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Generalized Cohen-Macaulay modules

Michinori SAKAGUCHI

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Introduction

The main purpose of this paper is to establish a notion of Cohen-Macaulay modules over an arbitrary commutative ring which generalizes that of Cohen-Macaulay modules over a noetherian, commutative ring. A finite module over a noetherian ring is said to be a Cohen-Macaulay module if its depth is equal to its Krull dimension (cf. [6]). Adapting M. Hochster's approach to a theory of grade, D. G. Northcott set up the concept of polynomial grade of modules over a commutative ring in [7] which is a generalization of the notion of depth. The author showed in [8] a relation between the polynomial grade of a module and the valuative dimension of it which was defined by P. Jaffard in [5]. Namely, let A be a quasi-local ring and M a non-zero, finite A-module. Then the polynomial grade $\mathrm{Gr}(M)$ of M is equal to or less than the valuative dimension $\mathrm{Dim}\, M$ of M. This fact suggests to us giving a definition of a Cohen-Macaulay module over an arbitrary ring in terms of polynomial grade and valuative dimension.

However it seems that many nice properties of Cohen-Macaulay modules over a noetherian ring come from the following inequality: depth $M \leqq \dim A/\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ in $\mathrm{Ass}(M)$, where M is a non-zero, finite module over a noetherian local ring A. In particular it follows from this fact that a noetherian, Cohen-Macaulay ring is universally catenarian. First the author has guessed that a generalization of this inequality could be obtained. However S. Itoh has recently pointed out to the author that it does not hold in general, i.e., we can find a non-zero, finite module M over a quasi-local ring A and an attached prime ideal $\mathfrak{p}$ of M such that $\mathrm{Gr}(M) > \mathrm{Dim}(A/\mathfrak{p})$ (see Appendix). Therefore if we would define a Cohen-Macaulay module M over an arbitrary ring A by $\mathrm{Gr}(M) = \mathrm{Dim}\, M$, many nice properties of the Cohen-Macaulay modules over a noetherian ring may not be accomplished. For this reason, adding the condition that the ring $A/\mathrm{Ann}(M)$ is catenarian to the above one, we may introduce the following definition: A non-zero, finite module M over a ring A is said to be a *Cohen-Macaulay module* if $\mathrm{Dim}(M_{\mathfrak{p}})$ is finite for all $\mathfrak{p} \in \mathrm{Supp}(M)$ and $\mathrm{Gr}(M_{\mathfrak{q}}) + \mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \mathrm{Dim}\, M_{\mathfrak{p}}$ for all pairs of prime ideals $\mathfrak{p}$, $\mathfrak{q}$ in $\mathrm{Supp}(M)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ (see (4.4)). This would be a natural generalization of the notion of Cohen-Macaulay modules over a noetherian ring.

In section 1 we give the terminology and the notations which we will use in this article. Section 2 deals with the polynomial heights and includes a generalization of Theorem 2 in [8]. In section 3 we study the polynomial grade and the valuative dimension of the module $M[X]$. In section 4 we introduce the motion of Cohen-Macaulay modules over an arbitrary ring and basic facts about these modules are established. Some examples of Cohen-Macaulay rings in our wider sense are presented in section 5.

1. Terminology

Throughout this paper, all rings are assumed to be commutative with identity, and all modules are assumed to be unitary. If A is a ring and $\mathfrak{p}$ is a prime ideal of A, then $\mathrm{ht}(\mathfrak{p})$ stands for the height of $\mathfrak{p}$ and $\dim A$ for the Krull dimension of A. If $X_1,\dots, X_n$ are indeterminates over A, then $\mathfrak{p}[X_1,\dots, X_n]$ is a prime ideal of the polynomial ring $A[X_1,\dots, X_n]$. The limit of the sequence $\{\mathrm{ht}(\mathfrak{p}[X_1, \dots, X_n])\}$ $(n=0, 1,\dots)$ is called the polynomial height of $\mathfrak{p}$ and is denoted by $\mathrm{Ht}(\mathfrak{p})$ (see [8]).

If A is an integral domain, the valuative dimension of A, denoted by $\mathrm{Dim}\, A$, is defined to be $\mathrm{Sup}\,\{\dim V \mid V$ is a valuation overring of $A\}$, and more generally the valuative dimension of a ring A is defined to be $\mathrm{Sup}_{\mathfrak{p}\in\mathrm{Spec}(A)}\{\mathrm{Dim}\,(A/\mathfrak{p})\}$ (see [5]). If A is a ring and M is a non-zero A-module, then $\mathrm{Ann}\,(M)$ denotes the set of annihilators of M, and by the valuative dimension of M we understand the valuative dimension of the ring $A/\mathrm{Ann}\,(M)$. The valuative dimension of the module M is denoted by $\mathrm{Dim}\, M$ or $\mathrm{Dim}\,(M)$.

Let $\mathfrak{a}$ be an ideal of a ring A and M an A-module. Then we denote by $M[X_1,\dots, X_n]$ the $A[X_1,\dots, X_n]$-module $M\otimes_A A[X_1,\dots, X_n]$. A sequence $\{a_1, a_2,\dots, a_m\}$ of m elements of $\mathfrak{a}$ is called an A-sequence on M composed of elements of $\mathfrak{a}$ if the sequence

$$0 \longrightarrow M/(a_1, a_2,\dots, a_{i-1})M \xrightarrow{a_i} M/(a_1, a_2,\dots, a_{i-1})M$$

is exact for each i, $1\leqq i\leqq m$. The upper bound of the lengths of all such A-sequences on M is called the classical grade of $\mathfrak{a}$ on M and it is denoted by $\mathrm{gr}_A\{\mathfrak{a}; M\}$. Furthermore the limit of the sequence $\{\mathrm{gr}_{A[X_1,\dots,X_n]}\{\mathfrak{a}[X_1,\dots, X_n]; M[X_1,\dots, X_n]\}\}$ $(n=0, 1,\dots)$ is called the polynomial grade of $\mathfrak{a}$ on M and is denoted by $\mathrm{Gr}_A\{\mathfrak{a}; M\}$. A prime ideal $\mathfrak{p}$ of A is said to be attached to the zero submodule of M if $\mathrm{Gr}_{A_\mathfrak{p}}\{\mathfrak{p}A_\mathfrak{p}; M\}=0$ and the set of prime ideals attached to the zero submodule of M is denoted by $\mathrm{Att}\,(M)$ (see [7]).

If M is an A-module, we denote by $\mathrm{Supp}\,(M)$ the support of M, which is the set of prime ideals $\mathfrak{p}$ of A such that $M_\mathfrak{p}\neq 0$. $\mathrm{Min}\,(M)$ means the set of prime ideals which are minimal prime ideals of the ideal $\mathrm{Ann}\,(M)$. To simplify the notation, we write $\mathrm{Gr}\,(M_\mathfrak{p})$ in place of $\mathrm{Gr}_{A_\mathfrak{p}}\{\mathfrak{p}A_\mathfrak{p}; M_\mathfrak{p}\}$ and $\mathrm{Dim}\,(M_\mathfrak{p})$ stands for the valuative dimension of $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal in $\mathrm{Supp}(M)$.

2. Polynomial heights

The following theorem plays an important role in our theory.

(2.1) (J. Brewer, P. Montgomery, E. Rutter and W. Heinzer [2], Theorem 1) *Let A be a ring and $\mathfrak{P}$ a prime ideal of $A[X_1,\dots, X_n]$ with $\mathfrak{P}\cap A=\mathfrak{p}$. Then we have* $\mathrm{ht}(\mathfrak{P})=\mathrm{ht}(\mathfrak{p}[X_1,\dots, X_n])+\mathrm{ht}(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n])$.

We have a similar equality for polynomial heights.

(2.2) Theorem. *Let A be a ring and $\mathfrak{P}$ a prime ideal of $A[X_1,\dots, X_n]$ with $\mathfrak{P}\cap A=\mathfrak{p}$. Then* $\mathrm{Ht}(\mathfrak{P})=\mathrm{Ht}(\mathfrak{p})+\mathrm{Ht}(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n])$.

Proof. Let $Y_1,\dots, Y_m$ be indeterminates over $A[X_1,\dots, X_n]$. Then $\mathfrak{P}[Y_1,\dots, Y_m]$ is a prime ideal of $A[X_1,\dots, X_n, Y_1,\dots, Y_m]$ and we see $\mathfrak{P}[Y_1,\dots, Y_m]\cap A[Y_1,\dots, Y_m]=\mathfrak{p}[Y_1,\dots, Y_m]$. Hence, by (2.1) and the fact that the ring $A[X_1,\dots, X_n, Y_1,\dots, Y_m]/\mathfrak{p}[X_1,\dots, X_n, Y_1,\dots, Y_m]$ is isomorphic to $(A[X_1,\dots, X_n]/\mathfrak{p}[X_1,\dots, X_n])[Y_1,\dots, Y_m]$, we obtain

$$\begin{aligned}\mathrm{ht}(\mathfrak{P}[Y_1,\dots, Y_m]) &= \mathrm{ht}(\mathfrak{p}[Y_1,\dots, Y_m, X_1,\dots, X_n]) + \\ &\quad \mathrm{ht}(\mathfrak{P}[Y_1,\dots, Y_m]/\mathfrak{p}[Y_1,\dots, Y_m, X_1,\dots, X_n]) \\ &= \mathrm{ht}(\mathfrak{p}[X_1,\dots, X_n][Y_1,\dots, Y_m]) + \\ &\quad \mathrm{ht}((\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n])[Y_1,\dots, Y_m]).\end{aligned}$$

Therefore letting m tend to infinity, it follows from the definition of polynomial height that

$$\begin{aligned}\mathrm{Ht}(\mathfrak{P}) &= \mathrm{Ht}(\mathfrak{p}[X_1,\dots, X_n]) + \mathrm{Ht}(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n]) \\ &= \mathrm{Ht}(\mathfrak{p}) + \mathrm{Ht}(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n]). \qquad q.e.d.\end{aligned}$$

(2.3) Corollary. *Let the assumptions be as in* (2.2). *Then* $\mathrm{Ht}(\mathfrak{P})=\mathrm{Ht}(\mathfrak{p})+\mathrm{ht}(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n])$.

Proof. By (2.2), it is sufficient to show that $\mathrm{ht}(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n])=\mathrm{Ht}(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n])$. Put $S=A-\mathfrak{p}$. Then $A[X_1,\dots, X_n]_S$ is isomorphic to $A_{\mathfrak{p}}[X_1,\dots, X_n]$. Since the height and the polynomial height are not changed by any localization, we have $\mathrm{ht}(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n])=\mathrm{ht}(\mathfrak{P}_S/\mathfrak{p}[X_1,\dots, X_n]_S)$ and $\mathrm{Ht}(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n])=\mathrm{Ht}(\mathfrak{P}_S/\mathfrak{p}[X_1,\dots, X_n]_S)$. It is clear that $\mathfrak{P}_S/\mathfrak{p}[X_1,\dots, X_n]_S$ is a prime ideal of a noetherian ring $(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})[X_1,\dots, X_n]$. Accordingly it follows from Prop. 1, (7) of [8] that $\mathrm{ht}(\mathfrak{P}_S/\mathfrak{p}[X_1,\dots, X_n]_S)=\mathrm{Ht}(\mathfrak{P}_S/\mathfrak{p}[X_1,\dots,$

$X_n]_S)$. We may therefore conclude that ht $(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n]) = \mathrm{Ht}\,(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n])$. *q.e.d.*

(2.4) REMARK. We see the following statements by the proof of (2.3). Let A be a ring and $\mathfrak{P}$ a prime ideal of $A[X_1,\dots, X_n]$. Put $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\mathrm{Ht}\,(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n]) = \mathrm{ht}\,(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n]) \leqq n$. In particular if $n=1$ and $\mathfrak{P} \supsetneqq \mathfrak{p}[X_1]$, then $\mathrm{Ht}\,(\mathfrak{P}/\mathfrak{p}[X_1]) = \mathrm{ht}\,(\mathfrak{P}/\mathfrak{p}[X_1]) = 1$.

(2.5) COROLLARY. *Let the situation be as in the statement of* (2.2) *and suppose that* $\mathrm{ht}\,(\mathfrak{p}) = \mathrm{Ht}\,(\mathfrak{p})$. *Then* $\mathrm{ht}\,(\mathfrak{P}) = \mathrm{Ht}\,(\mathfrak{P})$.

