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Z-transforms and noetherian pairs

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Let A be a noetherian ring, and let Z be a subset of Spec(A) which is stable under specialization. Assume that every element of Z is a regular prime ideal. Let M be an A-module such that every A-regular element is M-regular. The Z-transform $T(Z, M)$ of M is a subset of $M \otimes_A Q(A)$ defined as follows:

$$T(Z, M) = \{x \in M \otimes_A Q(A) \mid V(M :_A x) \subseteq Z\},$$

where $Q(A)$ is the total quotient ring of A, $M :_A x = \{a \in A \mid ax \in M\}$, and $V(M :_A x)$ is the set of prime ideals of A containing $M :_A x$. Since $A :_A (x+y)$ and $A :_A xy$ contain $(A :_A x)(A :_A y)$ for every x and y in $Q(A)$, $T(Z, A)$ is a subring of $Q(A)$ which contains A. It is easy to see that $T(Z, M)$ is a $T(Z, A)$-module. Note that $T(Z, M) = \Gamma(X, \mathscr{H}^0_{X/Z}(\tilde{M}))$ where $X = \mathrm{Spec}(A)$ and $\tilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module associated to M (cf. [2], Chap. IV, (5.9)).

In this paper, we shall give necessary and sufficient conditions on A so that $(A, T(Z, A))$ is a noetherian pair. For noetherian rings R and S with $R \subseteq S$, we say that (R, S) is a noetherian pair if every ring T, $R \subseteq T \subseteq S$, is noetherian. If Z is the set of all regular maximal ideals of A, then $T(Z, A)$ is the global transform A^g of A introduced by Matijevic in [3]. He proved that (A, A^g) is a noetherian pair if A is reduced.

Let $B = A/I$ where I is an ideal of A. Assume that $\mathrm{Ass}_A(B) \subseteq \mathrm{Ass}_A(A)$. Let $Z' = \{\mathfrak{p}/I \mid \mathfrak{p} \in Z \text{ and } \mathfrak{p} \supseteq I\}$. Then it is clear that every element of Z' is a regular prime ideal of B and $T(Z, B) = T(Z', B)$. Moreover we have a natural ring homomorphism ϕ: $T(Z, A) \to T(Z, B)$ whose kernel is $T(Z, I) = T(Z, A) \cap IQ(A)$. It should be remarked that $\phi(x)z = xz$ for every $x \in T(Z, A)$ and $z \in T(Z, B)$. In the case that Z is the set of all regular maximal ideals of A, $T(Z, B)$ is not the global transform of B in general. However if every maximal ideal of A is regular, then $T(Z, B) = B^g$.

Our main result is the following

Theorem. *Let A be a noetherian ring, and let Z be a subset of* Spec(A) *which is stable under specialization. Assume that every element of Z is a regular prime ideal. Then the following conditions on A are equivalent.*

(1) *$(A, T(Z, A))$ is a noetherian pair.*

(2) (a) *$T(Z, A/\mathfrak{p})$ is a finite $A/\mathfrak{p}$-module for every $\mathfrak{p} \in \mathrm{Ass}_A(A)$ such that $A_\mathfrak{p}$ is not reduced, and*

(b) $(A/\mathfrak{p}, T(Z, A/\mathfrak{p}))$ *is a noetherian pair for every* $\mathfrak{p} \in \mathrm{Ass}_A(A)$.

If $A_\mathfrak{p}$ *are not reduced for all associated prime ideal* $\mathfrak{p}$ *of* A, *then the above conditions are equivalent to the following*:

(3) $T(Z, A)$ *is finite over* A.

If $(A/\mathfrak{p})'$ (*=the derived normal domain of* $A/\mathfrak{p}$) *is finite over* $A/\mathfrak{p}$ *for every* $\mathfrak{p} \in \mathrm{Spec}(A)$, *then the conditions* (1) *and* (2) *are also equivalent to the following*:

(4) (a) *If* $\mathfrak{p}$ *is an associated prime ideal of* A *such that* A *is not reduced, then* $(A/\mathfrak{p})'$ *has no maximal ideals* $\mathfrak{m}$ *of height one such that* $\mathfrak{m} \cap (A/\mathfrak{p}) \in Z \cap \mathrm{Spec}(A/\mathfrak{p})$, *and*

(b) $(A/\mathfrak{p}, T(Z, A/\mathfrak{p}))$ *is a noetherian pair for every* $\mathfrak{p} \in \mathrm{Ass}_A(A)$.

