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# The $L^p$ approach to the Navier-Stokes equations with the Neumann boundary condition

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## 1. Introduction

Let $D$ be a bounded open set in $R^n$, $n \geq 3$, with smooth boundary $S$, and $\nu$ be the unit exterior normal to $S$. The motion of a viscous incompressible fluid in $D$ is described by the Navier-Stokes equation:

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u, \text{grad})u + \text{grad}\, q = f & \text{in} \quad D \times (0, T), \\ \text{div}\, u = 0 & \text{in} \quad D \times (0, T), \\ u(x, 0) = a(x) & \text{in} \quad D, \end{cases}$$

with the boundary condition:

$$(2) \qquad u(x, t) = 0 \qquad \text{on} \quad S \times (0, T).$$

Here $u(x, t) = (u_1(x, t), \dots, u_n(x, t))$, $q(x, t)$ and $f(x, t) = (f_1(x, t), \dots, f_n(x, t))$ denote the velocity, the pressure and the external force respectively, and $(u, \text{grad}) = u_j \partial / \partial x_j$.

So far, the above problem has been attacked mainly within the framework of the Hilbert space $(L^2(D))^n$. In this framework the existence and uniqueness, local in time, of strong solutions were established, when $n = 3$, by Kiselev and Ladyzhenskaya [9] under some regularity assumptions on the initial data. Then Kato and Fujita [5], [8] made these assumptions weaker and also proved similar but stronger results by the method of evolution equations in Hilbert spaces. Inoue and Wakimoto [7] extended the results of [5], [8] to the case when $n = 4$, 5. But the case $n \geq 6$ still remains open.

On the other hand, in [5], Fujita and Kato suggested the possibility of removing the regularity assumptions noticed above by passing from $L^2$ to general $L^p$ spaces. However, the existence of strong solutions in $L^p$ spaces is still not known, mainly because of the lack of knowledge about the $L^p$-theory of the Stokes system, i.e. the linearized version of the problem (1) and (2).

In this paper we consider in $(L^p(D))^n$, $n < p < \infty$, the equation (1) under the following boundary condition (the Neumann condition for 1-forms, see [3]):

$$\sigma(\delta, \nu)u = 0, \quad \sigma(\delta, \nu)du = 0 \qquad \text{on} \quad S\times(0, T), \tag{3}$$

where $d$ and $\delta$ denote the exterior differentiation and its formal adjoint respectively, acting on differential forms on $D$, and $\sigma(\delta, \nu)$ denotes the value at $\nu$ of the principal symbol $\sigma(\delta)$ of $\delta$. (Throughout this paper we identify vector fields and 1-forms by means of the standard Euclidean metric.) In 3-dimensional case (3) means that $u$ is tangential and $\mathrm{rot}\, u$ is perpendicular to $S$ at each time $t$. We shall establish the local existence and uniqueness of strong solutions of the problem (1) and (3) without any regularity assumptions on the initial data.

In Section 2, we give a brief survey on the decomposition of $(L^p(D))^n$ into the direct sum of solenoidal vector fields and gradients of scalar functions, which is a generalization of the well-known orthogonal decomposition theorem of $(L^2(D))^n$ (see [17]), namely,

$$(L^p(D))^n = X_p(D) \oplus G_p(D) \qquad \text{(direct sum)},$$

where

$$X_p(D) = \{u \in (L^p(D))^n;\ \delta u = 0 \text{ in } D,\ \sigma(\delta, \nu)u = 0 \text{ on } S\},$$

$$G_p(D) = \{u \in (L^p(D))^n;\ u = dq \text{ for some } q \in W^{1,p}(D)\}.$$

Since the details of the subject are presented in [6], we shall omit the proofs.

Section 3 is devoted to the investigation of the following elliptic boundary value problem, the Neumann problem for 1-forms:

$$\begin{cases} -\Delta u = f & \text{in } D, \\ \sigma(\delta, \nu)u = 0, \quad \sigma(\delta, \nu)du = 0 & \text{on } S, \end{cases} \tag{4}$$

where $-\Delta = d\delta + \delta d$ denotes the Laplacian acting on 1-forms on $D$. It will be shown that the Laplacian with the Neumann condition on $(L^p(D))^n$ leaves the space $X_p(D)$ invariant and hence generates a holomorphic semigroup on $X_p(D)$, which enables us to discuss the problem (1) and (3) on $L^p$ spaces. It is to be noted that the corresponding result is not known for the Stokes system except when $D$ is a half-space of $R^3$. See [13] in this respect.

Using the results obtained in Sections 2 and 3, we consider in Section 4 the problem (1) and (3) in the form of the following evolution equation in $X_p(D)$, $n<p<\infty$:

$$\begin{aligned} &\frac{du}{dt} + Au + P(u, \mathrm{grad})u = Pf, \quad t > 0, \\ &\qquad\qquad u(0) = a \in X_p(D), \end{aligned} \tag{5}$$

where $A=A_p$ is the restriction to $X_p(D)$ of the Laplacian with the Neumann condition, and $P=P_p$: $(L^p(D))^n \to X_p(D)$ is the projection along $G_p(D)$.

We shall mainly follow the discussion of [8] and prove the local existence and uniqueness of solutions of (5), which we shall call strong solutions of (1) and (3), for any $a \in X_p(D)$ under some assumptions on $Pf$.

In [4], Fabes, Lewis and Riviere discussed the equation (1) in $L^p$ spaces under various boundary conditions including (2) and (3) and proved the local existence and uniqueness of weak solutions. But the regularity property of their solutions is not clear.

