. [10], [18]). We say a module M over a ring A is

P if $\mathrm{P}(M)=\mathrm{Spec}(A)$,

P–0 if $\mathrm{P}(M)$ is thick in $\mathrm{Spec}(A)$,

P–1 if $\mathrm{P}(M)$ is open (may be empty) in $\mathrm{Spec}(A)$, and

P–2 if for every finitely generated A-algebra B, $M\otimes_A B$ is P–1 as a B-module.

Definition 5. Let k be a field, A be a k-algebra and M be an A-module. We say M is *geometrically* P (with respect to k) if for every finitely generated extension field K of k, the $A\otimes_k K$-module $M\otimes_k K$ is P.

Definition 6. A homomorphism $A\to B$ is called a *weak* P-*morphism* (resp. a P-*morphism*) if it is flat and $B\otimes_A k(\mathfrak{p})$ is P (resp. geometrically P) for every $\mathfrak{p}\in\mathrm{Spec}(A)$.

For example, $A\to A[X]$ is a P-morphism for any ring A and any P. On the other hand, $A\to A[[X]]$ is not a weak P-morphism in general (cf. [17]).

Proposition 7. *Consider a weak* P-*morphism* $f\colon A\to B$ *and an* A-*module* M. *Let* $g\colon \mathrm{Spec}(B)\to\mathrm{Spec}(A)$ *be the mapping associated to* f. *Then, the following statements hold*:

(1) $g^{-1}(\mathrm{P}(M))=\mathrm{P}(M\otimes_A B)$.

(2) *If* M *is* P, *then so is* $M\otimes_A B$. (*If* f *is faithfully flat, then the converse holds.*)

(3) *If* M *is* P–0 *and* f *is faithfully flat, then* $M\otimes_A B$ *is* P–0.

(4) *If* $M\otimes_A B$ *is* P–0 *and* f *is of finite type, then* M *is* P–0.

(5) *If* M *is* P–1, *then so is* $M\otimes_A B$.

(6) *If* $M\otimes_A B$ *is* P–1 *and* f *is faithfully flat, then* M *is* P–1.

(7) *If* $M\otimes_A B$ *is* P–2 *and* f *is a faithfully flat* P-*morphism, then* M *is* P–2.

Proof. (1) Let $\mathfrak{q}\in\mathrm{Spec}(B)$ and $\mathfrak{p}=\mathfrak{q}\cap A$. Then, $\mathfrak{q}\in g^{-1}(\mathrm{P}(M))\Leftrightarrow\mathfrak{p}\in\mathrm{P}(M)\Leftrightarrow M_{\mathfrak{p}}$ is P $\Leftrightarrow M_{\mathfrak{p}}\otimes_{A_{\mathfrak{p}}}B_{\mathfrak{q}}=(M\otimes_A B)_{\mathfrak{q}}$ is P by (III) $\Leftrightarrow\mathfrak{q}\in\mathrm{P}(M\otimes_A B)$.

(2) M is P $\Leftrightarrow\mathrm{P}(M)=\mathrm{Spec}(A)\Rightarrow\mathrm{P}(M\otimes_A B)=g^{-1}(\mathrm{P}(M))=g^{-1}(\mathrm{Spec}(A))=\mathrm{Spec}(B)\Leftrightarrow M\otimes_A B$ is P.

(3) M is P–0 $\Leftrightarrow\mathrm{P}(M)$ contains a non-empty open set $U\Rightarrow\mathrm{P}(M\otimes_A B)=g^{-1}(\mathrm{P}(M))$ contains the non-empty open set $g^{-1}(U)\Rightarrow M\otimes_A B$ is P–0.

(4) $M\otimes_A B$ is P–0 $\Rightarrow$ $\mathrm{P}(M\otimes_A B)=g^{-1}(\mathrm{P}(M))$ contains a non-empty open set $V\Rightarrow \mathrm{P}(M)\supset g(g^{-1}(\mathrm{P}(M)))$ contains the non-empty open set $g(V)$ since g is an open map in this case (cf. [7], (2.4.6)) $\Rightarrow M$ is P–0.

(5) M is P–1 $\Leftrightarrow$ P(M) is open $\Rightarrow g^{-1}(\mathrm{P}(M))=\mathrm{P}(M\otimes_A B)$ is open $\Leftrightarrow M\otimes_A B$ is P–1.

(6) $M\otimes_A B$ is P–1 $\Leftrightarrow \mathrm{P}(M\otimes_A B)=g^{-1}(\mathrm{P}(M))$ is open $\Leftrightarrow$ P(M) is open since in this case the topology of Spec(A) is the quotient topology of Spec(B) by g (cf. [7], (2.3.12)) $\Leftrightarrow M$ is P–1.

(7) Let $A\to C$ be a homomorphism of finite type. Then, $B\to B\otimes_A C$ is a homomorphism of finite type, hence $(M\otimes_A B)\otimes_B(B\otimes_A C)$ is P–1 as a $B\otimes_A C$-module. Since $(M\otimes_A B)\otimes_B(B\otimes_A C)=M\otimes_A B\otimes_A C=(M\otimes_A C)\otimes_C(C\otimes_A B)$ and $C\to C\otimes_A B$ is a faithfully flat P-morphism (cf. Prop. 11 below), $M\otimes_A C$ is P–1 by (6). Therefore M is P–2.

COROLLARY 8. *Let M be an A-module. Then M is* P *(resp.* P–0, P–1, P–2) *if and only if so is* $M[X]$.

REMARK 9. Although $A\to A[[X]]$ is not a weak P-morphism in general, it is easy to show that $M[[X]]$ is P if only if M is P. Moreover if I is an ideal of A and M is P, then the I-adic completion of M is P.

The following Prop. 10 and Prop. 11 are similar to propositions in [15], and proofs are omitted.

PROPOSITION 10. *Let f: $A\to B$ and g: $B\to C$ be homomorphisms of rings.*

(1) *If f and g are* P*-morphisms (resp. weak* P*-morphisms), then so is $g\circ f$.*

(2) *If $g\circ f$ is a* P*-morphism (resp. a weak* P*-morphism) and g is faithfully flat, then f is a* P*-morphism (resp. a weak* P*-morphism).*

(3) *Suppose that* P *satisfies* (III'). *If f and g are flat and $g\circ f$ is a weak* P*-morphism, then g is a weak* P*-morphism.*

PROPOSITION 11. *Let $B\to C$ be a* P*-morphism of A-algebras and $A\to A'$ be a homomorphism essentially of finite type. Then, $B\otimes_A A'\to C\otimes_A A'$ is a* P*-morphism.*

DEFINITION 12 (cf. [10], [15]). Let P be a given property. We say a ring A is *well-fibered* for P if $A_{\mathfrak{p}}\to \hat{A}_{\mathfrak{p}}$ is a P-morphism for every $\mathfrak{p}\in \mathrm{Spec}(A)$. We say an A-module M is well-fibered for P if the ring $A/\mathrm{Ann}(M)$ is well-fibered for P.

It is easy to see that M is well-fibered for P if and only if for any $\mathfrak{p}$ and $\mathfrak{q}$ in Supp(M) such that $\mathfrak{p}\supset\mathfrak{q}$, the ring $\hat{A}_{\mathfrak{p}}\otimes_{A_{\mathfrak{p}}} k(\mathfrak{q})$ is geometrically P (with respect to $k(\mathfrak{q})$).