PROOF. By the definition of polynomial height, we see easily that $\mathrm{ht}\,(\mathfrak{p}) \leqq \mathrm{ht}\,(\mathfrak{p}[X_1,\dots, X_n]) \leqq \mathrm{Ht}\,(\mathfrak{p})$. Hence, by the assumption, $\mathrm{ht}\,(\mathfrak{p}[X_1,\dots, X_n]) = \mathrm{Ht}\,(\mathfrak{p})$. It therefore follows from (2.1) and (2.3) that $\mathrm{ht}\,(\mathfrak{P}) = \mathrm{ht}\,(\mathfrak{p}[X_1,\dots, X_n]) + \mathrm{ht}\,(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n]) = \mathrm{Ht}\,(\mathfrak{p}) + \mathrm{ht}\,(\mathfrak{P}/\mathfrak{p}[X_1,\dots, X_n]) = \mathrm{Ht}\,(\mathfrak{P})$. *q.e.d.*

To show Theorem 2.7 which is a generalization of Theorem 1 of [8], we need the following

(2.6) LEMMA. *Let A be a quasi-local domain and* $\mathfrak{m}$ *the maximal ideal of A. Suppose that* $\mathrm{Ht}\,(\mathfrak{m})$ *is finite. Then* $\mathrm{Dim}\, A = \mathrm{Ht}\,(\mathfrak{m})$.

PROOF. Assume that $\mathrm{Ht}\,(\mathfrak{m}) = k$ where k is a non-negative integer. Then, by the fact that $\mathrm{Dim}\, A = k$ if and only if $\dim A[X_1,\dots, X_k] = 2k$ ([1], Theorem 6), we have only to show that $\dim A[X_1,\dots, X_k] = 2k$. Put $\mathfrak{P} = (\mathfrak{m}[X_1,\dots, X_k], X_1,\dots, X_k)$. Then, since $\mathrm{ht}\,(\mathfrak{m}[X_1,\dots, X_k]) = k$ by Prop. 2 of [8], it follows from (2.1) that $\mathrm{ht}\,(\mathfrak{P}) = 2k$. Therefore $\dim A[X_1,\dots, X_k] \geqq 2k$.

Suppose next that $\mathfrak{Q}$ is a prime ideal of $A[X_1,\dots, X_k]$. Let $\mathfrak{q} = A \cap \mathfrak{Q}$. Then $\mathfrak{q} \subseteq \mathfrak{m}$. Hence, by Prop. 1, (3) of [8], $\mathrm{ht}\,(\mathfrak{q}[X_1,\dots, X_k]) \leqq \mathrm{Ht}\,(\mathfrak{q}) \leqq \mathrm{Ht}\,(\mathfrak{m})$. This shows that $\mathrm{ht}(\mathfrak{q}[X_1,\dots, X_k]) \leqq k$. However we see $\mathrm{ht}(\mathfrak{Q}/\mathfrak{q}[X_1,\dots, X_k]) \leqq k$ by (2.4). Consequently, by (2.1), we obtain that $\mathrm{ht}\,(\mathfrak{Q}) = \mathrm{ht}\,(\mathfrak{q}[X_1,\dots, X_k]) + \mathrm{ht}\,(\mathfrak{Q}/\mathfrak{q}[X_1,\dots, X_k]) \leqq 2k$. It thus follows that $\dim A[X_1,\dots, X_k] \leqq 2k$. Accordingly we establish the equation $\dim A[X_1,\dots, X_k] = 2k$. *q.e.d.*

(2.7) THEOREM. *Let A be a ring. Then we have* $\mathrm{Dim}\, A = \mathrm{Sup}\,\{\mathrm{Ht}\,(\mathfrak{p})\}$, *where the supremum is taken over all the prime ideals* $\mathfrak{p}$ *of A.*

PROOF. By virtue of Theorem 1 of [8], it is sufficient to show that if $\mathrm{Dim}\, A = \infty$, then $\mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\mathrm{Ht}\,(\mathfrak{p})\} = \infty$. For this purpose, we have to prove that if $\mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\mathrm{Ht}\,(\mathfrak{p})\} \leqq k$, then $\mathrm{Dim}\, A \leqq k$ for each non-negative integer k. Assume that $\mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\mathrm{Ht}\,(\mathfrak{p})\} \leqq k$. Suppose first that A is an integral domain. Then $\mathrm{Ht}\,(\mathfrak{p}) \leqq k$, where $\mathfrak{p}$ is a prime ideal of A. Hence, by Prop. 1, (5) of [8], $\mathrm{Ht}\,(\mathfrak{p}A_{\mathfrak{p}}) \leqq k$. Thus, it follows from (2.6) that $\mathrm{Dim}\, A_{\mathfrak{p}} \leqq k$. However, by the definition of valuative dimension, we see that $\mathrm{Dim}\, A =$

$\mathrm{Sup}_{\mathfrak{p}\in\mathrm{Spec}(A)}\{\mathrm{Dim}\, A_{\mathfrak{p}}\}$. Therefore we have $\mathrm{Dim}\, A \leqq k$.

Next we proceed to general case. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A with $\mathfrak{q}\subseteq\mathfrak{p}$. Then, by Prop. 1, (4) of [8], we have $\mathrm{Ht}(\mathfrak{p}/\mathfrak{q})\leqq \mathrm{Ht}(\mathfrak{p})$. Since $\mathrm{Ht}(\mathfrak{p})\leqq k$, we can conclude $\mathrm{Ht}(\mathfrak{p}/\mathfrak{q})\leqq k$. Thus $\mathrm{Sup}\{\mathrm{Ht}(\mathfrak{p}/\mathfrak{q})\}\leqq k$, where $\mathfrak{p}/\mathfrak{q}$ runs over all prime ideals of $A/\mathfrak{q}$. Accordingly $\mathrm{Dim}\, A = \mathrm{Sup}_{\mathfrak{q}\in\mathrm{Spec}(A)}\{\mathrm{Dim}\,(A/\mathfrak{q})\}\leqq k$. *q.e.d.*

(2.8) PROPOSITION. *Let A be a ring, and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A with $\mathfrak{q}\subsetneqq\mathfrak{p}$. Then* $\mathrm{Dim}\, A/\mathfrak{q}\geqq \mathrm{Dim}\, A/\mathfrak{p}+1$.

PROOF. Let $\mathfrak{m}$ be a prime ideal of A such that $\mathfrak{p}\subseteq\mathfrak{m}$. Then $\mathfrak{q}[X_1,\dots,X_n]\subsetneqq \mathfrak{p}[X_1,\dots,X_n]\subseteq\mathfrak{m}[X_1,\dots,X_n]$ in $A[X_1,\dots,X_n]$. Therefore

$$\begin{aligned}\mathrm{ht}\,((\mathfrak{m}/\mathfrak{q})[X_1,\dots,X_n]) &= \mathrm{ht}\,(\mathfrak{m}[X_1,\dots,X_n]/\mathfrak{q}[X_1,\dots,X_n])\\ &\geqq \mathrm{ht}\,(\mathfrak{m}[X_1,\dots,X_n]/\mathfrak{p}[X_1,\dots,X_n])+1\\ &= \mathrm{ht}\,((\mathfrak{m}/\mathfrak{p})[X_1,\dots,X_n])+1.\end{aligned}$$

Let n tend to infinity. Then we see $\mathrm{Ht}\,(\mathfrak{m}/\mathfrak{q})\geqq \mathrm{Ht}\,(\mathfrak{m}/\mathfrak{p})+1$. Denote by $\mathrm{V}(\mathfrak{p})$ the set of prime ideals $\mathfrak{m}$ of A such that $\mathfrak{m}\supseteq\mathfrak{p}$. It follows from (2.7) that

$$\mathrm{Dim}\, A/\mathfrak{q}\geqq \mathrm{Sup}_{\mathfrak{m}\in\mathrm{V}(\mathfrak{p})}\{\mathrm{Ht}\,(\mathfrak{m}/\mathfrak{q})\}\geqq \mathrm{Sup}_{\mathfrak{m}\in\mathrm{V}(\mathfrak{p})}\{\mathrm{Ht}\,(\mathfrak{m}/\mathfrak{p})\}+1=\mathrm{Dim}\, A/\mathfrak{p}+1.$$

q.e.d.

3. Polynomial grade of $M[X]$

The following (3.2) and (3.5) are due to S. Itoh but the proofs given here are slightly different from his original ones.

(3.1) (D. G. Northcott [7], Lemma 8 of Chapter 5) *Let A be a ring, $\mathfrak{p}$ a prime ideal of A and M an A-module. Then the following statements are equivalent*:

(i) *$\mathfrak{p}$ is attached to the zero submodule of M.*

(ii) *If $\mathfrak{a}$ is a finitely generated ideal contained in $\mathfrak{p}A_{\mathfrak{p}}$, then there exists a non-zero element m of $M_{\mathfrak{p}}$ such that $\mathfrak{a}m=0$.*

(3.2) LEMMA. *Let A be a quasi-local ring with the maximal ideal $\mathfrak{m}$ and M an A-module. Furthermore let $\mathfrak{P}$ be a prime ideal of $A[X]$ such that* $\mathfrak{P}\cap A=\mathfrak{m}$ and $\mathfrak{m}[X]\subsetneqq\mathfrak{P}$. *If* $\mathrm{Gr}\,(M)=0$, *then* $\mathrm{Gr}\,(M[X]_{\mathfrak{P}})=1$.

PROOF. The assumptions concerning A and $\mathfrak{P}$ ensure that $\mathfrak{P}=(\mathfrak{m}[X], f)$, where f is a monic polynomial of positive degree. Since the ideal generated by the coefficients of f is A, it follows from Theorem 7 of Chapter 5 in [7] that f is an $M[X]$-regular element. Thus we have an exact sequence

$$0 \longrightarrow M[X] \xrightarrow{f} M[X] \longrightarrow N \longrightarrow 0$$

where $N=M[X]/fM[X]$. We shall now show $\mathfrak{P}\in \mathrm{Att}_{A[X]}(N)$. Suppose that $\mathfrak{b}$ is a finitely generated ideal such that $\mathfrak{b}\subseteq\mathfrak{P}$. Then we may write $\mathfrak{b}=(a_1+b_1f, a_2+b_2f,\dots, a_n+b_nf)$, where $a_i\in\mathfrak{m}[X]$ and $b_i\in A[X]$ for $1\leqq i\leqq n$. Let $\mathfrak{a}$ be the ideal of A which is generated by the coefficients of $a_1, a_2,\dots, a_n$. Then $\mathfrak{a}$ is a finitely generated ideal contained in $\mathfrak{m}$. Hence, by the assumption that $\mathrm{Gr}(M)=0$ and (3.1), there exists a non-zero element m of M such that $\mathfrak{a}m=0$. We let $\bar{m}$ denote the image of m under the natural mapping $M\to N$. Then $\mathfrak{a}\bar{m}=0$. This shows $\mathfrak{b}\bar{m}=0$, whence we see that $\mathfrak{b}A[X]_{\mathfrak{P}}(\bar{m}/1)=0$ in $N_{\mathfrak{P}}$. On the other hand, since f is a monic polynomial of positive degree, m is not in $fM[X]$, and hence $\bar{m}\neq 0$. Next we shall show that any element of $A[X]$ which is not contained in $\mathfrak{P}$ does not annihilate the element $\bar{m}$. For this purpose we assume that g is an element of $A[X]$ such that $g\bar{m}=0$. Then $gm=fe$, where $e\in M[X]$. Since f is a monic polynomial, we find an element h of $A[X]$ such that $e=hm$. Hence $gm=fhm$, and so $(g-fh)m=0$. By the same theorem, $g-fh\in\mathfrak{m}[X]$ because $m\neq 0$. It thus follows that $g\in(\mathfrak{m}[X], f)=\mathfrak{P}$. Therefore we see that any element of $A[X]-\mathfrak{P}$ does not annihilate $\bar{m}$. Consequently $\bar{m}/1\neq 0$ in $N_{\mathfrak{P}}$. We may conclude that $\mathfrak{P}\in\mathrm{Att}_{A[X]}(N)$ by (3.1).

Now localizing the above sequence at $\mathfrak{P}$, we obtain the exact sequence

$$0 \longrightarrow M[X]_{\mathfrak{P}} \xrightarrow{f} M[X]_{\mathfrak{P}} \longrightarrow N_{\mathfrak{P}} \longrightarrow 0.$$

Accordingly, by Theorem 15 of Chapter 5 in [7], we establish that $\mathrm{Gr}(M[X]_{\mathfrak{P}})=\mathrm{Gr}(N_{\mathfrak{P}})+1=1$. *q.e.d.*

(3.3) (M. Hochster [4], Cor. 1 to Prop. 2) *Let $A\to B$ be a homomorphism of rings, $\mathfrak{a}$ an ideal of A and M an A-module. Suppose that B is a faithfully flat A-module. Then* $\mathrm{Gr}_A\{\mathfrak{a}; M\}=\mathrm{Gr}_B\{\mathfrak{a}B; M\otimes_A B\}$.

