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Euler integral transformations of hypergeometric functions of two variables

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§0. Introduction

It is known that the hypergeometric functions F_1, F_2, F_3 and F_4 of two variables have the following Euler integral representations

(0.0) $$F_1(\alpha,\beta,\beta',\gamma;x,y) = C_0\int_{D_0} u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}du,$$

(0.1) $$F_1(\alpha,\beta,\beta',\gamma;x,y) = C_1\iint_{D_1} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux-vy)^{-\alpha}dudv,$$

(0.2) $$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)$$
$$= C_2\iint_{D_2} u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha}dudv,$$

(0.3) $$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)$$
$$= C_3\iint_{D_3} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux)^{-\alpha}(1-vy)^{-\alpha'}dudv,$$

(0.4) $$F_4(\alpha,\beta,\gamma,\gamma';x,y) = C_4\iint_{D_4} u^{\alpha-\gamma'}v^{\alpha-\gamma}(u+v-uv)^{\gamma+\gamma'-\alpha-2}(1-ux-vy)^{-\beta}dudv,$$

where C_j are some constants and D_j are some cycles. (0.0) is discovered by E. Picard, (0.4) is discovered by K. Aomoto (see [3]) and the others are discovered by P. Appell.

In this paper, we establish a principle of Euler integral representations and give new integral formulae for the hypergeometric functions F_2, G_1, G_2, H_1 and H_2 (Horn's notation).

THEOREM.

(0.6) $$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)$$
$$= C_6\iint_{D_6} u^{\alpha-\gamma'}v^{\alpha-\gamma}(u+v-uv)^{\gamma+\gamma'-\alpha-2}(1-ux)^{-\beta}(1-vy)^{-\beta'}dudv,$$

(0.7) $$G_1(\alpha,\beta,\beta';x,y) = C_7\int_{D_7} u^{\beta'-1}(u+1)^{-\beta-\beta'}(1-ux-\frac{y}{u})^{-\alpha}du,$$

$$(0.8)\quad G_2(\alpha,\alpha',\beta,\beta';x,y) = C_8\int_{D_8} u^{\beta'-1}(u+1)^{-\beta-\beta'}(1-ux)^{-\alpha}\left(1-\frac{y}{u}\right)^{-\alpha'}du,$$

$$(0.9)\quad H_1(\alpha,\beta,\gamma,\varepsilon;x,y) = C_9\iint_{D_9} u^{\alpha+\gamma-1}v^{\gamma-1}(uv+u-1)^{\varepsilon-\alpha-\gamma-1}(1-ux-vy)^{-\beta}dudv,$$

$$(0.10)\quad H_2(\alpha,\beta,\beta',\gamma,\varepsilon;x,y)$$

$$= C_{10}\iint_{D_{10}} u^{\alpha+\gamma-1}v^{\gamma-1}(uv+u-1)^{\varepsilon-\alpha-\gamma-1}(1-ux)^{-\beta}(1-vy)^{-\beta'}dudv,$$

where

$$C_6(=C_4) = (-1)^{\gamma-\alpha}\frac{\sin\pi(\alpha-\gamma)}{\pi}\frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(\gamma+\gamma'-\alpha-1)},$$

$$C_7 = C_8 = (-1)^{\beta'+1}\frac{\sin\pi\beta}{\pi}\Gamma(\beta)\Gamma(\beta'),$$

$$C_9 = C_{10} = \frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon-\alpha-\gamma)\Gamma(\alpha)\Gamma(\gamma)},$$

and

$$D_6(=D_4) = \{(u,v)\in\mathbf{R}^2 \mid 0\leqq u\leqq 1,\ v\leqq 0,\ u+v-uv\geqq 0\},$$

$$D_7 = D_8 = \{u\in\mathbf{R} \mid -1\leqq u\leqq 0\},$$

$$D_9 = D_{10} = \{(u,v)\in\mathbf{R}^2 \mid 0\leqq u\leqq 1,\ v\geqq 0,\ uv+u-1\leqq 0\}.$$

COROLLARY. *The system of differential equations satisfied by the function* $y^{-\alpha'}G_2(\alpha,\alpha',\beta,\beta';-x,-\frac{1}{y})$ *is identical to that of* $F_1(\alpha'+\beta',\alpha,\alpha',\alpha'-\beta+1;x,y)$.