PROPOSITION 13 (cf. [15], Lemma 5.5). *Let $A \to B$ be a faithfully flat* P*-morphism and M be an A-module. If $M \otimes_A B$ is well-fibered for* P, *then so is M.*

PROOF. We may assume that $M = A$. In fact, $M \otimes_A B$ is well-fibered for $\mathrm{P} \Leftrightarrow B/\mathrm{Ann}(M \otimes_A B) = B \otimes_A (A/\mathrm{Ann}(M))$ is well-fibered for $\mathrm{P} \Rightarrow A/\mathrm{Ann}(M)$ is well-fibered for P by the assumption since $A/\mathrm{Ann}(M) \to B \otimes_A (A/\mathrm{Ann}(M))$ is a faithfully flat P-morphism $\Leftrightarrow M$ is well-fibered for P. The rest of the proof is the same as in [15].

COROLLARY 14. *A ring A is well-fibered for* P *if and only if $A_{\mathfrak{m}} \to \hat{A}_{\mathfrak{m}}$ is a* P*-morphism for all $\mathfrak{m} \in \mathrm{Max}(A)$.*

The proof can be done similarly to [15], Cor. 5.6.

DEFINITION 15. We say an A-module M is *catenary* if $A/\mathrm{Ann}(M)$ is a catenary ring, namely, if $\mathrm{Supp}(M)$ is a catenary topological space in the sense of [7], Chap. 0, (14.3). We say M is *universally catenary* if $M \otimes_A B$ is catenary for every homomorphism $A \to B$ of finite type.

PROPOSITION 16. (1) *M is catenary $\Leftrightarrow \dim(M_{\mathfrak{p}}) = \dim(M_{\mathfrak{p}}/\mathfrak{q}M_{\mathfrak{p}}) + \dim(M_{\mathfrak{q}})$ for any $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathrm{Supp}(M)$ such that $\mathfrak{q} \subset \mathfrak{p} \Leftrightarrow M_{\mathfrak{p}}$ is a catenary $A_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Supp}(M)$.*

(2) *M is universally catenary $\Leftrightarrow M[X_1, \ldots, X_n]$ is catenary for all $n \geqq 0$ $\Leftrightarrow M[X]$ is catenary.*

(3) *Cohen-Macaulay modules are universally catenary.*

PROOF. (1): Cf. [7], Chap. 0, (14.3). (2): Cf. [12], Th. 3.6. (3): Cf. [7], Chap. 0, (16.5.12).

DEFINITION 17. An A-module M is said to be P-*excellent* if it satisfies the following conditions:

(1) M is universally catenary,

(2) M is P–2, and

(3) M is well-fibered for P.

If M satisfies only (2) and (3), then we say M is *quasi* P-*excellent*. A ring A is said to be P-excellent (resp. quasi P-excellent) if A is P-excellent (resp. quasi P-excellent) as an A-module.

EXAMPLE 18. Reg-excellent = excellent (cf. [7], [9]) and Gor-excellent = acceptable in the sense of R. Y. Sharp (cf. [15], [16]).

THEOREM 19. *Let $A \to B$ be a homomorphism essentially of finite type. If an A-module M is well-fibered for* P, *then $M \otimes_A B$ is well-fibered for* P.

COROLLARY 20. *Let M be a* P*-excellent* (*resp. quasi* P*-excellent*) *A-module. Then for any homomorphism $A \to B$ essentially of finite type, the B-module $M \otimes_A B$ is* P*-excellent* (*resp. quasi* P*-excellent*).

PROOF OF TH. 19 (cf. [15]): We may assume that $M = A$. In fact, M is well-fibered for P $\Leftrightarrow A/\mathrm{Ann}(M)$ is well-fibered for P $\Rightarrow B/\mathrm{Ann}(M \otimes_A B)$ is well-fibered for P by the assumption since $A/\mathrm{Ann}(M) \to B/\mathrm{Ann}(M \otimes_A B)$ is essentially of finite type. Therefore $M \otimes_A B$ is well-fibered for P. It is enough to show the assertion in the following three special cases: (1) $B = A_S$ for a multiplicative subset S of A, (2) $B = A/I$ for an ideal I of A, and (3) $B = A[X]$. Case (1) is clear. Case (2): For every $\mathfrak{q} = \mathfrak{p}/I \in \mathrm{Spec}(A/I)$, $A_{\mathfrak{p}} \to \hat{A}_{\mathfrak{p}}$ is a P-morphism by the assumption, hence $B_{\mathfrak{q}} = A_{\mathfrak{p}}/IA_{\mathfrak{p}} \to \hat{A}_{\mathfrak{p}}/I\hat{A}_{\mathfrak{p}} = \hat{B}_{\mathfrak{q}}$ is a P-morphism by Prop. 11. Case (3): Let $\mathfrak{q} \in \mathrm{Spec}(A[X])$ and $\mathfrak{p} = \mathfrak{q} \cap A$. Then replacing A by $A_{\mathfrak{p}}$, we can assume that A is local. Since $A[X] \to \hat{A}[X]$ is faithfully flat and $\hat{A}[X]$ is well-fibered for Reg by [10], (30.D), Th. 68 and (33.G), Th. 77 (hence well-fibered for P), $A[X]$ is well-fibered for P by Prop. 13.

PROPOSITION 21. *Suppose* P *and* Q *are given properties,* Q *satisfies* (NC), *and* P *implies* Q. *If a ring A is* P–2, *then* Q(M) *is open for any A-module M.* (P *and* Q *don't need to satisfy* (III).)

PROOF. Since for every $\mathfrak{p}$ in Q(M) $\cap$ Supp (M), P($A/\mathfrak{p}$) $\subset$ Q($A/\mathfrak{p}$) and P($A/\mathfrak{p}$) is a non-empty open set in Spec ($A/\mathfrak{p}$), Q($A/\mathfrak{p}$) is thick in Spec ($A/\mathfrak{p}$). Hence Q(M) is open by (NC).

COROLLARY 22. *Let* P *and* Q *be as in* Prop. 21. *If a ring A is* P–2 (*resp. quasi* P*-excellent,* P*-excellent*), *then A is* Q–2 (*resp. quasi* Q*-excellent,* Q*-excellent*).

PROOF. It is enough to prove the assertion for Q–2. If $A \to B$ is a homomorphism of finite type, then B is P–2. Hence Q(B) is open by Prop. 21. Therefore A is Q–2.

COROLLARY 23. Reg*-excellent* (*i.e., excellent*) *ring*
$\Rightarrow$ CI*-excellent ring*
$\Rightarrow$ Gor*-excellent* (*i.e., acceptable*) *ring*
$\Rightarrow$ CM*-excellent ring.*

COROLLARY 24. *An excellent ring is* P*-excellent for any* P *which satisfies* (NC). (P *doesn't need to satisfy* (III).)

PROPOSITION 25. *Suppose that* P *satisfies* (NC). *If a ring A is quasi* P*-excellent* (*resp.* P*-excellent*), *then every A-module M is quasi* P*-excellent* (*resp.* P*-excellent*).

Proof. It is enough to show that if A is quasi P-excellent, then every A-module M is P–2, i.e., $M\otimes_A B$ is P–1 for any homomorphism $A\to B$ of finite type. Since B is quasi P-excellent by Cor. 20, we can assume $A=B$, and in this case the assertion is clear from Prop. 21.

Corollary 26. *If A is an excellent ring, then every A-module M is* P-*excellent for any* P *which satisfies* (NC).

Lemma 27. (1) *Suppose that* P *satisfies* (NC) *and* (QC) (P *need not satisfy* (III)). *If an A-module M is* P, *then the A/I-module $M\otimes_A(A/I)$ is* P–1 *for every ideal I of A.*

(2) *Suppose that* P *satisfies* (NC), (QC) *and* (III). *If an A-module M is* P, *then M is* P–2.