(3.4) Lemma. *Let A be a ring, $\mathfrak{p}$ a prime ideal of A and M an A-module. Then* $\mathrm{Gr}(M_{\mathfrak{p}})=\mathrm{Gr}(M[X_1,\dots,X_n]_{\mathfrak{p}[X_1,\dots,X_n]})$.

Proof. Since $A[X_1,\dots,X_n]_{\mathfrak{p}[X_1,\dots,X_n]}$ is a faithfully flat $A_{\mathfrak{p}}$-module, our lemma follows from (3.3). *q.e.d.*

(3.5) Theorem. *Let A be a ring, M an A-module and $\mathfrak{P}$ a prime ideal of $A[X]$. In addition, let $\mathfrak{p}$ denote $A\cap\mathfrak{P}$ and assume $\mathfrak{p}[X]\subsetneqq\mathfrak{P}$. Then* $\mathrm{Gr}(M[X]_{\mathfrak{P}})=\mathrm{Gr}(M_{\mathfrak{p}})+1$.

Proof. Without loss of generality we may suppose that A is a quasi-local ring and $\mathfrak{p}$ is the maximal ideal of A. Let S denote the complement of $\mathfrak{P}$ in $A[X]$. Then we see easily that for every non-negative integer n,

$$\begin{aligned}
&\mathrm{gr}_{A[Y_1,\dots,Y_n]}\{\mathfrak{p}[Y_1,\dots,Y_n];\ M[Y_1,\dots,Y_n]\}\\
&\quad\leqq \mathrm{gr}_{A[Y_1,\dots,Y_n,X]}\{\mathfrak{p}[Y_1,\dots,Y_n,X];\ M[Y_1,\dots,Y_n,X]\}\\
&\quad\leqq \mathrm{gr}_{A[X][Y_1,\dots,Y_n]}\{\mathfrak{P}[Y_1,\dots,Y_n];\ M[X][Y_1,\dots,Y_n]\}\\
&\quad\leqq \mathrm{gr}_{A[X][Y_1,\dots,Y_n]_S}\{\mathfrak{P}[Y_1,\dots,Y_n]_S;\ M[X][Y_1,\dots,Y_n]_S\}\\
&\quad= \mathrm{gr}_{A[X]_{\mathfrak{P}}[Y_1,\dots,Y_n]}\{\mathfrak{P}A[X]_{\mathfrak{P}}[Y_1,\dots,Y_n];\ M[X]_{\mathfrak{P}}[Y_1,\dots,Y_n]\}.
\end{aligned}$$

Let n tend to infinity. Then it follows that $\mathrm{Gr}(M) \leqq \mathrm{Gr}(M[X]_{\mathfrak{P}})$. We may therefore assume that $\mathrm{Gr}(M)$ is finite.

Now suppose $\mathrm{Gr}(M)=k$. Then, by the definition of polynomial grade, there exists a non-negative integer m such that $\mathrm{gr}_{A[Y_1,\dots,Y_m]}\{\mathfrak{p}[Y_1,\dots,Y_m];\ M[Y_1,\dots,Y_m]\}=k$. Put $A'=A[Y_1,\dots,Y_m]_{\mathfrak{p}[Y_1,\dots,Y_m]}$, $\mathfrak{p}'=\mathfrak{p}[Y_1,\dots,Y_m]_{\mathfrak{p}[Y_1,\dots,Y_m]}$ and $M'=M[Y_1,\dots,Y_m]_{\mathfrak{p}[Y_1,\dots,Y_m]}$. Furthermore let $\mathfrak{P}'$ denote $\mathfrak{P}A'[X]$. Then we see that $\mathfrak{P}'$ is a prime ideal of $A'[X]$, and we can show that $\mathfrak{P}' \cap A' = \mathfrak{p}'$ and $\mathfrak{p}'[X] \subsetneqq \mathfrak{P}'$. Since $A'[X]_{\mathfrak{P}'}$ is a faithfully flat $A[X]_{\mathfrak{P}}$-module, it follows from (3.3) and (3.4) that $\mathrm{Gr}(M')=\mathrm{Gr}(M)$ and $\mathrm{Gr}(M'[X]_{\mathfrak{P}'})=\mathrm{Gr}(M[X]_{\mathfrak{P}})$. Thus, by the choice of m, $k=\mathrm{gr}_{A[Y_1,\dots,Y_m]}\{\mathfrak{p}[Y_1,\dots,Y_m];\ M[Y_1,\dots,Y_m]\} \leqq \mathrm{gr}_{A'}\{\mathfrak{p}';\ M'\} \leqq \mathrm{Gr}(M')=k$. Hence $\mathrm{gr}_{A'}\{\mathfrak{p}';\ M'\}=\mathrm{Gr}(M')$. Consequently we may assume that $\mathrm{gr}_A\{\mathfrak{p};\ M\}=\mathrm{Gr}(M)=k$. We can therefore find a sequence $\{a_1, a_2,\dots, a_k\}$ which is an A-sequence on M composed of elements of $\mathfrak{p}$. Put $N=M/(a_1, a_2,\dots, a_k)M$. Accordingly, by Theorem 15 of Chapter 5 in [7], $\mathrm{Gr}(N)=0$. Hence, by (3.2), $\mathrm{Gr}(N[X]_{\mathfrak{P}})=1$. Since $\{a_1, a_2,\dots, a_k\}$ is also an $A[X]_{\mathfrak{P}}$-sequence on $M[X]_{\mathfrak{P}}$ composed of elements of $\mathfrak{P}A[X]_{\mathfrak{P}}$ and we have $N[X]_{\mathfrak{P}}=M[X]_{\mathfrak{P}}/(a_1, a_2,\dots, a_k)\cdot M[X]_{\mathfrak{P}}$, we conclude $\mathrm{Gr}(M[X]_{\mathfrak{P}})=k+1$ by the same theorem. *q.e.d.*

(3.6) Lemma. *Let A be a ring and M a non-zero, finite A-module. If $\mathfrak{p} \in \mathrm{Supp}(M)$, then $\mathrm{Dim}(M_{\mathfrak{p}})=\mathrm{Ht}(\mathfrak{p}/\mathrm{Ann}(M))$.*

Proof. Since M is a finite A-module, $\mathrm{Ann}(M_{\mathfrak{p}})=\mathrm{Ann}(M)_{\mathfrak{p}}$. Thus, by the definition of valuative dimension of M, we see that

$$\begin{aligned}
\mathrm{Dim}(M_{\mathfrak{p}}) &= \mathrm{Dim}(A_{\mathfrak{p}}/\mathrm{Ann}(M_{\mathfrak{p}})) = \mathrm{Dim}(A_{\mathfrak{p}}/\mathrm{Ann}(M)_{\mathfrak{p}})\\
&= \mathrm{Dim}((A/\mathrm{Ann}(M))_{\mathfrak{p}/\mathrm{Ann}(M)}).
\end{aligned}$$

On the other hand, by (2.7) and Prop. 1, (5) of [8],

$$\begin{aligned}
\mathrm{Dim}((A/\mathrm{Ann}(M))_{\mathfrak{p}/\mathrm{Ann}(M)} &= \mathrm{Ht}((\mathfrak{p}/\mathrm{Ann}(M))(A/\mathrm{Ann}(M))_{\mathfrak{p}/\mathrm{Ann}(M)})\\
&= \mathrm{Ht}(\mathfrak{p}/\mathrm{Ann}(M)).
\end{aligned}$$

This yields $\mathrm{Dim}(M_{\mathfrak{p}})=\mathrm{Ht}(\mathfrak{p}/\mathrm{Ann}(M))$.

(3.7) Theorem. *Let A be a ring and M a non-zero, finite A-module.*

Suppose $\mathfrak{p} \in \operatorname{Supp}(M)$ *satisfing* $\operatorname{Gr}(M_{\mathfrak{p}}) = \operatorname{Dim}(M_{\mathfrak{p}})$. *If* $\mathfrak{P}$ *is a prime ideal of* $A[X]$ *such that* $\mathfrak{P} \cap A = \mathfrak{p}$, *then* $\mathfrak{P} \in \operatorname{Supp}(M[X])$ *and* $\operatorname{Gr}(M[X]_{\mathfrak{P}}) = \operatorname{Dim}(M[X]_{\mathfrak{P}})$.

PROOF. We begin by noting that $\mathfrak{P} \supseteq \mathfrak{p}[X]$. Put $\mathfrak{a} = \operatorname{Ann}(M)$. Then it is clear that $\operatorname{Ann}(M[X]) = \mathfrak{a}[X]$. Since $\mathfrak{p} \in \operatorname{Supp}(M)$ if and only if $\mathfrak{p} \supseteq \mathfrak{a}$, we see $\mathfrak{P} \supseteq \mathfrak{p}[X] \supseteq \mathfrak{a}[X]$, and hence $\mathfrak{P} \in \operatorname{Supp}(M[X])$. First suppose $\mathfrak{P} = \mathfrak{p}[X]$. Then it follows from (3.4) that $\operatorname{Gr}(M[X]_{\mathfrak{P}}) = \operatorname{Gr}(M_{\mathfrak{p}})$. Moreover, by (3.6), we have $\operatorname{Dim}(M[X]_{\mathfrak{P}}) = \operatorname{Ht}(\mathfrak{p}[X]/\mathfrak{a}[X]) = \operatorname{Ht}((\mathfrak{p}/\mathfrak{a})[X]) = \operatorname{Ht}(\mathfrak{p}/\mathfrak{a}) = \operatorname{Dim}(M_{\mathfrak{p}})$. Thus we conclude in this case that $\operatorname{Gr}(M[X]_{\mathfrak{P}}) = \operatorname{Dim}(M[X]_{\mathfrak{P}})$.

Next assume that $\mathfrak{P} \supsetneqq \mathfrak{p}[X]$. Then (3.5) shows that $\operatorname{Gr}(M[X]_{\mathfrak{P}}) = \operatorname{Gr}(M_{\mathfrak{p}}) + 1$ and it follows from (3.6) that $\operatorname{Dim}(M[X]_{\mathfrak{P}}) = \operatorname{Ht}(\mathfrak{P}/\mathfrak{a}[X])$. Let $\mathfrak{P}'$ be the prime ideal of $(A/\mathfrak{a})[X]$ which corresponds to $\mathfrak{P}/\mathfrak{a}[X]$ under the natural isomorphism: $A[X]/\mathfrak{a}[X] \to (A/\mathfrak{a})[X]$. Then $\mathfrak{P}' \cap A/\mathfrak{a} = \mathfrak{p}/\mathfrak{a}$ and $\mathfrak{P}' \supsetneqq (\mathfrak{p}/\mathfrak{a})[X]$. Accordingly, by (2.3) and (2.4), we obtain $\operatorname{Ht}(\mathfrak{P}/\mathfrak{a}[X]) = \operatorname{Ht}(\mathfrak{P}') = \operatorname{Ht}(\mathfrak{p}/\mathfrak{a}) + \operatorname{ht}(\mathfrak{P}'/(\mathfrak{p}/\mathfrak{a})[X]) = \operatorname{Ht}(\mathfrak{p}/\mathfrak{a}) + 1$. Therefore, from (3.6), we deduce that $\operatorname{Dim}(M[X]_{\mathfrak{P}}) = \operatorname{Dim}(M_{\mathfrak{p}}) + 1$. Thus these observations show that $\operatorname{Gr}(M[X]_{\mathfrak{P}}) = \operatorname{Dim}(M[X]_{\mathfrak{P}})$. *q.e.d.*

(3.8) COROLLARY. *Let A be a ring and M a non-zero, finite A-module such that* $\operatorname{Gr}(M_{\mathfrak{p}}) = \operatorname{Dim}(M_{\mathfrak{p}})$ *for all prime ideals* $\mathfrak{p}$ *in* $\operatorname{Supp}(M)$. *Then* $\operatorname{Gr}(M[X_1, \dots, X_n]_{\mathfrak{P}}) = \operatorname{Dim}(M[X_1, \dots, X_n]_{\mathfrak{P}})$ *for all prime ideals* $\mathfrak{P}$ *in* $\operatorname{Supp}(M[X_1, \dots, X_n])$.

PROOF. This corollary is an immediate consequence of (3.7). *q.e.d.*

4. Cohen-Macaulay modules

(4.1) LEMMA. *Let A be a ring, and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A with* $\mathfrak{q} \subseteq \mathfrak{p}$. *Then* $\operatorname{Ht}(\mathfrak{q}) + \operatorname{Ht}(\mathfrak{p}/\mathfrak{q}) \leqq \operatorname{Ht}(\mathfrak{p})$.