If Z is the set of all regular maximal ideals of A, then $(A/\mathfrak{p}, T(Z, A/\mathfrak{p}))$ is a noetherian pair for every $\mathfrak{p} \in \mathrm{Ass}_A(A)$, because $A/\mathfrak{p} \subseteq T(Z, A/\mathfrak{p}) \subseteq (A/\mathfrak{p})^g$ and $(A/\mathfrak{p}, (A/\mathfrak{p})^g)$ is a noetherian pair.

Corollary. *Let* A *be a noetherian ring such that every maximal ideal of* A *is regular. Then* (A, A^g) *is a noetherian pair if and only if* $(A/\mathfrak{p})^g$ *is a finite* $A/\mathfrak{p}$*-module for every* $\mathfrak{p} \in \mathrm{Ass}_A(A)$ *such that* $A_\mathfrak{p}$ *is not reduced.*

In [1], D. D. Anderson proved that, for a noetherian ring A, if $A_\mathfrak{m}$ is reduced for every regular maximal ideal $\mathfrak{m}$ of A, then (A, A^g) is a noetherian pair. The above theorem gives us another proof of his result. In fact, let Z be the set of all regular maximal ideals of A. If $\mathfrak{p}$ is an associated prime ideal of A such that $A_\mathfrak{p}$ is not reduced, then $V(\mathfrak{p}) \cap Z = \phi$; hence $T(Z, A/\mathfrak{p}) = A/\mathfrak{p}$. This shows that the condition (2) in Theorem is satisfied. Therefore (A, A^g) is a noetherian pair.

To prove the theorem, we need several lemmas. The first one is a variance of [2], Chap. IV, (5.11.1.1).

Lemma 1. *Let* A *be a noetherian ring, and let* $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ *be the set of minimal prime ideals* $\mathfrak{p}$ *of* A *such that* $A_\mathfrak{p}$ *is not reduced. Then we have the following statements.*

(1) *There is a chain of nilpotent ideals* $M_n \supset \cdots \supset M_0 = 0$ *of* A *with the following properties*:

(a) *For each* j $(0 \leqq j < n)$ *there exists a* $\mathfrak{p}_i$ $(1 \leqq i \leqq r)$ *such that* $\mathfrak{p}_i M_{j+1} \subseteq M_j$ *and* M_{j+1}/M_j *is isomorphic to an ideal of* $A/\mathfrak{p}_i$ *as* A*-modules.*

(b) $\mathrm{Ass}_A(A) = \mathrm{Ass}_A(A/M_j)$ *for* $j = 0, \dots, n$.

(c) $(A/M_n)_\mathfrak{p}$ *is reduced for every minimal prime ideal* $\mathfrak{p}$ *of* A.

(2) *For each* $\mathfrak{p}_i$ $(1 \leqq i \leqq r)$, *there is a non-zero nilpotent ideal* N_i *of* A *such that* $\mathrm{Ass}_A(A) = \mathrm{Ass}_A(A/N_i)$, $\mathfrak{p}_i N_i = 0$ *and* N_i *is isomorphic to an ideal of* $A/\mathfrak{p}_i$ *as* A*-modules.*