Proof. (1) Put $B=A/I$. For any $\mathfrak{q}=\mathfrak{p}/I$ in $\mathrm{P}(M\otimes_A B)\cap\mathrm{Supp}(M\otimes_A B)$, we have $\mathfrak{p}\in\mathrm{Supp}(M)$, hence $\mathrm{P}(A/\mathfrak{p})=\mathrm{P}(B/\mathfrak{q})$ is thick in $\mathrm{Spec}(A/\mathfrak{p})=\mathrm{Spec}(B/\mathfrak{q})$ by (QC). Therefore $\mathrm{P}(M\otimes_A B)$ is open by (NC). (2) Let $B=A[X_1,\dots,X_n]/J$ be a finitely generated A-algebra. Then $M[X_1,\dots,X_n]$ is P by (III) and $M\otimes_A B$ is P–1 by (1). Hence M is P–2.

Next, we consider two more conditions for P:

(NZ) Let M be an A-module and a be an element of A which is A-regular and M-regular. If M is P, then the A/aA-module M/aM is P.

(G) Let k be a field, A be a k-algebra and M be an A-module. If M is P, then M is geometrically P (with respect to k).

Proposition 28. *If* P *satisfies* (III) *and* (NZ), *then* P *satisfies* (G).

Proof (cf. [21]). Let k, A and M be as in (G) and take a finitely generated extension field K of k. We have to show that if M is P, then the $A\otimes_k K$-module $M\otimes_k K$ is P. By induction, we may assume that K is a simple extension $k(x)$ of k. If x is transcendental over k, then since $M\otimes_k K$ is a localization of $M[X]$ (which is P by Cor. 8), $M\otimes_k K$ is P. If x is algebraic over k, put $K=k[X]/(f)$, $f\neq 0$. Since the ideal of A generated by the coefficients of f is A, f is $A[X]$-regular and $M[X]$-regular (cf. [11], Chap. 5, Th. 7). Therefore, $M\otimes_k K=M[X]/fM[X]$ is P by (NZ).

Corollary 29. (G) *holds for* P = CI, Gor *and* CM.

Proposition 30 (cf. [21]). *Suppose that* P *satisfies* (III′) *and* (G). *Let $A\to B$ be a flat homomorphism and $A\to C$ be a homomorphism of finite type. If B and C are* P, *then so is $B\otimes_A C$.*

Proof. Since B is P, the flat homomorphism $A\to B$ is a weak P-morphism by (III′), and hence is a P-morphism by (G). Thus $C \to B \otimes_A C$ is a P-morphism by Prop. 11. This implies that $B \otimes_A C$ is P by Prop. 7, (2).

THEOREM 31. *Suppose that* P *satisfies* (III′), (NC), (QC) *and* (G) *and assume that the property* P *implies the catenary property* (*e.g.*, P=CI, Gor *and* CM). *If an A-module M is* P, *then M is* P-*excellent.*

PROOF. (1) Since $M[X]$ is P by Cor. 8, $M[X]$ is catenary by the hypothesis. Hence M is universally catenary by Prop. 16, (2).

(2) M is P–2 by Lemma 27, (2).

(3) M is well-fibered for P: We have to show that $\hat{A}_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{q})$ is geometrically P for any $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathrm{Supp}(M)$ such that $\mathfrak{p} \supset \mathfrak{q}$. It is enough to show that when A is a local ring, $M \neq 0$ is P and $\mathfrak{p}$ is an element of $\mathrm{Supp}(M)$, then $B = \hat{A} \otimes_A k(\mathfrak{p})$ is P. Take any $\mathfrak{q} \in \mathrm{Spec}(\hat{A})$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Then $A_{\mathfrak{p}} \to (\hat{A})_{\mathfrak{q}}$ is a flat local homomorphism and $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} (\hat{A})_{\mathfrak{q}} = (M \otimes_A \hat{A})_{\mathfrak{q}}$ is P. Thus $B_{\mathfrak{q}B} = (\hat{A})_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$ is P by (III′). Therefore $B = \hat{A} \otimes_A k(\mathfrak{p})$ is P.

THEOREM 32. *Suppose that* P *satisfies the same conditions as in* Th. 31. *If an A-module M is* P *and I is an ideal of A such that* $\mathrm{V}(I) \subset \mathrm{Supp}(M)$, *then A/I is a* P-*excellent ring. In particular,* $A/\mathrm{Ann}(M)$ *is a* P-*excellent ring.*

PROOF. Clearly A/I is universally catenary and well-fibered for P. A/I is P–2: We have to show that $\mathrm{P}(B)$ is open for any homomorphism $A/I \to B$ of finite type. If we write $B = A[X_1, \dots, X_n]/J$, then $M[X_1, \dots, X_n]$ is P and $\mathrm{V}(J) \subset \mathrm{Supp}(M[X_1, \dots, X_n])$. Thus we may assume that $B = A/I$. For any $\mathfrak{q} = \mathfrak{p}/I \in \mathrm{P}(B)$, we have $\mathfrak{p} \in \mathrm{Supp}(M)$ by the assumption, hence $\mathrm{P}(B/\mathfrak{q}) = \mathrm{P}(A/\mathfrak{p})$ is thick in $\mathrm{Spec}(B/\mathfrak{q}) = \mathrm{Spec}(A/\mathfrak{p})$ by (QC). Therefore $\mathrm{P}(B)$ is open by (NC).

COROLLARY 33. *Suppose there is an A-module M which is* P. *If* $\mathrm{Supp}(M)$ *coincides with* $\mathrm{Spec}(A)$, *then A is* P-*excellent.*

REMARK 34. R. Y. Sharp proved that if A has finite dimension and there exists a Gorenstein A-module M with $\mathrm{Supp}(M) = \mathrm{Spec}(A)$, then A is acceptable; and he asked whether the requirement that A be finite-dimensional is deleted from the hypothesis (cf. [16], p. 200, Remark). Cor. 33 (when P=Gor) answers this question affirmatively. (Note that (NC) and (QC) for P=Gor for modules will be proved in Part II.)

PROPOSITION 35 (cf. [9], (34.C), Th. 79). *Let I be an ideal of a ring A and B be the I-adic completion of A. If A is well-fibered for* P, *then the canonical map* $A \to B$ *is a* P-*morphism.*

PROOF. It is enough to show that $A_{\mathfrak{m}} \to B_{\mathfrak{n}}$ is a P-morphism for any $\mathfrak{n} \in \mathrm{Max}(B)$ and $\mathfrak{m} = \mathfrak{n} \cap A$. In this case, $\mathfrak{m}$ is a maximal ideal of A and $\hat{A}_{\mathfrak{m}} = \hat{B}_{\mathfrak{n}}$.

Since $A_m \to B_n \to \hat{B}_n = \hat{A}_m$ is a P-morphism by the assumption and $B_n \to \hat{B}_n$ is faithfully flat, $A_m \to B_n$ is a P-morphism by Prop. 10, (2).

COROLLARY 36. *If a ring A is well-fibered for* P, *then $A \to A[[X]]$ is a* P-*morphism.*

PROOF. Since $A[X]$ is well-fibered for P by Th. 19, the homomorphism $A[X] \to A[[X]]$ is a P-morphism by Prop. 35. Thus the composition $A \to A[X] \to A[[X]]$ of P-morphisms is a P-morphism.

Let A be an excellent (resp. acceptable) ring. Then, using Prop. 35 and Cor. 36, we can show as in [9], (34.C), Th. 79 that B in Prop. 35 and $A[[X]]$ are n-normal (resp. n-quasi normal) if so is A.