PROOF. This lemma follows from the fact that, for each non-negative integer n, $\operatorname{ht}(\mathfrak{q}[X_1, \dots, X_n]) + \operatorname{ht}(\mathfrak{p}[X_1, \dots, X_n]/\mathfrak{q}[X_1, \dots, X_n]) \leqq \operatorname{ht}(\mathfrak{p}[X_1, \dots, X_n])$. *q.e.d.*

(4.2) LEMMA. *Let A be a ring and M a non-zero, finite A-module. Assume that* $\mathfrak{p} \in \operatorname{Supp}(M)$. *Then* $\mathfrak{p} \in \operatorname{Min}(M)$ *if and only if* $\operatorname{Dim}(M_{\mathfrak{p}}) = 0$.

PROOF. The assertion that $\mathfrak{p} \in \operatorname{Min}(M)$ means $\operatorname{ht}(\mathfrak{p}/\operatorname{Ann}(M)) = 0$. But, by Prop. 1, (8) of [8], this statement is equivalent to $\operatorname{Ht}(\mathfrak{p}/\operatorname{Ann}(M)) = 0$. Therefore our lemma follows from (3.6). *q.e.d.*

(4.3) PROPOSITION. *Let A be a quasi-local ring and M a non-zero, finite A-module such that* $\operatorname{Dim}(M) < \infty$. *Suppose that, for all* $\mathfrak{p} \in \operatorname{Supp}(M)$, $\operatorname{Gr}(M_{\mathfrak{p}}) + \operatorname{Dim}(A/\mathfrak{p}) = \operatorname{Dim}(M)$. *Then*:

(i) *For all* $\mathfrak{p} \in \mathrm{Supp}(M)$, $\mathrm{Gr}(M_{\mathfrak{p}}) = \mathrm{Dim}(M_{\mathfrak{p}})$.

(ii) $\mathfrak{p} \in \mathrm{Min}(M)$ *if and only if* $\mathfrak{p} \in \mathrm{Supp}(M)$ *and* $\mathrm{Dim}(A/\mathfrak{p}) = \mathrm{Dim}(M)$.

(iii) $\mathrm{Min}(M) = \mathrm{Att}(M)$.

PROOF. Let $\mathfrak{m}$ be the maximal ideal of A. Assume that $\mathfrak{p} \in \mathrm{Supp}(M)$. Then, by Theorem 2 of [8], $\mathrm{Gr}(M_{\mathfrak{p}}) \leqq \mathrm{Dim}(M_{\mathfrak{p}})$. Thus it follows from (3.6) and (4.1) that

$$\begin{aligned}\mathrm{Gr}(M_{\mathfrak{p}}) + \mathrm{Dim}(A/\mathfrak{p}) \leqq \mathrm{Dim}(M_{\mathfrak{p}}) + \mathrm{Dim}(A/\mathfrak{p}) &= \mathrm{Ht}(\mathfrak{p}/\mathrm{Ann}(M)) + \mathrm{Ht}(\mathfrak{m}/\mathfrak{p}) \\ &\leqq \mathrm{Ht}(\mathfrak{m}/\mathrm{Ann}(M)) = \mathrm{Dim}(M).\end{aligned}$$

Hence we see that $\mathrm{Gr}(M_{\mathfrak{p}}) + \mathrm{Dim}(A/\mathfrak{p}) = \mathrm{Dim}(M_{\mathfrak{p}}) + \mathrm{Dim}(A/\mathfrak{p})$ by the assumption. But, since $\mathrm{Dim}(M)$ is finite, $\mathrm{Dim}(A/\mathfrak{p})$ is also finite. This therefore shows that $\mathrm{Gr}(M_{\mathfrak{p}}) = \mathrm{Dim}(M_{\mathfrak{p}})$ and we have proved the first assertion.

Now, by Exercise 4 of Chapter 6 in [7], it is clear that $\mathrm{Min}(M) \subseteq \mathrm{Att}(M) \subseteq \mathrm{Supp}(M)$. Hence it follows from (4.2) and the assertion (i) that $\mathfrak{p} \in \mathrm{Min}(M)$ if and only if $\mathrm{Gr}(M_{\mathfrak{p}}) = 0$, which settles the assertion (iii). The hypothesis that $\mathrm{Gr}(M_{\mathfrak{p}}) + \mathrm{Dim}(A/\mathfrak{p}) = \mathrm{Dim}(M)$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$ shows that $\mathrm{Gr}(M_{\mathfrak{p}}) = 0$ if and only if $\mathfrak{p} \in \mathrm{Supp}(M)$ and $\mathrm{Dim}(A/\mathfrak{p}) = \mathrm{Dim}(M)$. Consequently these arguments lead us to the assertion (ii). *q.e.d.*

The following definition is motivated by (4.3).

(4.4) DEFINITION. Let A be a ring. Then a non-zero, finite A-module M is called a *Cohen-Macaulay module* if it satisfies the following conditions:

(a) For each prime ideal $\mathfrak{p}$ in $\mathrm{Supp}(M)$, $\mathrm{Dim}(M_{\mathfrak{p}})$ is finite.

(b) For each pair of prime ideals $\mathfrak{p}$, $\mathfrak{q}$ in $\mathrm{Supp}(M)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$, $\mathrm{Gr}(M_{\mathfrak{q}}) + \mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \mathrm{Dim}(M_{\mathfrak{p}})$.

Further a ring A is called a *Cohen-Macaulay ring* if A is a Cohen-Macaulay A-module.

(4.5) LEMMA. *Let A be a noetherian local ring and M a non-zero, finite A-module. Assume that M is a Cohen-Macaulay A-module in the classical sense, namely* $\mathrm{depth}\, M = \dim M$. *Then, for each prime ideal* $\mathfrak{q}$ *in* $\mathrm{Supp}(M)$, $\mathrm{depth}\, M_{\mathfrak{q}} + \dim A/\mathfrak{q} = \dim M$.

PROOF. Let $\mathfrak{q}$ be a prime ideal in $\mathrm{Supp}(M)$. Then $\mathfrak{q} \supseteq \mathrm{Ann}(M)$. Thus we use induction on $\mathrm{ht}(\mathfrak{q}/\mathrm{Ann}(M))$. Put $n = \mathrm{ht}(\mathfrak{q}/\mathrm{Ann}(M))$. If $n = 0$, then $\mathfrak{q} \in \mathrm{Ass}(M)$. Hence $\mathrm{depth}\, M_{\mathfrak{q}} = 0$ and $\dim A/\mathfrak{q} = \dim M$. Accordingly, we have the equality in this case. From now on we assume therefore that $n \geqq 1$ and make the obvious inductive hypothesis. Then $\mathfrak{q} \notin \mathrm{Ass}(M)$. It follows that there exists an element f of $\mathfrak{q}$ such that the sequence

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow M_1 \longrightarrow 0$$

is exact, where $M_1=M/fM$. However M_1 is also a Cohen-Macaulay A-module and $\mathrm{ht}(\mathfrak{q}/\mathrm{Ann}(M_1))=n-1$. Hence, by the inductive hypothesis, we obtain $\mathrm{depth}(M_1)_\mathfrak{q}+\dim A/\mathfrak{q}=\dim M_1$. Adding 1 to both sides, we have $\mathrm{depth}\, M_\mathfrak{q}+\dim A/\mathfrak{q}=\dim M$. *q.e.d.*

(4.6) PROPOSITION. *Let A be a noetherian ring and M a non-zero, finite A-module. Then the following conditions are equivalent*:

(i) *M is a Cohen-Macaulay A-module in the classical sense. That is to say, for each maximal ideal $\mathfrak{m}$ of A in* $\mathrm{Supp}(M)$, $\mathrm{depth}\, M_\mathfrak{m}=\dim M_\mathfrak{m}$.

(ii) *M is a Cohen-Macaulay A-module in the sense of* (4.4).

PROOF. Note that if A is a noetherian ring and $\mathfrak{p}$ is in $\mathrm{Supp}(M)$, then $\mathrm{Gr}(M_\mathfrak{p})=\mathrm{depth}\, M_\mathfrak{p}$ and $\mathrm{Dim}(M_\mathfrak{p})=\dim M_\mathfrak{p}$, and so $\mathrm{Dim}(M_\mathfrak{p})<\infty$. Hence this proposition follows from (4.5) and the fact that the assertion (i) implies $\mathrm{depth}\, M_\mathfrak{p}=\dim M_\mathfrak{p}$ for all $\mathfrak{p}\in\mathrm{Supp}(M)$. *q.e.d.*

(4.7) PROPOSITION. *Let A be a ring and M a non-zero, finite A-module. Then*:

(i) *If M is a Cohen-Macaulay A-module and S is a multiplicatively closed subset of A with $M_S\neq 0$, then M_S is also a Cohen-Macaulay A_S-module.*

(ii) *M is a Cohen-Macaulay A-module if and only if $M_\mathfrak{m}$ is a Cohen-Macaulay $A_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ in* $\mathrm{Supp}(M)$.

PROOF. This proposition is obvious by the definition. *q.e.d.*

(4.8) THEOREM. *Let A be a ring and M a non-zero, finite A-module. Then M is a Cohen-Macaulay A-module if and only if the following three statements hold*:

(i) *For each prime ideal $\mathfrak{p}$ in* $\mathrm{Supp}(M)$, $\mathrm{Dim}(M_\mathfrak{p})$ *is finite.*

(ii) *For each prime ideal $\mathfrak{p}$ in* $\mathrm{Supp}(M)$, $\mathrm{Gr}(M_\mathfrak{p})=\mathrm{Dim}(M_\mathfrak{p})$.

(iii) *For each pair of prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ in* $\mathrm{Supp}(M)$ *such that* $\mathfrak{q}\subseteq\mathfrak{p}$, $\mathrm{Ht}(\mathfrak{p}/\mathrm{Ann}(M))=\mathrm{Ht}(\mathfrak{q}/\mathrm{Ann}(M))+\mathrm{Ht}(\mathfrak{p}/\mathfrak{q})$.

PROOF. Suppose that M is a Cohen-Macaulay A-module. Clearly we have the first assertion by the definition. Let $\mathfrak{p}$ be in $\mathrm{Supp}(M)$. Then it follows from the definition that $\mathrm{Gr}(M_\mathfrak{p})=\mathrm{Dim}(M_\mathfrak{p})$ becasuse $\mathrm{Dim}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})=0$. This proves the second assertion. Now assume that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals in $\mathrm{Supp}(M)$ with $\mathfrak{q}\subseteq\mathfrak{p}$. Then, by the definition and the assertion (ii), we obtain $\mathrm{Dim}(M_\mathfrak{q})+\mathrm{Dim}(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p})=\mathrm{Dim}(M_\mathfrak{p})$. However, by (2.7) and (3.6), we see that $\mathrm{Dim}(M_\mathfrak{q})=\mathrm{Ht}(\mathfrak{q}/\mathrm{Ann}(M))$, $\mathrm{Dim}(M_\mathfrak{p})=\mathrm{Ht}(\mathfrak{p}/\mathrm{Ann}(M))$ and $\mathrm{Dim}(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p})=\mathrm{Ht}(\mathfrak{p}/\mathfrak{q})$. Therefore $\mathrm{Ht}(\mathfrak{q}/\mathrm{Ann}(M))+\mathrm{Ht}(\mathfrak{p}/\mathfrak{q})=\mathrm{Ht}(\mathfrak{p}/\mathrm{Ann}(M))$, which settles the third assertion.

Conversely assume that the three conditions hold, and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals in $\mathrm{Supp}(M)$ with $\mathfrak{q} \subseteq \mathfrak{p}$. Then the assumptions and the above arguments show that

$$\mathrm{Gr}(M_{\mathfrak{q}}) + \mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \mathrm{Dim}(M_{\mathfrak{q}}) + \mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}})$$

$$= \mathrm{Ht}(\mathfrak{q}/\mathrm{Ann}(M)) + \mathrm{Ht}(\mathfrak{p}/\mathfrak{q}) = \mathrm{Ht}(\mathfrak{p}/\mathrm{Ann}(M)) = \mathrm{Dim}(M_{\mathfrak{p}}).$$

Consequently M is a Cohen-Macaulay A-module. *q.e.d.*

(4.9) DEFINITION. A ring A is said to be *polynomially catenarian* if the following two conditions hold:

(a) For each prime ideal $\mathfrak{p}$ of A, $\mathrm{Ht}(\mathfrak{p})$ is finite.

(b) For each set of prime ideals, $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$ of A such that $\mathfrak{r} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$, $\mathrm{Ht}(\mathfrak{p}/\mathfrak{r}) = \mathrm{Ht}(\mathfrak{q}/\mathfrak{r}) + \mathrm{Ht}(\mathfrak{p}/\mathfrak{q})$.

It is clear that if A is an integral domain, the condition (b) is equivalent to the statement that $\mathrm{Ht}(\mathfrak{p}) = \mathrm{Ht}(\mathfrak{q}) + \mathrm{Ht}(\mathfrak{p}/\mathfrak{q})$ for each pair of prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of A such that $\mathfrak{q} \subseteq \mathfrak{p}$. Suppose next that A is a noetherian ring. Then A is polynomially catenarian if and only if A is catenarian, because we see that $\mathrm{Ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of A.