Proof. (1): Let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ be an irredundant primary decomposition of (0) in A. We may assume that $\sqrt{\mathfrak{q}_i}=\mathfrak{p}_i$ for $i=1,\dots,r$. We put $\mathfrak{p}_j=\sqrt{\mathfrak{q}_j}$ for $j=r+1,\dots,m$. We may also assume that $\mathrm{Min}(A)=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$ for some n with $r\leqq n\leqq m$. We put $Q=\mathfrak{q}_{n+1}\cap\cdots\cap\mathfrak{q}_m$. For each $\mathfrak{p}_i$ $(1\leqq i\leqq r)$, there is a chain of nilpotent ideals $\mathfrak{p}_iA_{\mathfrak{p}_i}=J_{i0}\supset\cdots\supset J_{ie_i}=0$ of $A_{\mathfrak{p}_i}$ such that $\mathfrak{p}_iJ_{ij}\subseteq J_{i\,j+1}$ and $J_{ij}/J_{i\,j+1}=Q(A/\mathfrak{p}_i)$. Let I_{ij} be the inverse image of J_{ij} by the map $A\to A_{\mathfrak{p}_i}$. Then I_{ij} is a $\mathfrak{p}_i$-primary ideal of A, $\mathfrak{q}_i\subseteq I_{ij}\subseteq\mathfrak{p}_i$ and $\mathfrak{q}_i=I_{ie_i}$. Moreover $I_{ij}\supset I_{i\,j+1}$, $\mathfrak{p}_iI_{ij}\subseteq I_{i\,j+1}$ and $I_{ij}/I_{i\,j+1}$ is isomorphic to an ideal of $A/\mathfrak{p}_i$ as A-modules. We now put $M_{ij}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_{i-1}\cap I_{ij}\cap\mathfrak{p}_{i+1}\cap\cdots\cap\mathfrak{p}_n\cap Q$ $(1\leqq i\leqq r,\ 1\leqq j\leqq e_i)$. In this way we have a chain of nilpotent ideals $\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n\cap Q=M_{10}\supset\cdots\supset M_{1e_1}=M_{20}\supset\cdots\supset M_{re_r}=0$. It is now easy to see that the above chain of ideals satisfies the properties (a), (b) and (c).

(2): For each $\mathfrak{p}_i$ $(1\leqq i\leqq r)$, $N_i=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_{i-1}\cap I_{ie_i-1}\cap\mathfrak{q}_{i+1}\cap\cdots\cap\mathfrak{q}_m$ is a required nilpotent ideal.

LEMMA 2. *With the same A and Z as in Theorem, let $\mathfrak{p}$ be a minimal prime ideal of A. If N is a non-zero nilpotent ideal of A such that $\mathfrak{p}N=0$ and N is isomorphic to an ideal of $A/\mathfrak{p}$ as A-modules, then $T(Z, N)$ is isomorphic to an ideal of $T(Z, A/\mathfrak{p})$ as $T(Z, A)$-modules.*

PROOF. Let f: $N\to A/\mathfrak{p}$ be an injective homomorphism of A-modules such that $f(N)$ is an ideal of $A/\mathfrak{p}$. It is clear that $T(Z, N)$ is isomorphic to $T(Z, f(N))$ as $T(Z, A)$-modules and $T(Z, f(N))$ is an ideal of $T(Z, A/\mathfrak{p})$.

The following lemma is essentially proved in the proof of [2], Chap. IV, (5.11.2).

LEMMA 3. *Let A be a noetherian domain such that A' is finite over A. Let Z be a proper subset of* Spec(A) *which is stable under specialization. Then the following conditions on A are equivalent.*

(1) *$T(Z, A)$ is finite over A.*

(2) *If $\mathfrak{P}$ is a prime ideal of A' such that $\mathfrak{P}\cap A\in Z$, then* $\mathrm{ht}(\mathfrak{P})\geqq 2$.

PROOF. (2)$\Rightarrow$(1): Let $U=\mathrm{Spec}(A)-Z$, and let V be the set of height one prime ideals of A'. Since $\mathfrak{Q}\cap A\notin Z$ for every $\mathfrak{Q}\in V$, we have $T(Z, A)=\bigcap_{\mathfrak{p}\in U}A_{\mathfrak{p}}\subseteq\bigcap_{\mathfrak{Q}\in V}A'_{\mathfrak{Q}}=A'$.