PROPOSITION 37. *Suppose that* P *satisfies* (NC). *Let A be a semi-local ring. If A is well-fibered for* P, *then A is* P–2, *hence quasi* P-*excellent.*

PROOF. Since $\hat{A}$ is excellent (cf. [9], (34.B)), $\hat{A}$ is P–2 by Cor. 26. The homomorphism $A \to \hat{A}$ is a faithfully flat P-morphism by the assumption. Hence A is P–2 by Prop. 7, (7).

Finally, we shall treat the finite descent of P-excellent rings.

PROPOSITION 38. (1) *Let $A \to B$ be a finite homomorphism such that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective and $A \to C$ be a flat homomorphism. Then $A \to C$ is a* P-*morphism if and only if $B \to B \otimes_A C$ is a* P-*morphism.*

(2) *Let $A \to B$ be a finite homomorphism and M be an A-module such that* $\mathrm{Supp}(M \otimes_A B) \to \mathrm{Supp}(M)$ *is surjective. Then M is well-fibered for* P *if and only if $M \otimes_A B$ is well-fibered for* P.

PROOF. (1) Cf. [5], Prop. 1.2. (2) Since $A/\mathrm{Ann}(M) \to B/\mathrm{Ann}(M \otimes_A B)$ is finite and $\mathrm{Spec}(B/\mathrm{Ann}(M \otimes_A B)) \to \mathrm{Spec}(A/\mathrm{Ann}(M))$ is surjective, we may assume that $M = A$. In this case, see [5], Prop. 1.3.

LEMMA 39. *Suppose that $A \to B$ is a homomorphism of finite type such that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is dominant, and that A is an integral domain and* $\mathrm{Spec}(B)$ *is irreducible (e.g., $A \subset B$ is a finitely generated extension of integral domains). Let M be an A-module. If $M \otimes_A B$ is* P–0, *then so is M.*

The proof is similar to that of [5], Prop. 2.1.

PROPOSITION 40. *Suppose that* P *satisfies* (NC). *If $A \to B$ is a homomorphism of finite type such that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective, then A is* P–2 *if and only if B is* P–2.

PROOF. It is enough to show the 'if' part. We have to show that for any homomorphism $A \to C$ of finite type, $\mathrm{P}(C)$ is open. We can assume that $A=C$, since $C \to B \otimes_A C$ is a homomorphism of finite type such that $\mathrm{Spec}(B \otimes_A C) \to \mathrm{Spec}(C)$ is surjective and $B \otimes_A C$ is P–2. For any $\mathfrak{p} \in \mathrm{P}(A)$, take $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Since $A/\mathfrak{p} \subset B/\mathfrak{q}$ is of finite type and $\mathrm{P}(B/\mathfrak{q})$ is a non-empty open set by the assumption, $\mathrm{P}(A/\mathfrak{p})$ is thick by Lemma 39. Therefore $\mathrm{P}(A)$ is open by (NC).

THEOREM 41. *Suppose that* P *satisfies* (NC). *Let $A \to B$ be a finite homomorphism such that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective* (*e.g.*, $A \subset B$ *is a finite extension*). *Then A is quasi* P*-excellent if and only if so is B.*

PROOF. It is clear from Prop. 38, (2) and Prop. 40.

Part II. Nagata's criterion

In this part, we shall examine Nagata's criterion for certain properties. The following lemma is useful to verify the openness of a subset of $\mathrm{Spec}(A)$.

LEMMA 1. (Topological lemma of Nagata, cf. [9], (22.B).) *Let A be a noetherian ring and U be a subset of* $\mathrm{Spec}(A)$. *In order that U is open, it is necessary and sufficient that*

(a) *U is stable under generalization (i.e., if $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$, $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{q} \in U$, then $\mathfrak{p} \in U$), and*

(b) *for each $\mathfrak{p} \in U$, $U \cap \mathrm{V}(\mathfrak{p})$ is thick in* $\mathrm{V}(\mathfrak{p})$.

PROPOSITION 2. *Let n be a non-negative integer and M be an A-module. Then the following statements are equivalent:*

(1) $\mathrm{S}_n(M)$ *is open.*

(2) *For any $\mathfrak{p} \in \mathrm{CM}(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_\mathfrak{p}) < n-1$, $\mathfrak{p}$ belongs to $\mathrm{S}_n(M)^\circ$, the interior of $\mathrm{S}_n(M)$.*

(3) *For any $\mathfrak{p} \in \mathrm{CM}(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_\mathfrak{p}) < n-1$, $\mathrm{V}(\mathfrak{p}) \cap \mathrm{S}_n(M)$ is thick in* $\mathrm{V}(\mathfrak{p})$.

PROOF. It is enough to show that (3)⇒(1). We prove it by induction on n. If $n=1$, then $\mathrm{S}_1(M) = \mathrm{Spec}(A) - \cup \mathrm{V}(\mathfrak{p})$, where $\mathfrak{p}$ runs over the set $\mathrm{Ass}(M)$, is always open. Suppose that $n \geqq 2$. It is sufficient to prove that $\mathrm{V}(\mathfrak{p}) \cap \mathrm{S}_n(M)$ is thick in $\mathrm{V}(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{S}_n(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_\mathfrak{p}) \geqq n-1$. Since $M_\mathfrak{p}$ is S_n, $\mathrm{depth}(M_\mathfrak{p}) \geqq \mathrm{Inf}(n, \dim(M_\mathfrak{p})) \geqq n-1 > 0$. Hence there is an $M_\mathfrak{p}$-regular element a in $\mathfrak{p}$. Considering an open neighbourhood of $\mathfrak{p}$, we can assume that a is M-regular. Then $\mathrm{S}_n(M) \cap \mathrm{V}(a) = \mathrm{S}_{n-1}(M/aM)$ (where M/aM is considered as an A/aA-module and $\mathrm{Spec}(A/aA)$ is identified with the subspace $\mathrm{V}(a)$ of $\mathrm{Spec}(A)$) and the condition of (3) is satisfied if we replace M and n by M/aM

and $n-1$, respectively. Hence $S_{n-1}(M/aM)$ is open and $\mathfrak{p}/aA \in S_{n-1}(M/aM)$. Therefore $S_n(M) \cap V(\mathfrak{p}) = S_{n-1}(M/aM) \cap V(\mathfrak{p}/aA)$ is thick in $V(\mathfrak{p}) = V(\mathfrak{p}/aA)$.

COROLLARY 3. *Let M be an A-module and n be a non-negative integer. Then,*

$\mathrm{CM}(M)$ *is open* $\Rightarrow S_{n+1}(M)$ *is open* $\Rightarrow S_n(M)$ *is open.*

PROOF. Suppose that $S_{n+1}(M)$ is open. Then for any $\mathfrak{p} \in \mathrm{CM}(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_\mathfrak{p}) < n-1$, we have $\mathfrak{p} \in S_{n+1}(M) = S_{n+1}(M)^\circ \subset S_n(M)^\circ$. Therefore $S_n(M)$ is open by Prop. 2. The proof of the first implication is similar.