(4.10) COROLLARY. *Let A be a ring and M a Cohen-Macaulay A-module. Then $A/\mathrm{Ann}(M)$ is polynomially catenarian.*

PROOF. We can easily prove our corollary by the assertion (iii) of (4.8). *q.e.d.*

(4.11) LEMMA. *Let A be a quasi-local ring with the maximal ideal $\mathfrak{m}$ and M a non-zero, finite A-module such that $\mathrm{Gr}(M) = \mathrm{Dim}(M)$. Furthermore let f be an M-regular element of $\mathfrak{m}$. Then $\mathrm{Dim}(M) = \mathrm{Dim}(M/fM) + 1$.*

PROOF. Since f is an M-regular element of $\mathfrak{m}$, f is not contained in any minimal prime ideal of $\mathrm{Ann}(M)$. Thus, by Prop. 1, (9) of [8], $\mathrm{Ht}(\mathfrak{m}/\mathrm{Ann}(M)) \geqq \mathrm{Ht}(\mathfrak{m}/(\mathrm{Ann}(M), f)) + 1$. Therefore, by Prop. 1, (4) of [8], $\mathrm{Ht}(\mathfrak{m}/\mathrm{Ann}(M)) \geqq \mathrm{Ht}(\mathfrak{m}/\mathrm{Ann}(M/fM)) + 1$, because $\mathrm{Ann}(M/fM) \supseteq (\mathrm{Ann}(M), f)$. Hence, by (3.6), $\mathrm{Dim}(M) \geqq \mathrm{Dim}(M/fM) + 1$. Therefore $\mathrm{Gr}(M/fM) + 1 = \mathrm{Gr}(M) = \mathrm{Dim}(M) \geqq \mathrm{Dim}(M/fM) + 1$ by Theorem 15 of Chapter 5 in [7]. It consequently follows from Theorem 2 of [8] that $\mathrm{Gr}(M/fM) + 1 = \mathrm{Gr}(M) = \mathrm{Dim}(M) = \mathrm{Dim}(M/fM) + 1$. *q.e.d.*

(4.12) PROPOSITION. *Let A be a ring and M a non-zero, finite A-module. Further, let f be an element of A such that the sequence*

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0$$

is exact and $M/fM \neq 0$. *If* M *is a Cohen-Macaulay* A*-module, then* M/fM *is also a Cohen-Macaulay* A*-module.*

PROOF. Put $M_1 = M/fM$. Let $\mathfrak{p}$ be a prime ideal in $\mathrm{Supp}(M_1)$. Then $\mathfrak{p} \in \mathrm{Supp}(M)$ and $f \in \mathfrak{p}$. Now localizing the exact sequence at $\mathfrak{p}$, we have an exact sequence

$$0 \longrightarrow M_{\mathfrak{p}} \xrightarrow{f} M_{\mathfrak{p}} \longrightarrow M_{1\mathfrak{p}} \longrightarrow 0.$$

However, by (4.8), $\mathrm{Gr}(M_{\mathfrak{p}}) = \mathrm{Dim}(M_{\mathfrak{p}})$ and $\mathrm{Dim}(M_{\mathfrak{p}}) < \infty$. It therefore follows from (4.11) that $\mathrm{Dim}(M_{1\mathfrak{p}})$ is finite. Next let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals in $\mathrm{Supp}(M_1)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Then we see that

$$\mathrm{Gr}(M_{1\mathfrak{q}}) + \mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \mathrm{Gr}(M_{\mathfrak{q}}) + \mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) - 1 = \mathrm{Dim}(M_{\mathfrak{p}}) - 1,$$

because M is a Cohen-Macaulay A-module and $\mathrm{Gr}(M_{\mathfrak{q}}) = \mathrm{Gr}(M_{1\mathfrak{q}}) + 1$. Hence, by (4.11), $\mathrm{Gr}(M_{1\mathfrak{q}}) + \mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \mathrm{Dim}(M_{1\mathfrak{p}})$. Accordingly M_1 is a Cohen-Macaulay A-module. *q.e.d.*

(4.13) THEOREM. *Let* A *be a ring and* M *a Cohen-Macaulay* A*-module. If* $(A/\mathrm{Ann}(M))[X]$ *is polynomially catenarian, then* $M[X]$ *is a Cohen-Macaulay* $A[X]$-module.

PROOF. Put $\mathfrak{a} = \mathrm{Ann}(M)$. Then it is clear that $\mathrm{Ann}(M[X]) = \mathfrak{a}[X]$. Let $\mathfrak{P}$ be a prime ideal in $\mathrm{Supp}(M[X])$, and put $\mathfrak{p} = A \cap \mathfrak{P}$. Then $\mathfrak{P} \supseteq \mathfrak{a}[X]$ and $\mathfrak{p} \in \mathrm{Supp}(M)$. Since M is a Cohen-Macaulay A-module, we see that $\mathrm{Dim}(M_{\mathfrak{p}})$ is finite, and therefore, by (3.6), $\mathrm{Ht}(\mathfrak{p}/\mathfrak{a})$ is finite. On the other hand, since $\mathfrak{P}/\mathfrak{a}[X] \cap A/\mathfrak{a} = \mathfrak{p}/\mathfrak{a}$, it follows from (2.3) and (2.4) that $\mathrm{Ht}(\mathfrak{P}/\mathfrak{a}[X]) = \mathrm{Ht}(\mathfrak{p}/\mathfrak{a}) + \mathrm{ht}(\mathfrak{P}/\mathfrak{p}[X]) \leqq \mathrm{Ht}(\mathfrak{p}/\mathfrak{a}) + 1 < \infty$. Consequently we conclude that $\mathrm{Dim}(M[X]_{\mathfrak{P}})$ is finite because $\mathrm{Dim}(M[X]_{\mathfrak{P}}) = \mathrm{Ht}(\mathfrak{P}/\mathfrak{a}[X])$.

Let now $\mathfrak{P}$ and $\mathfrak{Q}$ be prime ideals of $A[X]$ in $\mathrm{Supp}(M[X])$ such that $\mathfrak{Q} \subseteq \mathfrak{P}$. Put $\mathfrak{p} = A \cap \mathfrak{P}$ and $\mathfrak{q} = A \cap \mathfrak{Q}$. Then $\mathfrak{p}$ and $\mathfrak{q}$ are in $\mathrm{Supp}(M)$ and $\mathfrak{q} \subseteq \mathfrak{p}$. Therefore, since M is a Cohen-Macaulay A-module, $\mathrm{Gr}(M_{\mathfrak{q}}) + \mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \mathrm{Dim}(M_{\mathfrak{p}})$. To complete the proof we shall distinguish between the following four possibilities: (i) $\mathfrak{P} = \mathfrak{p}[X]$ and $\mathfrak{Q} = \mathfrak{q}[X]$; (ii) $\mathfrak{P} \neq \mathfrak{p}[X]$ and $\mathfrak{Q} = \mathfrak{q}[X]$; (iii) $\mathfrak{P} = \mathfrak{p}[X]$ and $\mathfrak{Q} \neq \mathfrak{q}[X]$; (iv) $\mathfrak{P} \neq \mathfrak{p}[X]$ and $\mathfrak{Q} \neq \mathfrak{q}[X]$. We can observe that if $\mathfrak{Q} = \mathfrak{q}[X]$, then $\mathrm{Gr}(M[X]_{\mathfrak{Q}}) = \mathrm{Gr}(M_{\mathfrak{q}})$ by (3.4) and that if $\mathfrak{Q} \neq \mathfrak{q}[X]$, then $\mathrm{Gr}(M[X]_{\mathfrak{Q}}) = \mathrm{Gr}(M_{\mathfrak{q}}) + 1$ in view of (3.5). Furthermore we see that if $\mathfrak{P} = \mathfrak{p}[X]$, then $\mathrm{Dim}(M[X]_{\mathfrak{P}}) = \mathrm{Ht}(\mathfrak{P}/\mathfrak{a}[X]) = \mathrm{Ht}((\mathfrak{p}/\mathfrak{a})[X]) = \mathrm{Ht}(\mathfrak{p}/\mathfrak{a}) = \mathrm{Dim}(M_{\mathfrak{p}})$ by (3.6) and that if $\mathfrak{P} \neq \mathfrak{p}[X]$, then $\mathrm{Dim}(M[X]_{\mathfrak{P}}) = \mathrm{Dim}(M_{\mathfrak{p}}) + 1$ by the proof of (3.7). In addition, (2.7) yields $\mathrm{Dim}(A[X]_{\mathfrak{P}}/\mathfrak{Q}A[X]_{\mathfrak{P}}) = \mathrm{Ht}(\mathfrak{P}/\mathfrak{Q})$ and $\mathrm{Dim}(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \mathrm{Ht}(\mathfrak{p}/\mathfrak{q})$.

These arguments show the following equalities. In case (i), it is clear that $\mathrm{Ht}(\mathfrak{P}/\mathfrak{Q}) = \mathrm{ht}(\mathfrak{p}/\mathfrak{q})$. Thus,

$$\mathrm{Gr}\,(M[X]_{\mathfrak{Q}}) + \mathrm{Dim}\,(A[X]_{\mathfrak{P}}/\mathfrak{Q}A[X]_{\mathfrak{P}}) = \mathrm{Gr}\,(M_{\mathfrak{p}}) + \mathrm{Ht}\,(\mathfrak{P}/\mathfrak{Q})$$
$$= \mathrm{Gr}\,(M_{\mathfrak{q}}) + \mathrm{Ht}\,(\mathfrak{p}/\mathfrak{q}) = \mathrm{Gr}\,(M_{\mathfrak{q}}) + \mathrm{Dim}\,(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}})$$
$$= \mathrm{Dim}\,(M_{\mathfrak{p}}) = \mathrm{Dim}\,(M[X]_{\mathfrak{P}}).$$

In case (ii), by (2.3) and (2.4), $\mathrm{Ht}\,(\mathfrak{P}/\mathfrak{Q}) = \mathrm{Ht}\,(\mathfrak{p}/\mathfrak{q}) + 1$, whence

$$\mathrm{Gr}\,(M[X]_{\mathfrak{Q}}) + \mathrm{Dim}\,(A[X]_{\mathfrak{P}}/\mathfrak{Q}A[X]_{\mathfrak{P}}) = \mathrm{Gr}\,(M_{\mathfrak{q}}) + \mathrm{Ht}\,(\mathfrak{P}/\mathfrak{Q})$$
$$= \mathrm{Gr}\,(M_{\mathfrak{q}}) + \mathrm{Ht}\,(\mathfrak{p}/\mathfrak{q}) + 1 = \mathrm{Gr}\,(M_{\mathfrak{q}}) + \mathrm{Dim}\,(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) + 1$$
$$= \mathrm{Dim}\,(M_{\mathfrak{p}}) + 1 = \mathrm{Dim}\,(M[X]_{\mathfrak{P}}).$$

Since $(A/\mathrm{Ann}\,(M))[X]$ is polynomially catenarian, $\mathrm{Ht}\,(\mathfrak{P}/\mathfrak{Q}) = \mathrm{Ht}\,(\mathfrak{P}/\mathfrak{q}[X]) - \mathrm{Ht}\,(\mathfrak{Q}/\mathfrak{q}[X])$. In case (iii), we see that $\mathrm{Ht}\,(\mathfrak{P}/\mathfrak{q}[X]) = \mathrm{Ht}\,(\mathfrak{p}/\mathfrak{q})$ and $\mathrm{Ht}\,(\mathfrak{Q}/\mathfrak{q}[X]) = 1$ by (2.4). Therefore

$$\mathrm{Gr}\,(M[X]_{\mathfrak{Q}}) + \mathrm{Dim}\,(A[X]_{\mathfrak{P}}/\mathfrak{Q}A[X]_{\mathfrak{P}}) = \mathrm{Gr}\,(M_{\mathfrak{q}}) + 1 + \mathrm{Ht}\,(\mathfrak{P}/\mathfrak{Q})$$
$$= \mathrm{Gr}\,(M_{\mathfrak{q}}) + 1 + \mathrm{Ht}\,(\mathfrak{P}/\mathfrak{q}[X]) - \mathrm{Ht}\,(\mathfrak{Q}/\mathfrak{q}[X])$$
$$= \mathrm{Gr}\,(M_{\mathfrak{q}}) + 1 + \mathrm{Ht}\,(\mathfrak{p}/\mathfrak{q}) - 1 = \mathrm{Gr}\,(M_{\mathfrak{q}}) + \mathrm{Dim}\,(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}})$$
$$= \mathrm{Dim}\,(M_{\mathfrak{p}}) = \mathrm{Dim}\,(M[X])_{\mathfrak{P}}.$$