We transform the hypergeometric differential equations, by *Euler kernels*, so that the transformed differential equations have elementary function solutions. This method, though quite simple, has not been applied to hypergeometric functions of two variables. Moreover, A. Erdélyi mentioned in [2] that "..., *by transformation of the systems of partial differential equations satisfied by the hypergeometric functions. This method, though simple in theory, is rather laborious in practice and not very useful for discovering new transformations*".

For other hypergeometric functions $G_3, H_3,\ldots, H_7$, we have not yet succeeded to discover "*elementary*" Euler integral formulae.

§1. Review

Since hypergeometric functions and equations of two variables are not very familiar, we list up the series $\sum A_{mn}x^m y^n$ and the systems of differential operators P_1, P_2 which annihilate them.

	A_{mn}	$P_1,\ P_2$
F_1	$\dfrac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)}$	$\frac{1}{x}\delta(\gamma-1+\delta+\delta')-(\alpha+\delta+\delta')(\beta+\delta)$ $\frac{1}{y}\delta'(\gamma-1+\delta+\delta')-(\alpha+\delta+\delta')(\beta'+\delta')$
F_2	$\dfrac{(\alpha,m+n)(\beta,m)(\beta',n)}{(\gamma,m)(\gamma',n)(1,m)(1,n)}$	$\frac{1}{x}\delta(\gamma-1+\delta)-(\alpha+\delta+\delta')(\beta+\delta)$ $\frac{1}{y}\delta'(\gamma'-1+\delta')-(\alpha+\delta+\delta')(\beta'+\delta')$
F_3	$\dfrac{(\alpha,m)(\alpha',n)(\beta,m)(\beta',n)}{(\gamma,m+n)(1,m)(1,n)}$	$\frac{1}{x}\delta(\gamma-1+\delta+\delta')-(\alpha+\delta)(\beta+\delta)$ $\frac{1}{y}\delta(\gamma-1+\delta+\delta')-(\alpha'+\delta')(\beta'+\delta')$
F_4	$\dfrac{(\alpha,m+n)(\beta,m+n)}{(\gamma,m)(\gamma',n)(1,m)(1,n)}$	$\frac{1}{x}\delta(\gamma-1+\delta)-(\alpha+\delta+\delta')(\beta+\delta+\delta')$ $\frac{1}{y}\delta'(\gamma'-1+\delta')-(\alpha+\delta+\delta')(\beta+\delta+\delta')$
G_1	$\dfrac{(\alpha,m+n)(\beta,n-m)(\beta',m-n)}{(1,m)(1,n)}$	$\frac{1}{x}\delta(\beta-\delta+\delta')-(\alpha+\delta+\delta')(\beta'+\delta-\delta')$ $\frac{1}{y}\delta'(\beta'+\delta-\delta')-(\alpha+\delta+\delta')(\beta-\delta+\delta')$
G_2	$\dfrac{(\alpha,m)(\alpha',n)(\beta,n-m)(\beta',m-n)}{(1,m)(1,n)}$	$\frac{1}{x}\delta(\beta-\delta+\delta')-(\alpha+\delta)(\beta'+\delta-\delta')$ $\frac{1}{y}\delta'(\beta'+\delta-\delta')-(\alpha'+\delta')(\beta-\delta+\delta')$
H_1	$\dfrac{(\alpha,m-n)(\beta,m+n)(\gamma,n)}{(\varepsilon,m)(1,m)(1,n)}$	$\frac{1}{x}\delta(\varepsilon-1+\delta)-(\alpha+\delta-\delta')(\beta+\delta+\delta')$ $\frac{1}{y}\delta'(\alpha+\delta-\delta')-(\gamma+\delta')(\beta+\delta+\delta')$
H_2	$\dfrac{(\alpha,m-n)(\beta,m)(\beta',n)(\gamma,n)}{(\varepsilon,m)(1,m)(1,n)}$	$\frac{1}{x}\delta(\varepsilon-1+\delta)-(\alpha+\delta-\delta')(\beta+\delta)$ $\frac{1}{y}\delta'(\alpha+\delta-\delta')-(\gamma+\delta')(\beta'+\delta')$