THEOREM 4 (cf. [8], Th. 6). *If $S_n(A/\mathfrak{p})$ is thick in $\mathrm{Spec}(A/\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{CM}(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_\mathfrak{p}) < n-1$, then $S_n(M)$ is open. In particular,* (NC) *holds for S_n.*

PROOF. By Prop. 2, it is enough to show that $S_n(M) \cap V(\mathfrak{p})$ is thick in $V(\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{CM}(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_\mathfrak{p}) < n-1$. By [7], (6.10.6), there is an open neighbourhood U of $\mathfrak{p}$ such that

$$\dim(M_\mathfrak{q}) = \dim(M_\mathfrak{p}) + \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}),$$

and
$$\mathrm{depth}(M_\mathfrak{q}) = \mathrm{depth}(M_\mathfrak{p}) + \mathrm{depth}(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q})$$

for all $\mathfrak{q} \in V(\mathfrak{p}) \cap U$. By the assumption, there is an open set V of $\mathrm{Spec}(A)$ such that $\emptyset \neq V \cap V(\mathfrak{p}) \subset S_n(A/\mathfrak{p})$. Pur $W = V \cap U \cap V(\mathfrak{p})$. This is a non-empty open subset of $V(\mathfrak{p})$ and for every $\mathfrak{q} \in W$, we have

$$\begin{aligned} \mathrm{depth}(M_\mathfrak{q}) &= \mathrm{depth}(M_\mathfrak{p}) + \mathrm{depth}(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}) \\ &\geqq \dim(M_\mathfrak{p}) + \mathrm{Inf}(n, \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q})) \\ &\geqq \mathrm{Inf}(n, \dim(M_\mathfrak{p}) + \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q})) \\ &= \mathrm{Inf}(n, \dim(M_\mathfrak{q})). \end{aligned}$$

Thus $W \subset S_n(M) \cap V(\mathfrak{p})$ and this shows that $S_n(M) \cap V(\mathfrak{p})$ is thick in $V(\mathfrak{p})$.

PROPOSITION 5. *Suppose that P is a given property and n is a non-negative integer. Then for an A-module M, the following statements are equivalent:*

(1) *$P_n(M)$ is open.*

(2) *For any $\mathfrak{p} \in P(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_\mathfrak{p}) < n$, $\mathfrak{p}$ belongs to $P_n(M)^\circ$.*

(3) *For any $\mathfrak{p} \in P(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_\mathfrak{p}) < n$, $V(\mathfrak{p}) \cap P_n(M)$ is thick in $V(\mathfrak{p})$.*

The following lemma is in [6], Proof of Th. 2.3.

LEMMA 6. *Let Y be a non-empty subset of* $\mathrm{Spec}(A)$ *and put* $I=\cap\{\mathfrak{q}\mid\mathfrak{q}\in Y\}$. *Then for any* $\mathfrak{p}\in\mathrm{Ass}(A/I)$, *it holds that* $\mathfrak{p}=\cap\{\mathfrak{q}\mid\mathfrak{q}\in Y, \mathfrak{p}\subset\mathfrak{q}\}$. *In particular, there is a* $\mathfrak{q}\in Y$ *such that* $\mathfrak{p}\subset\mathfrak{q}$.

PROOF OF PROP. 5. It is enough to show that (3)⇒(1). We can assume that $\mathrm{P}_n(M)\neq\mathrm{Spec}(A)$. Put $Z=\mathrm{Supp}(M)-\mathrm{P}_n(M)$, $Y=\mathrm{Min}(Z)$ and $I=\cap\{\mathfrak{q}\mid\mathfrak{q}\in Y\}$. If we can show that $\mathrm{Ass}(A/I)\subset Y$, then we have $Z=\mathrm{V}(I)$, which is a closed set. In fact, if $\mathfrak{q}\in Z$, then there is a $\mathfrak{p}\in Y$ such that $\mathfrak{p}\subset\mathfrak{q}$. Since $\mathfrak{p}\supset I$, we have $\mathfrak{q}\in\mathrm{V}(I)$. Conversely if $\mathfrak{q}$ is in $\mathrm{V}(I)$, then there is a $\mathfrak{p}\in\mathrm{Ass}(A/I)$ such that $\mathfrak{p}\subset\mathfrak{q}$. By the assumption, $\mathfrak{p}\in Y\subset Z$ and since Z is stable under specialization, we have $\mathfrak{q}\in Z$. Assume that $\mathrm{Ass}(A/I)\not\subset Y$ and take a $\mathfrak{p}\in\mathrm{Ass}(A/I)-Y$. By Lemma 6, there is a $\mathfrak{q}\in Y$ such that $\mathfrak{q}\supsetneqq\mathfrak{p}$. Then $\dim(M_\mathfrak{q})\leqq n$. In fact, since $M_\mathfrak{q}$ is not P_n, there is an $\mathfrak{r}A_\mathfrak{q}\in\mathrm{Supp}(M_\mathfrak{q})$ such that $\dim(M_\mathfrak{r})\leqq n$ and $M_\mathfrak{r}$ is not P. Hence $\mathfrak{r}\notin\mathrm{P}_n(M)$, and this implies $\mathfrak{r}\in Z$. Since $\mathfrak{r}\subset\mathfrak{q}$, we get $\mathfrak{r}=\mathfrak{q}$ by the minimality of $\mathfrak{q}$. Thus $\dim(M_\mathfrak{q})\leqq n$. Since $\dim(M_\mathfrak{p})<\dim(M_\mathfrak{q})\leqq n$, $\mathrm{V}(\mathfrak{p})\cap\mathrm{P}_n(M)$ is thick in $\mathrm{V}(\mathfrak{p})$ by the assumption. Therefore there is an $f\notin\mathfrak{p}$ such that $\mathrm{V}(\mathfrak{p})\cap\mathrm{D}(f)\subset\mathrm{V}(\mathfrak{p})\cap\mathrm{P}_n(M)$. If we assume that $f\notin\mathfrak{q}$ for some $\mathfrak{q}\in Y\cap\mathrm{V}(\mathfrak{p})$, then $\mathfrak{q}\in\mathrm{V}(\mathfrak{p})\cap\mathrm{D}(f)\subset\mathrm{P}_n(M)$ contrary to the assumption $\mathfrak{q}\in Y$. Hence $f\in\cap\{\mathfrak{q}\mid\mathfrak{p}\subset\mathfrak{q}\in Y\}=\mathfrak{p}$ (cf. Lemma 6), again a contradiction.

COROLLARY 7. *Let M be an A-module and n be a non-negative integer. Then, for a given property* P,

$\mathrm{P}(M)$ *is open* ⇒ $\mathrm{P}_{n+1}(M)$ *is open* ⇒ $\mathrm{P}_n(M)$ *is open.*

PROOF. Suppose that $\mathrm{P}_{n+1}(M)$ is open. Then for any $\mathfrak{p}\in\mathrm{P}(M)\cap\mathrm{Supp}(M)$ such that $\dim(M_\mathfrak{p})<n$, we have $\mathfrak{p}\in\mathrm{P}_{n+1}(M)=\mathrm{P}_{n+1}(M)^\circ\subset\mathrm{P}_n(M)^\circ$. Therefore $\mathrm{P}_n(M)$ is open by Prop. 5. The proof of the first implication is similar.

THEOREM 8. *Suppose that a given property* P *satisfies the following conditions*:

(a) *If* $\mathfrak{p}\in\mathrm{P}(M)$ *and* $\mathrm{Supp}(M)=\mathrm{V}(\mathfrak{p})$, *then there is an open neighbourhood U of $\mathfrak{p}$ such that* $\mathrm{P}(M)\cap\mathrm{V}(\mathfrak{p})\cap U=\mathrm{P}(A/\mathfrak{p})\cap U$. *(Here $A/\mathfrak{p}$ is considered as an $A/\mathfrak{p}$-module and* $\mathrm{Spec}(A/\mathfrak{p})$ *is identified with the subspace* $\mathrm{V}(\mathfrak{p})$ *of* $\mathrm{Spec}(A)$.)

(b) *Let $(A, \mathfrak{m})$ be a local ring and M be an A-module. If $a\in\mathfrak{m}$ is M-regular, then M is* P *if and only if the A/aA-module M/aM is* P.