Finally, in case (iv), by (2.3) and (2.4), $\mathrm{Ht}\,(\mathfrak{P}/\mathfrak{q}[X]) = \mathrm{Ht}\,(\mathfrak{p}/\mathfrak{q}) + 1$ and $\mathrm{Ht}\,(\mathfrak{Q}/\mathfrak{q}[X]) = 1$. Hence it follows by a similar method that

$$\mathrm{Gr}\,(M[X]_{\mathfrak{Q}}) + \mathrm{Dim}\,(A[X]_{\mathfrak{P}}/\mathfrak{Q}A[X]_{\mathfrak{P}}) = \mathrm{Gr}\,(M_{\mathfrak{q}}) + 1 + \mathrm{Ht}\,(\mathfrak{P}/\mathfrak{Q})$$
$$= \mathrm{Gr}\,(M_{\mathfrak{q}}) + 1 + \mathrm{Ht}\,(\mathfrak{P}/\mathfrak{q}[X]) - \mathrm{Ht}\,(\mathfrak{Q}/\mathfrak{q}[X])$$
$$= \mathrm{Gr}\,(M_{\mathfrak{q}}) + 1 + \mathrm{Ht}\,(\mathfrak{p}/\mathfrak{q}) + 1 - 1 = \mathrm{Gr}\,(M_{\mathfrak{q}}) + \mathrm{Dim}\,(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) + 1$$
$$= \mathrm{Dim}\,(M_{\mathfrak{p}}) + 1 = \mathrm{Dim}\,(M[X]_{\mathfrak{P}}).$$

Consequently $M[X]$ is a Cohen-Macaulay $A[X]$-module. *q.e.d.*

5. Examples of Cohen-Macaulay rings

(5.1) PROPOSITION. *Let A be a quasi-local ring. Then we have the following statements*:

(i) *If* $\mathrm{Dim}\,A = 0$, *then A is a Cohen-Macaulay ring.*

(ii) *Suppose* $\mathrm{Dim}\,A = 1$. *Then A is a Cohen-Macaulay ring if and only if* $\mathrm{Gr}\,(A)$ *is positive.*

(iii) *Suppose* $\mathrm{Dim}\,A = 2$. *Then A is a Cohen-Macaulay ring if and only* *if* $\mathrm{Gr}(A)=\mathrm{Dim}(A/\mathfrak{p})$ *for each prime ideal* $\mathfrak{p}$ *in* $\mathrm{Att}(A)$. *In particular a quasi-local domain A with* $\mathrm{Gr}(A)=\mathrm{Dim}\,A=2$ *is a Cohen-Macaulay ring.*

PROOF. Since the assertions (i) and (ii) are obvious, we shall only show the assertion (iii). The "only if" part of (iii) follows from (4.3). We prove the "if" part of (iii). Suppose now that, for all $\mathfrak{p}\in\mathrm{Att}(A)$, $\mathrm{Gr}(A)=\mathrm{Dim}(A/\mathfrak{p})$. Then, by the facts that $\mathrm{Dim}\,A=\mathrm{Sup}_{\mathfrak{p}\in\mathrm{Min}(A)}\{\mathrm{Dim}(A/\mathfrak{p})\}$, $\mathrm{Dim}\,A=2$ and $\mathrm{Min}(A)\subseteq\mathrm{Att}(A)$, we see that $\mathrm{Dim}(A/\mathfrak{p})=2$ for all prime ideals $\mathfrak{p}$ in $\mathrm{Att}(A)$ and that $\mathrm{Gr}(A)=\mathrm{Dim}(A)$. Hence, by (2.8), $\mathrm{Min}(A)=\mathrm{Att}(A)$. Let $\mathfrak{p}$ be a prime ideal of A. Then we have $\mathrm{Gr}(A_\mathfrak{p})\leqq\mathrm{Dim}(A_\mathfrak{p})$ by Theorem 2 of [8]. It therefore follows that if $\mathrm{Ht}(\mathfrak{p})=0$, then $\mathrm{Gr}(A_\mathfrak{p})=\mathrm{Dim}(A_\mathfrak{p})=0$ and that if $\mathrm{Ht}(\mathfrak{p})=1$, then $\mathrm{Gr}(A_\mathfrak{p})=\mathrm{Dim}(A_\mathfrak{p})=1$.

Next we assume that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of A such that $\mathfrak{q}\subseteq\mathfrak{p}$ and we shall show the equality $\mathrm{Gr}(A_\mathfrak{q})+\mathrm{Dim}(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p})=\mathrm{Dim}(A_\mathfrak{p})$. Thus, we may also assume that $\mathfrak{q}\subsetneqq\mathfrak{p}$ by the above arguments. Accordingly, since $\mathrm{Ht}(\mathfrak{q})<\mathrm{Ht}(\mathfrak{p})\leqq 2$, we can distinguish three cases: (a) $\mathrm{Ht}(\mathfrak{q})=0$ and $\mathrm{Ht}(\mathfrak{p})=1$; (b) $\mathrm{Ht}(\mathfrak{q})=1$ and $\mathrm{Ht}(\mathfrak{p})=2$; (c) $\mathrm{Ht}(\mathfrak{q})=0$ and $\mathrm{Ht}(\mathfrak{p})=2$. In case (a), $\mathrm{Gr}(A_\mathfrak{q})+\mathrm{Dim}(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p})=0+1=\mathrm{Dim}(A_\mathfrak{p})$. We can easily show the equality in cases (b) and (c) by the same method. Consequently A is a Cohen-Macaulay ring.

The second statement of (iii) follows from the fact that if A is an integral domain, then $\mathrm{Att}(A)=\{0\}$. *q.e.d.*

(5.2) EXAMPLE. Let B be a polynomial ring $k[X_1, X_2,\dots, X_n,\dots]$ over a field k, and let m be a positive integer. Further let $\mathfrak{a}=(X_{m+1}^2, X_{m+2}^2,\dots)$ and let $\mathfrak{p}=(X_{m+1}, X_{m+2},\dots)$. Then it is clear that $\mathfrak{p}=\sqrt{\mathfrak{a}}$. Put $A=B/\mathfrak{a}$ and $\mathfrak{P}=\mathfrak{p}/\mathfrak{a}$. Then A is a non-noetherian ring and $\mathfrak{P}=\sqrt{0}$. Since the ring $A/\mathfrak{P}$ is isomophic to $k[X_1, X_2,\dots, X_m]$, $A/\mathfrak{P}$ is a noetherian, Cohen-Macaulay ring of dimension m and $A/\mathfrak{P}$ is catenarian. However, by Lemma 2 of [8] and (2.8), $\mathrm{Ht}(\mathfrak{Q})=\mathrm{Ht}(\mathfrak{Q}/\mathfrak{P})$ for all prime ideals $\mathfrak{Q}$ of A, becasue $\mathfrak{P}=\sqrt{0}$. Thus $\mathrm{Dim}\,A=\mathrm{Dim}(A/\mathfrak{P})=\dim(A/\mathfrak{P})=m$ since $A/\mathfrak{P}$ is noetherian. Furthermore A is a polynomially catenarian ring, because, for any pair of prime ideals $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ of A with $\mathfrak{Q}_1\subseteq\mathfrak{Q}_2$, we obtain $\mathrm{Ht}(\mathfrak{Q}_1)+\mathrm{Ht}(\mathfrak{Q}_2/\mathfrak{Q}_1)=\mathrm{Ht}(\mathfrak{Q}_1/\mathfrak{P})+\mathrm{Ht}((\mathfrak{Q}_2/\mathfrak{P})/(\mathfrak{Q}_1/\mathfrak{P}))=\mathrm{Ht}(\mathfrak{Q}_2/\mathfrak{P})=\mathrm{Ht}(\mathfrak{Q}_2)$. On the other hand A is isomorphic to $C[X_1,\dots, X_m]$, where $C=k[X_{m+1}, X_{m+2},\dots]/(X_{m+1}^2, X_{m+2}^2,\dots)$. Since $\mathrm{Dim}\,C=0$, C is a Cohen-Macaulay ring. It thus follows from (3.8) that $\mathrm{Gr}(A_\mathfrak{Q})=\mathrm{Dim}(A_\mathfrak{Q})$ for all prime ideals $\mathfrak{Q}$ of A. Consequently, in view of (4.8), we can conclude that A is a nonnoetherian, Cohen-Macaulay ring with $\mathrm{Dim}\,A=m$.

(5.3) LEMMA. *Let A be a Krull domain and* $\mathfrak{p}$ *a prime ideal of A. Then* $\mathrm{ht}(\mathfrak{p})=1$ *if and only if* $\mathrm{Ht}(\mathfrak{p})=1$.

PROOF. Assume that $\mathrm{ht}(\mathfrak{p})=1$. Then we see that $\mathrm{ht}(\mathfrak{p})=\mathrm{ht}(\mathfrak{p}A_\mathfrak{p})=$

$\mathrm{Ht}(\mathfrak{p}A_{\mathfrak{p}})=\mathrm{Ht}(\mathfrak{p})$, because $A_{\mathfrak{p}}$ is a noetherian ring. Thus $\mathrm{Ht}(\mathfrak{p})=1$. Conversely suppose that $\mathrm{Ht}(\mathfrak{p})=1$. Since $\mathrm{ht}(\mathfrak{p})\leqq \mathrm{Ht}(\mathfrak{p})$, $\mathrm{ht}(\mathfrak{p})\leqq 1$. If $\mathrm{ht}(\mathfrak{p})=0$, then $\mathrm{Ht}(\mathfrak{p})=0$ by Prop. 1, (8) of [8]. Therefore we have $\mathrm{ht}(\mathfrak{p})=1$. *q.e.d.*

The following theorem is a generalization of Serre's criterion of normality and its proof is quite similar to Fossum's one of Theorem 4.1 in [3].

(5.4) THEOREM. *Let A be an integral domain. Then A is a Krull domain if and only if A satisfies the following three conditions*:

(FC) *Given f in A, $f\neq 0$, there is at most a finite number of prime ideals $\mathfrak{p}$ of A such that $f\in\mathfrak{p}$ and* $\mathrm{Ht}(\mathfrak{p})=1$.

(R_1) *If $\mathfrak{p}$ is a prime ideal of A such that* $\mathrm{Ht}(\mathfrak{p})=1$, *then $A_{\mathfrak{p}}$ is a discrete valuation ring of rank one.*

(S_2) *For each prime ideal $\mathfrak{q}$ of A*, $\mathrm{Gr}(A_{\mathfrak{q}})\geqq\inf\{2, \mathrm{Ht}(\mathfrak{q})\}$.

PROOF. First assume that A is a Krull domain. Then the conditions (FC) and (R_1) are well known by (5.3). Hence we shall prove the condition (S_2). It will suffice to show that if $\mathfrak{q}$ is a prime ideal of A such that $\mathrm{Ht}(\mathfrak{q})\geqq 2$, then $\mathrm{Gr}(A_{\mathfrak{q}})\geqq 2$. Now assume that $\mathfrak{q}$ is a prime ideal with $\mathrm{Ht}(\mathfrak{q})\geqq 2$. Then $\mathrm{ht}(\mathfrak{q})\geqq 2$ by (5.3). It follows from the proof of Theorem 4.1 in [3] that $\mathrm{gr}_{A_{\mathfrak{q}}}\{\mathfrak{q}A_{\mathfrak{q}}; A_{\mathfrak{q}}\}\geqq 2$. Thus $\mathrm{Gr}(A_{\mathfrak{q}})\geqq 2$.