Here we used the following symbols.

$$(\alpha,\ m)=\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)},\quad \delta=x\frac{\partial}{\partial x},\quad \delta'=y\frac{\partial}{\partial y}.$$

This table reads as follows: (e.g. H_1)

$$z = H_1(\alpha,\beta,\gamma,\varepsilon;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha,m-n)(\beta,m+n)(\gamma,n)}{(\varepsilon,m)(1,m)(1,n)} x^m y^n$$

satisfies

$$P_1 z = \frac{1}{x}\delta(\varepsilon-1+\delta)z - (\alpha+\delta-\delta')(\beta+\delta+\delta')z = 0,$$

$$P_2 z = \frac{1}{y}\delta'(\alpha+\delta-\delta')z - (\gamma+\delta')(\beta+\delta+\delta')z = 0.$$

§ 2. Euler kernels

Put

$$K(\lambda) := (1-ux-vy)^{-\lambda},$$

$$K(\lambda, \lambda') := (1-ux)^{-\lambda}(1-vy)^{-\lambda'}$$

which shall be called Euler kernels of the first and the second kind. The following lemma, though obvious, is essential.

LEMMA. *Let*

$$\theta = u\frac{\partial}{\partial u}, \quad \theta' = v\frac{\partial}{\partial v}.$$

Then, for $K(\lambda)$, *we have*

(2.1) $$\delta K(\lambda) = \theta K(\lambda), \quad \delta' K(\lambda) = \theta' K(\lambda),$$

(2.2) $$(a+\delta+\delta')K(\lambda) = \lambda K(\lambda+1) + (a-\lambda)K(\lambda),$$

(2.3) $$\frac{1}{x}\delta(a+b\delta+c\delta')K(\lambda) = \lambda u(a+b+b\delta+c\delta')K(\lambda+1),$$

(2.4) $$\frac{1}{y}\delta'(a+b\delta+c\delta')K(\lambda) = \lambda v(a+c+b\delta+c\delta')K(\lambda+1),$$

and, for $K(\lambda, \lambda')$, *we have*

(2.5) $$\delta K(\lambda, \lambda') = \theta K(\lambda, \lambda'), \quad \delta' K(\lambda, \lambda') = \theta' K(\lambda, \lambda'),$$

(2.6) $$(a+\delta)K(\lambda, \lambda') = \lambda K(\lambda+1,\lambda') + (a-\lambda)K(\lambda,\lambda'),$$

(2.7) $$(a'+\delta')K(\lambda,\lambda') = \lambda' K(\lambda,\lambda'+1) + (a'-\lambda')K(\lambda,\lambda'),$$

(2.8) $$\frac{1}{x}\delta(a+b\delta+c\delta')K(\lambda,\lambda') = \lambda u(a+b+b\delta+c\delta')K(\lambda+1,\lambda'),$$

(2.9) $$\frac{1}{y}\delta'(a+b\delta+c\delta')K(\lambda,\lambda') = \lambda' v(a+c+b\delta+c\delta')K(\lambda,\lambda'+1).$$

§3. Formal transformations and their principle

Let P_1 and P_2 be one of the pairs of differential operators introduced in §1. We want to represent the solution $z=z(x, y)$ of the system

$$P_i z = 0 \qquad (i = 1,2)$$

by the integral

$$z = \iint w(u, v)K\,dudv$$

where K is the Euler kernel of the first or second kind and $w=w(u,v)$ is some elementary function of u and v. If we can differentiate under the sign of integral, then we have

$$P_i z = \iint w(u, v)P_i K\,dudv. \qquad (i = 1,2)$$

If there exist a function L and differential operators Q_i $(i=1,2)$ with respect to u and v which do not include x and y, such that

$$P_i K = Q_i L, \qquad (i = 1,2)$$

then w is determined by the system

$$Q_i^* w = 0, \qquad (i = 1,2)$$

where Q_i^* is the adjoint operator of Q_i. And if Q_i $(i=1, 2)$ are of the first order, since Q_i^* $(i=1,2)$ are also of the first order, w will be an elementary function of u and v.