(c) P ⟹ CM.

Then,

(1) (NC) *and* (QC) *hold for* P.

(2) *Let M be an A-module and n be a non-negative integer. If* $\mathrm{P}_n(A/\mathfrak{p})$ *is thick in* $\mathrm{Spec}(A/\mathfrak{p})$ *for all* $\mathfrak{p}\in\mathrm{P}(M)\cap\mathrm{Supp}(M)$ *such that* $\dim(M_\mathfrak{p})<n$, *then* $\mathrm{P}_n(M)$ *is open. In particular,* (NC) *holds for* P_n.

(3) *If an A-module M is* P_n, *then* $P_n(A/\mathfrak{p})$ *is thick in* $\operatorname{Spec}(A/\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Supp}(M)$ *such that* $\dim(M_\mathfrak{p}) \leqq n$.

Proof. (1) (NC) for P: We have to show that if $P(A/\mathfrak{p})$ is thick in $\operatorname{Spec}(A/\mathfrak{p})$ for all $\mathfrak{p} \in P(M) \cap \operatorname{Supp}(M)$, then $P(M)$ is open. By Lemma 1, it is enough to show that $V(\mathfrak{p}) \cap P(M)$ is thick in $V(\mathfrak{p})$ for all $\mathfrak{p} \in P(M) \cap \operatorname{Supp}(M)$. Since $M_\mathfrak{p}$ is P, it is CM by (c) and we can take a maximal $M_\mathfrak{p}$-regular sequence $(x_1,\dots, x_r)$ such that $x_i \in \mathfrak{p}$ and $r=\dim(M_\mathfrak{p})$. Considering an open neighbourhood of $\mathfrak{p}$, we can assume that the sequence $(x_1,\dots, x_r)$ is M-regular. By the condition (b), we can replace M by $M/(x_1,\dots, x_r)M$ and assume that $\dim(M_\mathfrak{p})=0$. Moreover, again considering an open neighbourhood of $\mathfrak{p}$, we can assume that $\operatorname{Supp}(M)=V(\mathfrak{p})$. Then, by the condition (a), there is an open set U such that $\mathfrak{p} \in U$ and $P(M) \cap V(\mathfrak{p}) \cap U = P(A/\mathfrak{p}) \cap U$. Therefore $P(M) \cap V(\mathfrak{p})$ is thick in $V(\mathfrak{p})$.

(QC) for P: We have to show that if M is P, then $P(A/\mathfrak{p})$ is thick in $\operatorname{Spec}(A/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Supp}(M)$. By the same reduction as above, we can assume that there is an open set U such that $\mathfrak{p} \in U$ and $P(A/\mathfrak{p}) \cap U = P(M) \cap V(\mathfrak{p}) \cap U = V(\mathfrak{p}) \cap U$ since $P(M) = \operatorname{Spec}(A)$. Therefore $P(A/\mathfrak{p})$ is thick in $V(\mathfrak{p})$.

(2) By Prop. 5, it is enough to show that $P_n(M) \cap V(\mathfrak{p})$ is thick in $V(\mathfrak{p})$ for all $\mathfrak{p} \in P(M) \cap \operatorname{Supp}(M)$ such that $\dim(M_\mathfrak{p}) < n$. In this case, $M_\mathfrak{p}$ is P and the reduction in (1) can be applied. The rest of proof is the same as in (1).

(3) also can be proved by using a similar reduction as that in (1). We omit the proof.

Proposition 9. *The condition* (a) *of* Th. 8 *is satisfied for* P=CM, Gor *and* CI.

Proof. P=CM: Let M and $\mathfrak{p}$ be as in (a). Then, by [7], (6.10.6), there is an open set U such that $\mathfrak{p} \in U$ and $\operatorname{coprof}(M_\mathfrak{q}) = \operatorname{coprof}(M_\mathfrak{p}) + \operatorname{coprof}((A/\mathfrak{p})_\mathfrak{q}) = \operatorname{coprof}((A/\mathfrak{p})_\mathfrak{q})$ (since $M_\mathfrak{p}$ is CM) for all $\mathfrak{q} \in V(\mathfrak{p}) \cap U$, where $\operatorname{coprof}(M) = \dim(M) - \operatorname{depth}(M)$. Thus $M_\mathfrak{q}$ is CM if and only if $(A/\mathfrak{p})_\mathfrak{q}$ is CM for all $\mathfrak{q} \in V(\mathfrak{p}) \cap U$.

P=Gor (cf. [6], [10]): If $M_\mathfrak{p}$ is a zero-dimensional Gorenstein module of rank r, then $(\operatorname{Ext}^1_A(A/\mathfrak{p}, M))_\mathfrak{p}=0$ and $(\operatorname{Hom}_A(A/\mathfrak{p}, M))_\mathfrak{p} = (A/\mathfrak{p})^r_\mathfrak{p}$. Considering an open neighbourhood of $\mathfrak{p}$, we can assume that $\operatorname{Ext}^1_A(A/\mathfrak{p}, M)=0$ and $\operatorname{Hom}_A(A/\mathfrak{p}, M) = (A/\mathfrak{p})^r$. Then, using long exact sequences for Ext, we have $\operatorname{Ext}^i_A(A/\mathfrak{p}, M)=0$ for all $i>0$. Then, as in [6], Lemma 2.1, we can show that $M_\mathfrak{q}$ is Gor if and only if $(A/\mathfrak{p})_\mathfrak{q}$ is Gor for all $\mathfrak{q} \in V(\mathfrak{p})$.

P=CI (cf. [6]): We may assume that $M=A$. If $A_\mathfrak{p}$ is CI, then $H_i(A_\mathfrak{p}, k(\mathfrak{p}), k(\mathfrak{p}))=0$ for $i=3$ and 4. Since $H_i(A_\mathfrak{p}, k(\mathfrak{p}), k(\mathfrak{p})) = (H_i(A, A/\mathfrak{p}, A/\mathfrak{p}))_\mathfrak{p}$ and $H_i(A, A/\mathfrak{p}, A/\mathfrak{p})$ is a finite $A/\mathfrak{p}$-module (cf. [1]), we can assume that $H_i(A, A/\mathfrak{p}, A/\mathfrak{p})$ is zero for $i=3, 4$ and free for $0 \leqq i < 3$. Therefore, for any $\mathfrak{q} \in V(\mathfrak{p})$, $H_i(A_\mathfrak{q}, (A/\mathfrak{p})_\mathfrak{q}, (A/\mathfrak{p})_\mathfrak{q})$ is zero for $i=3$, 4 and free for $0 \leqq i < 3$. Hence, by [6], Lemma 3.2, $A_{\mathfrak{q}}$ is CI if and only if $(A/\mathfrak{p})_{\mathfrak{q}}$ is CI for all $\mathfrak{q} \in V(\mathfrak{p})$.

PROPOSITION 10. *The condition* (b) *of* Th. 8 *is satisfied for* P=CM, Gor *and* CI.

PROOF FOR P=CM. It is clear from the following equalities:

$$\dim_{A/aA}(M/aM) = \dim_A(M/aM) = \dim_A(M) - 1,$$

$$\operatorname{depth}_{A/aA}(M/aM) = \operatorname{depth}_A(M/aM) = \operatorname{depth}_A(M) - 1.$$

PROOF FOR P=CI. When $M=A$, the assertion is already known (cf. [21]), and is also shown by the next lemma. For a general M, the proposition follows from Lemma 12 below.