Next, to prove the converse, assume that the three conditions (FC), (R_1) and (S_2) hold. We shall show that (S_2) implies $A=\bigcap_{\mathrm{Ht}(\mathfrak{p})=1}A_{\mathfrak{p}}$. Suppose that $x\in\bigcap_{\mathrm{Ht}(\mathfrak{p})=1}A_{\mathfrak{p}}$ and $x\neq 0$. Let $\mathfrak{a}=A:_A x$. If $\mathfrak{a}=A$, then $x\in A$. Assume the contrary, that is to say $\mathfrak{a}\neq A$. Then there exists a prime ideal $\mathfrak{q}$ of A such that $\mathfrak{q}$ is a minimal prime ideal of $\mathfrak{a}$. Since $\mathfrak{a}_{\mathfrak{q}}=A_{\mathfrak{q}}:_{A_{\mathfrak{q}}}x$, we see $\mathrm{Ht}(\mathfrak{q})\geqq 2$. Therefore the assumption (S_2) implies $\mathrm{Gr}(A_{\mathfrak{q}})\geqq 2$. However, since $\mathfrak{q}$ is a minimal prime ideal of $\mathfrak{a}$, we see that $\sqrt{\mathfrak{a}A_{\mathfrak{q}}}=\mathfrak{q}A_{\mathfrak{q}}$. This shows $\mathrm{Gr}_{A_{\mathfrak{q}}}\{\mathfrak{a}A_{\mathfrak{q}}; A_{\mathfrak{q}}\}\geqq 2$ by Theorem 12 of Chapter 5 in [7]. Accordingly, there exists an integer n such that we can find an $A_{\mathfrak{q}}[X_1,\dots, X_n]$-sequence $\{u, v\}$ composed of elements of $\mathfrak{a}A_{\mathfrak{q}}[X_1,\dots, X_n]$. Put $r=xu$ and $s=xv$. Then r and s are in $A_{\mathfrak{q}}[X_1,\dots, X_n]$ because $\mathfrak{a}A_{\mathfrak{q}}=A_{\mathfrak{q}}:_{A_{\mathfrak{q}}}x$. Thus $us=vr$ and hence we can write $r=uw$ where $w\in A_{\mathfrak{q}}[X_1,\dots, X_n]$, since $\{u, v\}$ is an $A_{\mathfrak{q}}[X_1,\dots, X_n]$-sequence. Therefore $x=w\in K\cap A_{\mathfrak{q}}[X_1,\dots, X_n]=A_{\mathfrak{q}}$ where K is the quotient field of A. This is a contradiction. Consequently we have established that the condition (S_2) implies $A=\bigcap_{\mathrm{Ht}(\mathfrak{p})=1}A_{\mathfrak{p}}$. Now our theorem follows from (FC) and (R_1). *q.e.d.*

(5.5) COROLLARY. *If A is a Krull domain such that* $\mathrm{Dim}\,A\leqq 2$, *then A is a Cohen-Macaulay ring.*

PROOF. This follows from (5.1) and (5.4). *q.e.d.*

(5.6) PROPOSITION. *Let V be a non-trivial valuation ring. Then* $\mathrm{Gr}(V)=1$.

PROOF. Let $\mathfrak{m}$ be the maximal ideal of V and let x be a non-zero element of $\mathfrak{m}$. Then we have an exact sequence

$$0 \longrightarrow V \xrightarrow{x} V \longrightarrow V/xV \longrightarrow 0.$$

Therefore, to prove the proposition, we have only to show $\mathrm{Gr}(V/xV)=0$, namely $\mathfrak{m} \in \mathrm{Att}(V/xV)$. By (3.1), it is enough to see that, for any finitely generated ideal $\mathfrak{a}$ contained in $\mathfrak{m}$, there exists a non-zero element m, of V/xV such that $\mathfrak{a}m=0$. Now assume that $\mathfrak{a}$ is a finitely generated ideal of V contained in $\mathfrak{m}$. Since V is a valuation ring, $\mathfrak{a}$ is a principal ideal aV and hence either $aV \subseteq xV$ or $aV \supsetneqq xV$. We denote by $\bar{z}$ the image of z under the natural mapping $V \to V/xV$. If $aV \subseteq xV$, then $\bar{1} \neq 0$ and $aV\bar{1}=0$ in V/xV. Thus we may assume that $aV \supsetneqq xV$. Then $x=ay$ where $y \in \mathfrak{m}$. Note that $y \notin xV$. Hence $\bar{y} \neq 0$ and $aV\bar{y}=0$ in V/xV. Consequently we see that $\mathfrak{m} \in \mathrm{Att}(V/xV)$. *q.e.d.*

(5.7) COROLLARY. *Let V be a non-trivial valuation ring. Then V is a Cohen-Macaulay ring if and only if the rank of V is one.*

PROOF. Since $\mathrm{Dim}\, V = \dim V$, our corollary follows from (5.6). *q.e.d.*

(5.8) LEMMA. *Let V be a valuation ring. If $\mathfrak{P}$ is a prime ideal of $V[X_1, \ldots, X_n]$, then* $\mathrm{Ht}(\mathfrak{P}) = \mathrm{ht}(\mathfrak{P})$.

PROOF. Let $\mathfrak{p}$ be a prime ideal of V. Then $V_{\mathfrak{p}}$ is a valuation ring. It thus follows from (2.7) and the definition of valuative dimension that $\mathrm{Ht}(\mathfrak{p}) = \mathrm{Dim}\, V_{\mathfrak{p}} = \dim V_{\mathfrak{p}} = \mathrm{ht}(\mathfrak{p})$. Therefore we can prove our lemma by (2.5). *q.e.d.*

(5.9) LEMMA. *Let V be a valuation ring with the maximal ideal $\mathfrak{m}$ and let $\mathfrak{Q}$ be a prime ideal of $V[X_1, \ldots, X_n]$ such that $\mathrm{ht}(\mathfrak{Q})=1$ and $\mathfrak{Q} \cap V=0$. Then $\mathfrak{Q}$ is a principal ideal generated by a polynomial which is not contained in $\mathfrak{m}[X_1, \ldots, X_n]$.*

PROOF. Let S be the set of non-zero elements of V, and let K be the quotient field of V. Then $\mathfrak{Q}_S$ is a prime ideal of $K[X_1, \ldots, X_n]$ and the height of $\mathfrak{Q}_S$ is one. Thus $\mathfrak{Q}_S$ is a principal ideal because $K[X_1, \ldots, X_n]$ is a unique factorization domain. Since V is a valuation ring, we may assume that $\mathfrak{Q}_S=(f)$, where f is an element of $V[X_1, \ldots, X_n]$ which is not contained in $\mathfrak{m}[X_1, \ldots, X_n]$. Accordingly $(f) \subseteq \mathfrak{Q}$. Now let v be the valuation on K associated with V and let v^* be the trivial extension of v to $K(X_1, \ldots, X_n)$. That is to say, $v^*(\sum a_{i_1 i_2 \cdots i_n} X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n}) = \inf\{v(a_{i_1 i_2 \cdots i_n}) \mid a_{i_1 i_2 \cdots i_n} \neq 0\}$. Assume $g \in \mathfrak{Q}$. Then we can write $g=(h/s)f$ where $h \in V[X_1, \ldots, X_n]$ and $s \in S$. Hence it follows that $v^*(g) = v^*(h/s) + v^*(f) = v^*(h/s)$, and so $v^*(h/s) \geqq 0$. Consequently $h/s \in V[X_1, \ldots, X_n]$ and hence $g \in (f)$. We therefore conclude that $\mathfrak{Q}=(f)$. *q.e.d.*

(5.10) PROPOSITION. *Let V be a valuation ring of rank one. Then $V[X_1]$ and $V[X_1, X_2]$ are Cohen-Macaulay rings.*

PROOF. First, by (2.3) and (2.7), note that $\mathrm{Dim}\, V[X_1]=2$ and $\mathrm{Dim}\, V[X_1, X_2]=3$. Since V is a Cohen-Macaulay ring, it follows from (3.8) and (5.1) that $V[X_1]$ is a Cohen-Macaulay ring. Moreover, again by (3.8), we see that $\mathrm{Gr}(V[X_1, X_2]_{\mathfrak{P}})=\mathrm{Dim}(V[X_1, X_2]_{\mathfrak{P}})$ for all prime ideals $\mathfrak{P}$ of $V[X_1, X_2]$. Thus, by (4.8), to prove that $V[X_1, X_2]$ is a Cohen-Macaulay ring, it is sufficient to show that $V[X_1, X_2]$ is polynomially catenarian. In view of (5.8), this is equivalent to the assertion that $V[X_1, X_2]$ is catenarian. Therefore we have only to prove that $\mathrm{ht}(\mathfrak{P})=\mathrm{ht}(\mathfrak{Q})+\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})$ for all pairs of prime ideals $\mathfrak{P}, \mathfrak{Q}$ of $V[X_1, X_2]$ such that $\mathfrak{Q}\subsetneq\mathfrak{P}$.

Now assume that $\mathfrak{P}$ and $\mathfrak{Q}$ are prime ideals of $V[X_1, X_2]$ with $\mathfrak{Q}\subsetneq\mathfrak{P}$ and $\mathfrak{m}$ is the maximal ideal of V. For the remainder of our discussion we separate the cases: $\mathfrak{Q}\cap V=\mathfrak{m}$; $\mathfrak{P}\cap V=0$; $\mathfrak{Q}\cap V=0$ and $\mathfrak{P}\cap V=\mathfrak{m}$.

Case (i) $\mathfrak{Q}\cap V=\mathfrak{m}$. For this situation we see that $\mathfrak{m}[X_1, X_2]\subseteq\mathfrak{Q}\subsetneq\mathfrak{P}$. It thus follows from (2.1) that $\mathrm{ht}(\mathfrak{P})=\mathrm{ht}(\mathfrak{m}[X_1, X_2])+\mathrm{ht}(\mathfrak{P}/\mathfrak{m}[X_1, X_2])$ and $\mathrm{ht}(\mathfrak{Q})=\mathrm{ht}(\mathfrak{m}[X_1, X_2])+\mathrm{ht}(\mathfrak{Q}/\mathfrak{m}[X_1, X_2])$. However the ring $V[X_1, X_2]/\mathfrak{m}[X_1, X_2]$ is isomorphic to $(V/\mathfrak{m})[X_1, X_2]$, whence $V[X_1, X_2]/\mathfrak{m}[X_1, X_2]$ is catenarian. Hence $\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})=\mathrm{ht}(\mathfrak{P}/\mathfrak{m}[X_1, X_2])-\mathrm{ht}(\mathfrak{Q}/\mathfrak{m}[X_1, X_2])$. Accordingly we have $\mathrm{ht}(\mathfrak{P})=\mathrm{ht}(\mathfrak{Q})+\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})$ and the desired result follows.

Case (ii) $\mathfrak{P}\cap V=0$. This time it is clear that $\mathfrak{Q}\cap V=0$. Put $S=V-\{0\}$. Then $\mathfrak{Q}\cap S=\phi$ and $\mathfrak{P}\cap S=\phi$. Therefore $\mathfrak{Q}_S$ and $\mathfrak{P}_S$ are prime ideals of $K[X_1, X_2]$ with $\mathfrak{Q}_S\subsetneq\mathfrak{P}_S$, where K is the quotient field of V. Since $K[X_1, X_2]$ is catenarian, we obtain $\mathrm{ht}(\mathfrak{P})=\mathrm{ht}(\mathfrak{P}_S)=\mathrm{ht}(\mathfrak{Q}_S)+\mathrm{ht}(\mathfrak{P}_S/\mathfrak{Q}_S)=\mathrm{ht}(\mathfrak{Q})+\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})$.

Case (iii) $\mathfrak{Q}\cap V=0$ and $\mathfrak{P}\cap V=\mathfrak{m}$. In this case we assume that $\mathfrak{Q}\neq 0$, because if $\mathfrak{Q}=0$, then the equality holds clearly. Further it is sufficient to show that $\mathrm{ht}(\mathfrak{P})\leqq\mathrm{ht}(\mathfrak{Q})+\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})$. Since $\mathrm{Dim}\, V[X_1, X_2]=3$, $\mathrm{ht}(\mathfrak{P})\leqq 3$. If $\mathrm{ht}(\mathfrak{P})\leqq 2$, then this inequality is clear because $\mathrm{ht}(\mathfrak{Q})\geqq 1$ and $\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})\geqq 1$. Hence from now on we assume $\mathrm{ht}(\mathfrak{P})=3$. If either $\mathrm{ht}(\mathfrak{Q})\geqq 2$ or $\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})\geqq 2$, then we can see that $\mathrm{ht}(\mathfrak{P})\leqq\mathrm{ht}(\mathfrak{Q})+\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})$. Thus suppose that $\mathrm{ht}(\mathfrak{Q})=1$ and $\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})=1$. Then it follows from (5.9) that $\mathfrak{Q}$ is a principal ideal. Put $\mathfrak{Q}=(f)$, where f is an element of $V[X_1, X_2]$ not contained in $\mathfrak{m}[X_1, X_2]$. Considering the natural mapping $V[X_1, X_2]\rightarrow V[X_1, X_2]/\mathfrak{m}[X_1, X_2]$, we put $\bar{\mathfrak{Q}}=(\mathfrak{Q}+\mathfrak{m}[X_1, X_2])/\mathfrak{m}[X_1, X_2]$ and $\bar{\mathfrak{P}}=\mathfrak{P}/\mathfrak{m}[X_1, X_2]$, and let $\bar{f}$ be the image of f. Then $\bar{\mathfrak{P}}$ is a prime ideal of $V[X_1, X_2]/\mathfrak{m}[X_1, X_2]$ and $\bar{\mathfrak{Q}}=(\bar{f})$. Moreover $\bar{\mathfrak{P}}$ is a minimal prime ideal of $\bar{\mathfrak{Q}}$ because $\mathrm{ht}(\mathfrak{P}/\mathfrak{Q})=1$. However $V[X_1, X_2]/\mathfrak{m}[X_1, X_2]$ is a noetherian ring by the proof of case (i). We now apply Krull's principal ideal theorem, which shows us that $\mathrm{ht}(\bar{\mathfrak{P}})=1$. It thus follows from (2.1) that $\mathrm{ht}(\mathfrak{P})=\mathrm{ht}(\mathfrak{m}[X_1, X_2])+\mathrm{ht}(\bar{\mathfrak{P}})=2$, because $\mathrm{ht}(\mathfrak{m}[X_1, X_2])=\mathrm{Ht}(\mathfrak{m})=\mathrm{Dim}\, V=1$. This leads us to a contradiction. Therefore we have proved the equality in case (iii). *q.e.d.*

As we state in Introduction, it is important for a generalization of the notion of Cohen-Macaulay modules to ask the following question: Let A be a quasi-local ring and M a non-zero, finite A-module. Then does the inequality $\mathrm{Gr}(M) \leq \mathrm{Dim}(A/\mathfrak{p})$ hold for all $\mathfrak{p} \in \mathrm{Att}(M)$? Mr. S. Itoh has showed me a negative answer. He has kindly allowed me to include his counterexample in this paper as appendix.