We shall determine K so that a function L and operators Q_i $(i=1,2)$ of the first order exist. By virtue of the lemma in §2, we conclude that *if the right hand sides of P_1 and P_2 contain a term of the form $(a+\delta+\delta')$ in common, then we let $K=K(a)$, and if the right hand sides of P_1 and P_2 contain terms of the form $(a+\delta)$ and $(a'+\delta')$ respectively, then we let $K=K(a,a')$.* Remark that every system P_1, P_2 in §1 satisfies one or both of the above conditions.

For example, we shall apply this principle to H_1. Since the right hand sides of P_1 and P_2 have a term $(\beta+\delta+\delta')$ in common, we let $K=K(\beta)$. Then by (2.1), ..., (2.4), we have

$$Q_1 = \beta\{u(\varepsilon+\theta) - (a+\theta-\theta')\},$$

$$Q_2 = \beta\{v(\alpha-1+\theta-\theta') - (\gamma-\theta')\}$$

and $$L = K(\beta+1).$$

Since $\theta^* = -\theta-1$ and $(\theta')^* = -\theta'-1$, we have

$$Q_1^* = \beta\{(\varepsilon-\theta-1)u - (\alpha-\theta+\theta')\},$$

$$Q_2^* = \beta\{(\alpha-1-\theta+\theta')v - (\gamma-\theta'-1)\}.$$

An easy calculation shows that the system $Q_1^* w = Q_2^* w = 0$ has a solution

$$w = u^{\alpha+\gamma-1}v^{\gamma-1}(uv+u-1)^{-\alpha-\gamma+\varepsilon-1}.$$

For F_1 $(K=K(\beta,\beta'))$, G_1 $(K=K(\alpha))$ and G_2 $(K=K(\alpha,\alpha'))$, the supports of any solutions of the corresponding equations $Q_1^* w = Q_2^* w = 0$ are on the varieties $\{u=v\}$, $\{uv=1\}$ and $\{uv=1\}$ respectively. That is, in the usual sense, the system $Q_1^* w = Q_2^* w = 0$ is incompatible. In these cases, we modify the respective Euler kernels as follows:

$$K(\beta,\beta') \longrightarrow (1-ux)^{-\beta}(1-vy)^{-\beta'},$$

$$K(\alpha) \longrightarrow (1-ux-\frac{y}{u})^{-\alpha},$$

$$K(\alpha,\alpha') \longrightarrow (1-ux)^{-\alpha}(1-\frac{y}{u})^{-\alpha'}.$$

Then, if we replace the double integral by the simple integral, the argument in the beginning of this section leaves valid and moreover we have $Q_1^* \equiv Q_2^*$.

In this way, we obtained (formal) Euler integral representations for the solutions of the systems in § 1.

§ 4. Proof of the theorem

Once the formal integral representations are obtained, the definite integrals (0.1),..., (0.10) are obtained by routine method. That is, by making use of the power series expansion of the Euler kernels

$$(1-ux)^{-\lambda}(1-vy)^{-\lambda'} = \sum_{m,n=0}^{\infty} \frac{(\lambda,m)(\lambda',n)}{(1,m)(1,n)} u^m v^n x^m y^n,$$

$$(1-ux-vy)^{-\lambda} = \sum_{m,n=0}^{\infty} \frac{(\lambda,m+n)}{(1,m)(1,n)} u^m v^n x^m y^n$$

and the formulae

$$\int_0^1 u^{p-1}(1-u)^{q-1}du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

and

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin \pi x}.$$

The corollary to the theorem can be deduced from a comparison of the integrals (0.0) and (0.8), or by a transformation of dependent and independent variables in the system of G_2.

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