LEMMA 11. *Let $(A, \mathfrak{m})$ be a noetherian local ring and I be an ideal of A. If $a \in \mathfrak{m}$ is A/I-regular, then $J=(I, a)$ is generated by an A-regular sequence if and only if I is generated by an A-regular sequence.*

PROOF. Put $k=A/\mathfrak{m}$, $B=A/I$ and $C=A/J$. Then, the next exact sequence holds (cf. [1], V, Th. 1 and VI, Th. 25): $0=H_3(B, C, k)\to H_2(A, B, k)\to H_2(A, C, k)\to H_2(B, C, k)=0$. Hence, $H_2(A, B, k)=0$ if and only if $H_2(A, C, k)=0$ and this implies our assertion (cf. [1], VI, Th. 25).

LEMMA 12. *Let $(A, \mathfrak{m})$ be a (not necessarily noetherian) local ring, M be a finitely presented A-module and $a \in \mathfrak{m}$ be an M-regular element. Then M is free if and only if the A/aA-module M/aM is free.*

PROOF. It is enough to show the 'if' part. Take an exact sequence $0\to K \to F\to M\to 0$ such that F is free with $\dim_k(F/\mathfrak{m}F)=\dim_k(M/\mathfrak{m}M)$, where $k=A/\mathfrak{m}$. Then K is finitely generated and since a is M-regular, the sequence $0\to K/aK \to F/aF\to M/aM\to 0$ is exact by the snake lemma. Because F/aF and M/aM are free A/aA-modules of the same rank, the surjection $F/aF\to M/aM$ is an isomorphism. Hence $K/aK=0$ and this implies that $K=0$ by Nakayama's lemma. Thus $M=F$ is free.

PROOF OF PROP. 10 FOR P=Gor. If M is Gor and a is M-regular, then M/aM is Gor by [13], Th. 4.11. Conversely, suppose that M/aM is Gor as an A/aA-module and $n=\dim(M)$. It is known that M is Gor if and only if M is CM and $H^n_{\mathfrak{m}}(M)$ is injective. Since M is CM by the assertion for P=CM, it is enough to show that $N=H^n_{\mathfrak{m}}(M)$ is injective. From the exact sequence $0\to M \xrightarrow{a} M\to M/aM\to 0$, we have the following exact sequence:

$$0 = H^{n-1}_{\mathfrak{m}}(M) \longrightarrow H^{n-1}_{\mathfrak{m}}(M/aM) \longrightarrow H^n_{\mathfrak{m}}(M) \xrightarrow{a} H^n_{\mathfrak{m}}(M) \longrightarrow H^n_{\mathfrak{m}}(M/aM) = 0.$$

Therefore N is an artinian A-module such that $N=aN$ and $\operatorname{Hom}_A(A/aA, N)$ is

A/aA-injective. By the Matlis duality, an artinian A-module N is A-injective if and only if its Matlis dual $D_A(N)=\mathrm{Hom}_A(N, E_A(k))$ is an $\hat{A}$-free module. Since $\mathrm{Hom}_A(A/aA, N)$ is A/aA-injective, $D_{A/aA}(\mathrm{Hom}_A(A/aA, N))$ is $\hat{A}/a\hat{A}$-free and

$$\begin{aligned} D_{A/aA}(\mathrm{Hom}_A(A/aA, N)) &= \mathrm{Hom}_{A/aA}(\mathrm{Hom}_A(A/aA, N), E_{A/aA}(k)) \\ &= \mathrm{Hom}_{A/aA}(\mathrm{Hom}_A(A/aA, N), \mathrm{Hom}_A(A/aA, E_A(k))) \\ &= \mathrm{Hom}_A(\mathrm{Hom}_A(A/aA, N), E_A(k)) \\ &= A/aA \otimes_A D_A(N) = \hat{A}/a\hat{A} \otimes_{\hat{A}} D_A(N). \end{aligned}$$

Therefore $D_A(N)$ is a finitely generated $\hat{A}$-module, a is $D(N)$-regular and $D(N)/aD(N)$ is $\hat{A}/a\hat{A}$-free. Hence $D_A(N)$ is $\hat{A}$-free by Lemma 12. Therefore N is A-injective and this completes the proof of Prop. 10.

THEOREM 13. (1) (NC) *holds for* P=CI, Gor, CM, CI_n, Gor_n *and* CM_n, *where n is a non-negative integer.*

(2) (QC) *holds for* P=CI, Gor *and* CM.

PROOF. Clear from Th. 8, Prop. 9 and Prop. 10.

PROPOSITION 14. $\mathrm{P}_0(M)$ *is open for any property* P *and any A-module M.*

PROOF. From the fact $\mathrm{P}_0(M)=\mathrm{Spec}(A)-\cup\{V(\mathfrak{p})\mid \mathfrak{p}\in \mathrm{Min}(M)-\mathrm{P}(M)\}$, where $\mathrm{Min}(M)=\mathrm{Min}(\mathrm{Supp}(M))$, the proposition is clear.

COROLLARY 15. $\mathrm{QR}(M)$ (cf. Part I, Example 2, (9)) *is open for any A-module M.*

PROOF. Since $S_1(M)$ and $\mathrm{Gor}_0(M)$ are open, $\mathrm{QR}(M)=S_1(M)\cap \mathrm{Gor}_0(M)$ is open.

PROPOSITION 16. (1) *For a ring A, the following statements are equivalent*:

$(a)_n$ n-$\mathrm{Nor}(A)$ *is open.*

$(b)_n$ *For all* $\mathfrak{p}\in \mathrm{Reg}(A)$ *such that* $\dim(A_\mathfrak{p})<n$, $\mathfrak{p}$ *belongs to* $(n\text{-}\mathrm{Nor}(A))^\circ$.

$(c)_n$ *For all* $\mathfrak{p}\in \mathrm{Reg}(A)$ *such that* $\dim(A_\mathfrak{p})<n$, $V(\mathfrak{p})\cap(n\text{-}\mathrm{Nor}(A))$ *is thick in* $V(\mathfrak{p})$.

(2) *For an A-module M, the following statements are equivalent*:

$(a)_n$ n-$\mathrm{QN}(M)$ *is open.*

$(b)_n$ *For all* $\mathfrak{p}\in \mathrm{Gor}(M)\cap \mathrm{Supp}(M)$ *such that* $\dim(M_\mathfrak{p})<n$, $\mathfrak{p}$ *belongs to* $(n\text{-}\mathrm{QN}(M))^\circ$.

$(c)_n$ *For all* $\mathfrak{p}\in \mathrm{Gor}(M)\cap \mathrm{Supp}(M)$ *such that* $\dim(M_\mathfrak{p})<n$, $V(\mathfrak{p})\cap(n\text{-}\mathrm{QN}(M))$ *is thick in* $V(\mathfrak{p})$.

PROOF. We prove only (2) since the proof of (1) is similar. It is enough to show that $(c)_n \Rightarrow (a)_n$ by induction on n. If $n=0$, then 0-QN(M)=QR(M) is open by Cor. 15. Suppose that $n>0$. Since $(c)_n \Rightarrow (c)_{n-1} \Rightarrow (a)_{n-1}$, $(n-1)$-QN(M) is an open set by the assumption. Hence we can assume that M is $(n-1)$-QN. If $\mathfrak{p}$ is in $\mathrm{CM}(M) \cap \mathrm{Supp}(M)$ and $\dim(M_{\mathfrak{p}})<n$, then $M_{\mathfrak{p}}$ is $(n-1)$-QN by the assumption. Hence $\mathfrak{p} \in \mathrm{Gor}(M) \subset n$-QN$(M)$. Therefore $V(\mathfrak{p}) \cap (n\text{-QN}(M))$ is thick in $V(\mathfrak{p})$ and this implies that $V(\mathfrak{p}) \cap S_{n+1}(M)$ is thick in $V(\mathfrak{p})$. Thus $S_{n+1}(M)$ is open by Prop. 2. Similarly, using Prop. 5, we can show that $\mathrm{Gor}_n(M)$ is open. Therefore n-QN$(M)=S_{n+1}(M) \cap \mathrm{Gor}_n(M)$ is open.