The following appendix is due to Mr. S. Itoh and the author would like to thank him for his kindness.

Appendix

Let A be a non-noetherian, quasi-local ring and $\mathfrak{m}$ the maximal ideal of A. Suppose that $\mathrm{Dim}\,A$ is finite. We fix a sequence $c_0, c_1,\dots, c_n,\dots$ of non-zero elements of $\mathfrak{m}$ such that $c_0A \subsetneqq (c_0, c_1)A \subsetneqq \cdots \subsetneqq (c_0, c_1,\dots, c_n)A \subsetneqq \cdots$. Put $B = A[X]_{(\mathfrak{m},X)}$ where X is an indeterminate. Let $\mathfrak{J}$ be an ideal of B generated by $\{c_0 + c_nX^n \mid n=1, 2,\dots\}$.

LEMMA 1. $\bigcap_n (X^n, \mathfrak{J})B \supsetneqq \mathfrak{J}$.

PROOF. It is clear that $c_0 \in \bigcap_n (X^n, \mathfrak{J})B$. We shall show that $c_0 \notin \mathfrak{J}$. Suppose contrarily that $c_0 \in \mathfrak{J}$. Then

$$(*) \qquad hc_0 = \textstyle\sum_{i=1}^m (c_0 + c_iX^i)f_i$$

for some $h, f_i \in A[X]$ such that $h(0)=1$. Comparing the constant terms of the equation $(*)$, we see $c_0 = \sum_{i=1}^m c_0 f_i(0)$. This shows that $f_i(0) \notin \mathfrak{m}$ for some i. Put $n = \mathrm{Min}\{i \mid f_i(0) \notin \mathfrak{m}\}$. We also put $h = \sum_j a_jX^j$ and $f_i = \sum_j b_{ij}X^j$ $(i=1, \dots, m)$ where $a_j, b_{ij} \in A$ and $a_0 = 1$. Comparing the coefficients of X^n of the equation $(*)$, we have $c_0a_n = \sum_{i=1}^n (b_{i,n-i}c_i + b_{in}c_0) + \sum_{i=n+1}^m b_{in}c_0$. Therefore $b_{n0}c_n \in (c_0,\dots, c_{n-1})A$. Since $b_{n0}(=f_n(0)) \notin \mathfrak{m}$, $c_n \in (c_0,\dots, c_{n-1})A$. This is a contradiction. Therefore $c_0 \notin \mathfrak{J}$. *q.e.d.*

Let Y, Z be indeterminates over $A[X]$ and put $C = A[X, Y, Z]_{(\mathfrak{m},X,Y,Z)}$. Note that the ring C is naturally isomorphic to $B[Y, Z]_{(\mathfrak{n},Y,Z)}$ where $\mathfrak{n}$ is the maximal ideal of B. We now put $B' = B/\mathfrak{J}$ and let $\mathfrak{n}'$ be the maximal ideal of B'. Further let f be a ring homomorphism $B[Y, Z] \to B'[Y, xY^{-1}]$ such that $f(Y) = Y$ and $f(Z) = xY^{-1}$ where $x = X \bmod \mathfrak{J}$. Then $(\mathfrak{n}', Y, xY^{-1})$ is a maximal ideal of $B'[Y, xY^{-1}]$ and $f^{-1}((\mathfrak{n}', Y, xY^{-1})) = (\mathfrak{n}, Y, Z)$ because $f((\mathfrak{n}, Y, Z)) \subseteq (\mathfrak{n}', Y, xY^{-1})$ and $B[Y, Z]/(\mathfrak{n}, Y, Z) \cong B'[Y, xY^{-1}]/(\mathfrak{n}', Y, xY^{-1}) \cong A/\mathfrak{m}$. Put $C' = B'[Y, xY^{-1}]_{(\mathfrak{n}',Y,xY^{-1})}$. Then we obtain a surjective ring homomorphism $C \to C'$ which sends X, Y and Z to x, Y and xY^{-1} respectively. We denote by $\mathfrak{J}_0$ the kernel of $C \to C'$.

LEMMA 2. *Y is $C/\mathfrak{J}_0$-regular and* $\bigcap_n (Y^n, \mathfrak{J}_0)C \supsetneqq \mathfrak{J}_0$.

PROOF. Since Y is $B'[Y, xY^{-1}]$-regular, Y is also C'-regular. Hence Y is $C/\mathfrak{I}_0$-regular because $C/\mathfrak{I}_0=C'$. We shall show $B'\subseteq C'$. Assume that b is an element of B' such that $b=0$ in C'. Then there is an element $\sum_{i\geq 0,j\geq 0} b_{ij}Y^i(xY^{-1})^j$ of $B'[Y, xY^{-1}]$ not contained in $(\mathfrak{n}', Y, xY^{-1})$ such that $(\sum_{i,j} b_{ij}Y^i(xY^{-1})^j)b=0$, where $b_{ij}\in B'$. We see that $b_{00}\notin \mathfrak{n}'$ and $(\sum_i b_{ii}x^i)b=0$. Since $\sum_i b_{ii}x^i\notin\mathfrak{n}'$, $b=0$ in B'. Therefore $B'\subseteq C'$. It thus follows from Lemma 1 and the equality $x=(xY^{-1})Y$ that $\bigcap_n Y^nC'\supseteq\bigcap_n x^nB'\neq 0$. This shows that $\bigcap_n(Y^n, \mathfrak{I}_0)C\supsetneqq\mathfrak{I}_0$.
q.e.d.

Let M be an A-module. We denote by $\mathrm{Ass}_A(M)$ the set of prime ideals $\mathfrak{p}$ of A which are minimal prime ideals of $\mathrm{Ann}(Am)$ for some $m\in M$. Note $\mathrm{Ass}_A(M)\subseteq\mathrm{Att}_A(M)$.

Let $Z_1,\dots, Z_r$ (where $r>\mathrm{Dim}\, A$) be distinct indeterminates over $A[X, Y, Z]$ and put $R=A[X, Y, Z, Z_1,\dots, Z_r]_{(\mathfrak{m},X,Y,Z,Z_1,\dots,Z_r)}$. Furthermore let $\mathfrak{M}$ be the maximal ideal of C. Then the ring R is isomorphic to $C[Z_1,\dots, Z_r]_{(\mathfrak{M},Z_1,\dots,Z_r)}$. Put $\mathfrak{I}'=\bigcap_n(Y^n, \mathfrak{I}_0)C$ and $\mathfrak{K}=(\mathfrak{I}_0, Z_1\mathfrak{I}',\dots, Z_r\mathfrak{I}')R$.

LEMMA 3. *There exists* $\mathfrak{P}\in\mathrm{Att}_R(R/\mathfrak{K})$ *such that* $\mathrm{Dim}\,(R/\mathfrak{P})<\mathrm{Gr}\,(R/\mathfrak{K})$. (*Therefore the R-module* $R/\mathfrak{K}$ *gives a counterexample to our question.*)

PROOF. Let c be an element of C. If $cY\in\mathfrak{I}_0$, then $c\in\mathfrak{I}_0$ because Y is $C/\mathfrak{I}_0$-regular. If $cY\in\mathfrak{I}'$, then $c\in\mathfrak{I}'$. In fact, we can write $cY=a_nY^n+b_n$ $(n=1, 2,\dots)$ where $a_n\in C$ and $b_n\in\mathfrak{I}_0$. Since $b_n=(c-a_nY^{n-1})Y\in\mathfrak{I}_0$ and Y is $C/\mathfrak{I}_0$-regular, we have $c-a_nY^{n-1}\in\mathfrak{I}_0$. Therefore $c\in\mathfrak{I}'$. It is now easy to see that Y is $C[Z_1,\dots, Z_r]/(\mathfrak{I}_0, Z_1\mathfrak{I}',\dots, Z_r\mathfrak{I}')$-regular. In particular, Y is $R/\mathfrak{K}$-regular.

We next prove that $\mathrm{Gr}\,(R/\mathfrak{K})=\mathrm{Gr}\,(C/\mathfrak{I}_0)+r$, that is, $\mathrm{Gr}_R\{\mathfrak{N}; R/\mathfrak{K}\}=\mathrm{Gr}_C\{\mathfrak{M}; C/\mathfrak{I}_0\}+r$ where $\mathfrak{N}$ is the maximal ideal of R. In fact, $R/(Y, \mathfrak{K})R=R/(Y, \mathfrak{I}_0)R=(C/(Y, \mathfrak{I}_0)C)[Z_1,\dots, Z_r]_{(\mathfrak{M}/(Y,\mathfrak{I}_0)C,Z_1,\dots,Z_r)}$. Therefore $\mathrm{Gr}_R\{\mathfrak{N}; R/\mathfrak{K}\}=\mathrm{Gr}_R\{\mathfrak{N}; R/(Y, \mathfrak{K})R\}+1$. Put $R'=(C/(Y, \mathfrak{I}_0)C)[Z_1,\dots, Z_r]_{(\mathfrak{M}/(Y,\mathfrak{I}_0)C,Z_1,\dots,Z_r)}$. Since $\{Z_1,\dots, Z_r\}$ is an R-sequence on R' composed of elements of $\mathfrak{N}$,

$$\mathrm{Gr}_R\{\mathfrak{N}; R/(Y, \mathfrak{K})R\}=\mathrm{Gr}_R\{\mathfrak{N}; R'/(Z_1,\dots, Z_r)R'\}+r=\mathrm{Gr}_C\{\mathfrak{M}; C/(Y, \mathfrak{I}_0)C\}+r,$$

and hence

$$\mathrm{Gr}_R\{\mathfrak{N}; R/\mathfrak{K}\} = \mathrm{Gr}_C\{\mathfrak{M}; C/(Y, \mathfrak{I}_0)C\} + r + 1 = \mathrm{Gr}_C\{\mathfrak{M}; C/\mathfrak{I}_0\} + r.$$

Finally, by Lemma 2, we can find an element u of $\mathfrak{I}'$ not contained in $\mathfrak{I}_0$. Then $\mathfrak{K}:_Ru\ni Z_1,\dots, Z_r$ and so $\mathfrak{K}:_Ru=(\mathfrak{I}_0:_Cu, Z_1,\dots, Z_r)R$. Let now $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{I}_0:_Cu$. Then $\mathfrak{P}=(\mathfrak{p}, Z_1,\dots, Z_r)R$ is a minimal prime ideal of $\mathfrak{K}:_Ru$ and therefore $\mathfrak{P}\in\mathrm{Ass}_R(R/\mathfrak{K})\subseteq\mathrm{Att}_R(R/\mathfrak{K})$. Since $R/\mathfrak{P}=A/\mathfrak{p}$, we conclude that $\mathrm{Dim}\,(R/\mathfrak{P})=\mathrm{Dim}\,(A/\mathfrak{p})\leqq\mathrm{Dim}\,A<r\leqq\mathrm{Gr}_R\{\mathfrak{N}; R/\mathfrak{K}\}=\mathrm{Gr}\,(R/\mathfrak{K})$.
q.e.d.

Summarizing the above discussion, we have the following:

THEOREM. *Let A be a non-noetherian quasi-local ring and $\mathfrak{m}$ the maximal ideal of A. Suppose that* $\operatorname{Dim} A$ *is finite. Then for sufficiently many distinct indeterminates $Z_1,\dots, Z_r$, there exists an ideal $\mathfrak{K}$ of $R=A[Z_1,\dots, Z_r]_{(\mathfrak{m},Z_1,\dots,Z_r)}$ such that* $\operatorname{Dim}(R/\mathfrak{P})<\operatorname{Gr}(R/\mathfrak{K})$ *for some* $\mathfrak{P}\in \operatorname{Att}_R(R/\mathfrak{K})$.

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Department of Management Science,
Faculty of Business and Commerce,
Hiroshima Shudo University