(1)$\Rightarrow$(2): Since A' is finite over A, there is a non-zero element t of A such that $tA'\subseteq A$. It is easy to see that $tT(Z, A')\subseteq T(Z, A)$. Therefore $T(Z, A')$ is finite over A, and hence $A'=T(Z, A')$. Let $Z'=\{\mathfrak{Q}\in\mathrm{Spec}(A')\mid\mathfrak{Q}\cap A\in Z\}$. Then it is also easy to see that $T(Z, A')=T(Z', A')$. Suppose that there exists a prime ideal $\mathfrak{p}$ of A' such that $\mathrm{ht}(\mathfrak{p})=1$ and $\mathfrak{p}\in Z'$. Then there exist s $(\neq 0)$ and a in A' such that $\mathfrak{p}=sA':_{A'}a$. In particular $a/s\notin A'$. On the other hand, $a/s\in T(Z', A')=A'$ because $\mathfrak{p}\in Z'$ and $(a/s)\mathfrak{p}\subseteq A'$. This is a contradiction.

LEMMA 4. *Let A, B and C be domains with the same field of fractions such that (A, B) is a noetherian pair and C is finite over B. Then (A, C) is a noetherian pair.*

PROOF. Let R be a ring such that $A \subseteq R \subseteq C$, and let t be a non-zero element of B such that $tC \subseteq B$. Since $Q(A)=Q(B)$, we may assume that t is an element of A. Then $tR \subseteq B \cap R$, and $B \cap R$ is noetherian; hence R is a finite $B \cap R$-module. Therefore R is noetherian.

LEMMA 5. *Let (A, B) be a noetherian pair. Then every nilpotent ideal of B is a finite A-module.*

PROOF. Let J be a nilpotent ideal of B. Since $A[J]$ is noetherian, $J=\sum_{i=1}^{e} A[J]x_i=\sum_{i=1}^{e} Ax_i+J\sum_{i=1}^{e} Ax_i$. Therefore J is a finite A-module, because J is a nilpotent ideal.

We now prove the theorem: Let $(0)=\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ be an irredundant primary decomposition of (0) in A. Assume that $\mathrm{Min}(A)=\{\sqrt{\mathfrak{q}_1},\dots,\sqrt{\mathfrak{q}_s}\}$. We put $I=\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ and $J=T(Z, A) \cap IQ(A)$. It is easy to see that J is the kernel of the homomorphism $T(Z, A) \to T(Z, A/I)$. We first show that (1) is equivalent to the following:

(2′) *J is a finite A-module and $T(Z, A/\mathfrak{p})$ is finite over $A/\mathfrak{p}$ for every minimal prime ideal $\mathfrak{p}$ of A such that $A_\mathfrak{p}$ is not reduced. Moreover $(A/\mathfrak{p}, T(Z, A/\mathfrak{p}))$ is a noetherian pair for every $\mathfrak{p} \in \mathrm{Ass}_A(A)$.*

(1)⇒(2′): By Lemma 5, J is a finite A-module. Let $\mathfrak{p}$ be an associated prime ideal of A. Then $A/\mathfrak{p}$ is isomorphic to an ideal of A as A-modules; hence $T(Z, A/\mathfrak{p})$ is isomorphic to an ideal of $T(Z, A)$ as $T(Z, A)$-modules. Therefore $T(Z, A/\mathfrak{p})$ is a finite $T(Z, A)$-module. This shows that the ring homomorphism $\phi: T(Z, A) \to T(Z, A/\mathfrak{p})$ is finite, because $\phi(x)z=xz$ for every $x \in T(Z, A)$ and $z \in T(Z, A/\mathfrak{p})$. On the other hand, since $(A, T(Z, A))$ is a noetherian pair, so is $(A/\mathfrak{p}, \mathrm{Im}(\phi))$. Therefore, by Lemma 4, $(A/\mathfrak{p}, T(Z, A/\mathfrak{p}))$ is a noetherian pair. Let now $\mathfrak{p}$ be a minimal prime ideal of A such that $A_\mathfrak{p}$ is not reduced. By Lemma 1, there is a non-zero nilpotent ideal K of A such that $\mathfrak{p}K=0$, $\mathrm{Ass}_A(A)=\mathrm{Ass}_A(A/K)$ and K is isomorphic to an ideal of $A/\mathfrak{p}$. We put $K'=T(Z, A) \cap KQ(A)$ $(=T(Z, K))$. Then, by Lemma 5, K' is a finite A-module and, by Lemma 2, we may consider that K' is an ideal of $T(Z, A/\mathfrak{p})$. Since $tT(Z, A/\mathfrak{p}) \subseteq K'$ for every $t \in K'$ and $T(Z, A/\mathfrak{p}) \cong tT(Z, A/\mathfrak{p})$ if $t \neq 0$, $T(Z, A/\mathfrak{p})$ is a finite A-module.