THEOREM 17. (1) *If* n-Nor$(A/\mathfrak{p})$ *is thick in* $\mathrm{Spec}(A/\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Reg}(A)$ *such that* $\dim(A_{\mathfrak{p}})<n$, *then* n-Nor(A) *is open. In particular,* (NC) *holds for* n-Nor.

(2) *If* n-QN$(A/\mathfrak{p})$ *is thick in* $\mathrm{Spec}(A/\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Gor}(M) \cap \mathrm{Supp}(M)$ *such that* $\dim(M_{\mathfrak{p}})<n$, *then* n-QN(M) *is open. In particular,* (NC) *holds for* n-QN.

PROOF. We prove only (2) since the proof of (1) is similar. By induction on n, we can assume that $n>0$ and M is $(n-1)$-QN as in the proof of Prop. 16. If $\mathfrak{p}$ is $S_{n+1}(M) \cap \mathrm{Supp}(M)$ and $\dim(M_{\mathfrak{p}})<n$, then $M_{\mathfrak{p}}$ is $(n-1)$-QN by the assumption. Hence $\mathfrak{p} \in \mathrm{Gor}(M) \subset n$-QN$(M)$. Therefore n-QN$(A/\mathfrak{p})$ is thick by the assumption. Since $S_{n+1}(A/\mathfrak{p}) \supset n$-QN$(A/\mathfrak{p})$, $S_{n+1}(A/\mathfrak{p})$ is thick for all $\mathfrak{p} \in \mathrm{CM}(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_{\mathfrak{p}})<n$. Hence $S_{n+1}(M)$ is open by Th. 4. Similarly, using Th. 8, (2) for P=Gor, we can show that $\mathrm{Gor}_n(M)$ is open. Therefore n-QN$(M)=S_{n+1}(M) \cap \mathrm{Gor}_n(M)$ is open.

PROPOSITION 18. (1) Reg(A) *is open* $\Rightarrow$ $(n+1)$-Nor(A) *is open* $\Rightarrow$ n-Nor(A) *is open.*

(2) Gor(M) *is open* $\Rightarrow$ $(n+1)$-QN(M) *is open* $\Rightarrow$ n-QN(M) *is open.*

PROOF. We prove only the right side of (2), since other assertions are similarly proved. Suppose that $(n+1)$-QN(M) is open. If $\mathfrak{p}$ is in $\mathrm{Gor}(M) \cap \mathrm{Supp}(M)$ and $\dim(M_{\mathfrak{p}})<n$, then $\mathfrak{p} \in (n+1)\text{-QN}(M)=((n+1)\text{-QN}(M))^{\circ} \subset (n\text{-QN}(M))^{\circ}$. Hence n-QN(M) is open by Prop. 16.

PROPOSITION 19. *If A is* $(n-1)$-Nor *and* n-Nor(A) *is open, then* n-QN(A) *is open.*

PROOF. If $\mathfrak{p}$ is in $\mathrm{Gor}(A)$ and $\dim(A_{\mathfrak{p}})<n$, then $\mathfrak{p} \in \mathrm{Reg}(A) \subset n$-Nor$(A)$ $=(n\text{-Nor}(A))^{\circ} \subset (n\text{-QN}(A))^{\circ}$. Hence n-QN(A) is open by Prop. 16, (2).

COROLLARY 20. *If A is reduced and* Nor(A) *is open (e.g., if the integral closure of A is finite over A), then* QN(A) *is open.*

PROPOSITION 21. (1) $S_n(M)$ *is open* $\Rightarrow$ $CM_n(M)$ *is open.*

(2) n-$\mathrm{Nor}(A)$ *is open* $\Rightarrow$ $R_n(A)$ *is open.*

(3) n-$\mathrm{QN}(M)$ *is open* $\Rightarrow$ $\mathrm{Gor}_n(M)$ *is open.*

PROOF. We prove only (3), since other assertions are similarly proved. Take a $\mathfrak{p} \in \mathrm{Gor}(M) \cap \mathrm{Supp}(M)$ such that $\dim(M_{\mathfrak{p}}) < n$. Then $\mathfrak{p} \in n$-$\mathrm{QN}(M)$ $= (n\text{-}\mathrm{QN}(M))^{\circ} \subset \mathrm{Gor}_n(M)^{\circ}$. Therefore $\mathrm{Gor}_n(M)$ is open by Prop. 5.

References

[1] M. André, Homologie des algèbres commutatives, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[2] L. L. Avramov, Flat morphisms of complete intersections, Soviet Math. Dokl. **16** (1975), 1413–1417.

[3] W. Bruns, Eine Charakterisierung der (R_k, S_{k+1})-Ringe, Arch. Math. **30** (1978), 118–121.

[4] H.-B. Foxby, n-Gorenstein rings, Proc. Amer. Math. Soc. **42** (1974), 67–72.

[5] S. Greco, Two theorems on excellent rings, Nagoya Math. J. **60** (1976), 139–149.

[6] S. Greco and M. G. Marinari, Nagata's criterion and openness of loci for Gorenstein and complete intersection, Math. Z. **160** (1978), 207–216.

[7] A. Grothendieck et J. Dieudonné, Éléments de géométrie algébrique, Chapitre IV, Première partie, Seconde partie, Publ. Math. IHES. **20** (1964), **24** (1965).

[8] C. Massaza e P. Valabrega, Sull'apertura di luoghi in uno schema localmente noetheriano, Boll. U. M. I. **14** (1977), 564–574.

[9] H. Matsumura, Commutative algebra, Benjamin, New York, 1970.

[10] H. Matsumura, Excellent rings, acceptable rings etc., Lecture Notes in RIMS, Kyoto Univ. **328** (1978), 130–138.

[11] D. G. Northcott, Finite free resolutions, Cambridge Univ. Press, 1976.

[12] L. J. Ratliff, On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals, (I), Amer. J. Math. **91** (1969), 508–528.

[13] R. Y. Sharp, Gorenstein modules, Math. Z. **115** (1970), 117–139.

[14] R. Y. Sharp, The Euler characteristic of a finitely generated module of finite injective dimension, Math. Z. **130** (1973), 79–93.

[15] R. Y. Sharp, Acceptable rings and homomorphic images of Gorenstein rings, J. of Algebra **44** (1977), 246–261.

[16] R. Y. Sharp, A commutative Noetherian ring which possesses a dualizing complex is acceptable, Math. Proc. Camb. Phil. Soc. **82** (1977), 197–213.

[17] R. Y. Sharp, On the fibre rings of a formal power series extension, Quart. J. Math. Oxford **28** (1977), 487–494.

[18] P. Valabrega, Formal fibers and openness of loci, J. Math. Kyoto Univ. **18** (1978), 199–208.

[19] W. V. Vasconcelos, Reflexive modules over Gorenstein rings, Proc. Amer. Math. Soc. **19** (1968), 1349–1355.

[20] W. V. Vasconcelos, Quasi-normal rings, Illinois J. Math. **14** (1970), 268–273.

[21] K. Watanabe, T. Ishikawa, S. Tachibana and K. Otsuka, On tensor products Gorenstein rings, J. Math. Kyoto Univ. **18** (1969), 413–423.

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