(2′)⇒(1): It is clear that $A/I \subseteq T(Z, A)/J \subseteq T(Z, A/I)$. Let R be a ring such that $A \subseteq R \subseteq T(Z, A)$. Since $J \cap R$ is a finite A-module, it is a finitely generated nilpotent ideal of R. Thus R is noetherian if and only if so is $R/(J \cap R)$ by the theorem of Cohen. Therefore it is sufficient to show that $(A/I, T(Z, A/I))$ is a noetherian pair; hence we may assume that $\mathrm{Min}(A)=\mathrm{Ass}_A(A)$. Let $\{\mathfrak{p}_1,\dots,\mathfrak{p}_r\}$

$=\mathrm{Min}(A)$. Then the canonical embedding $A_{\mathrm{red}}\rightarrow A/\mathfrak{p}_1\times\cdots\times A/\mathfrak{p}_r$ induces an embedding $T(Z, A_{\mathrm{red}})\rightarrow T(Z, A/\mathfrak{p}_1)\times\cdots\times T(Z, A/\mathfrak{p}_r)$. Since each $(A/\mathfrak{p}_i, T(Z, A/\mathfrak{p}_i))$ is a noetherian pair, by Eakin-Nagata's theorem, $(A_{\mathrm{red}}, T(Z, A_{\mathrm{red}}))$ is also a noetherian pair. There exists a chain of nilpotent ideals $M_n\supset\cdots\supset M_0=0$ of A which satisfies the properties (a), (b) and (c) of Lemma 1 (1). We use induction on n in order to show that $(A, T(Z, A))$ is a noetherian pair. If $n=0$, then $A=A_{\mathrm{red}}$. This case is proved already. We then assume that $n\geqq 1$ and $(A/M_1, T(Z, A/M_1))$ is a noetherian pair. Let $N=T(Z, A)\cap M_1Q(A)$ $(=T(Z, M_1))$. Let $\mathfrak{p}$ be a minimal prime ideal of A such that $A_{\mathfrak{p}}$ is not reduced, $\mathfrak{p}M_1=0$ and M_1 is isomorphic to an ideal of $A/\mathfrak{p}$ as A-modules. By Lemma 2, N is isomorphic to an ideal of $T(Z, A/\mathfrak{p})$ as $T(Z, A)$-modules. Therefore N is a finite A-module. Let now R be a ring such that $A\subseteq R\subseteq T(Z, A)$. Since $(A/M_1, T(Z, A/M_1))$ is a noetherian pair and $A/M_1\subseteq R/(N\cap R)\subseteq T(Z, A/M_1)$, $R/(N\cap R)$ is noetherian. On the other hand, $N\cap R$ is a nilpotent ideal of R and is a finite A-module. Therefore every prime ideal of R is finitely generated; hence, by the theorem of Cohen, R is noetherian. This shows that $(A, T(Z, A))$ is a noetherian pair.

(2)$\Leftrightarrow$(2′): Note that $\mathrm{Ass}_A(I)=\mathrm{Ass}_A(A)-\mathrm{Min}(A)$ and $J=T(Z, I)$. Then by [2], Chap. IV (5.11.1), J is a finite A-module if and only if $T(Z, A/\mathfrak{p})$ is a finite $A/\mathfrak{p}$-module for every $\mathfrak{p}\in\mathrm{Ass}_A(I)$. Therefore the assertion is clear.

If $A_{\mathfrak{p}}$ is not reduced for every $\mathfrak{p}\in\mathrm{Ass}_A(A)$, then the equivalence between (2) and (3) follows from [2], Chap. IV (5.11.1).

If $(A/\mathfrak{p})'$ is finite over $A/\mathfrak{p}$ for every $\mathfrak{p}\in\mathrm{Spec}(A)$, then the equivalence between (2) and (4) follows easily from Lemma 3.

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