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## 2. The decomposition theorem

In this section we review some results due to Fujiwara and Morimoto [6] on the direct sum decomposition of $(L^p(D))^n$, $1<p<\infty$, which will be needed later. In what follows we denote by $W^{s,p}(D)$ the Sobolev space of order $s$, with the norm $\|\cdot\|_{s,p}$. All functions considered in this paper are assumed to be real, unless otherwise specified. For the sake of convenience, elements in $(L^p(D))^n$ will be regarded as 1-forms. Thus, for example, $\delta u = -\operatorname{div} u$, $\sigma(\delta, \nu)u = -u \cdot \nu$, in the notation of vector analysis.

THEOREM 2.1. *Suppose that* $u \in (L^p(D))^n$ *and* $\delta u \in L^p(D)$. *Then the boundary value* $\sigma(\delta, \nu)u$ *makes sense and belongs to* $W^{-1/p,p}(S)$. *Further, there exists a constant* $C>0$ *independent of* $u$ *such that*

$$\|\sigma(\delta, \nu)u\|_{-1/p,p} \le C(\|u\|_{0,p} + \|\delta u\|_{0,p}). \tag{6}$$

Now, we set

$$X_p(D) = \{u \in (L^p(D))^n;\ \delta u = 0 \text{ in } D,\ \sigma(\delta, \nu)u = 0 \text{ on } S\}. \tag{7}$$

By the above theorem one can easily see that $X_p(D)$ is a closed subspace of $(L^p(D))^n$.

LEMMA 2.2. *The space of all* $u \in (C_0^\infty(D))^n$ *satisfying* $\delta u = 0$ *in* $D$ *is dense in* $X_p(D)$.

Let us now construct a bounded operator $P_p$: $(L^p(D))^n \to X_p(D)$ as follows.

For each $u \in (L^p(D))^n$ we can choose $q_j \in W^{1,p}(D)$, $j=1, 2$, such that

$$\text{(8)}\quad \text{(i)} \begin{cases} -\Delta q_1 = \delta u & \text{in } D, \\ q_1 = 0 & \text{on } S, \end{cases} \qquad \text{(ii)} \begin{cases} -\Delta q_2 = 0 & \text{in } D, \\ \partial q_2/\partial \nu = \sigma(\delta, \nu)(dq_1 - u) & \text{on } S. \end{cases}$$

The existence and uniqueness of $q_j$, $j=1, 2$, (up to an additive constant for $q_2$) is assured by the $L^p$-theory of elliptic boundary value problems (see [10], [11]). Thus, $P_p u \equiv u - d(q_1 + q_2)$ is well-defined and belongs to $X_p(D)$. The boundedness of $P_p$ follows from the well-known estimates of elliptic problems ([10], [11]). If we set

$$G_p(D) = \{u \in (L^p(D))^n;\ u = dq \quad \text{for some} \quad q \in W^{1,p}(D)\}, \tag{9}$$

then it is obvious from our construction of $P_p$ that $(L^p(D))^n$ is the sum of $X_p(D)$ and $G_p(D)$. More precisely we have

THEOREM 2.3. (i) $(L^p(D))^n = X_p(D) \oplus G_p(D)$, *(direct sum)*.
(ii) $P_p$ *is the projection onto* $X_p(D)$ *along* $G_p(D)$.

From this it follows that $G_p(D)$ is also a closed subspace of $(L^p(D))^n$. For the dual operator $P_p^*$ we have

THEOREM 2.4. $P_p^* = P_{p'}$, $p' = p/(p-1)$.

Using all these results one can obtain the following

THEOREM 2.5. (i) $X_p(D)^\perp = G_{p'}(D)$. (ii) $X_p(D)^* = X_{p'}(D)$.

Here $X_p(D)^\perp$ denotes the annihilator of $X_p(D)$.

## 3. The Neumann problem for 1-forms

Our aim in this section is to investigate the relation between the decomposition theorem of the preceding section and the boundary value problem:

$$\begin{cases} -\Delta u = f & \text{in} \quad D, \\ \sigma(\delta, \nu)u = 0, \quad \sigma(\delta, \nu)du = 0 & \text{on} \quad S. \end{cases} \tag{4}$$

Let us begin with the variational formulation of (4). We set

$$V = \{u \in (W^{1,2}(D))^n;\ \sigma(\delta, \nu)u = 0 \ \text{on} \ S\},$$

and consider the bilinear form:

$$a(u, v) = (du, dv) + (\delta u, \delta v), \qquad u, v \in V,$$

where $(u, v)$ denotes the $L^2$-inner product. Then a direct calculation leads us to

PROPOSITION 3.1. $a(u, v)$ *is coercive on* $V$, *i.e., there exist positive constants* $C_0$ *and* $C_1$ *such that*

$$a(u, u) \geq C_0\|u\|_{1,2}^2 - C_1\|u\|_{0,2}^2 \qquad \textit{for any} \quad u \in V. \tag{10}$$

Thus it follows from the regularity theorem for coercive forms (see [12]) that if $u \in V$ and $f \in (L^2(D))^n$ satisfy the equation

$$(11) \qquad a(u, v) = (f, v) \qquad \text{for any} \quad v \in V,$$

then $u$ is in $(W^{2,2}(D))^n$, and satisfies

$$(12) \qquad -\Delta u = f \qquad \text{in} \quad D,$$

with $-\Delta = d\delta + \delta d$. Multiplying both sides of (12) with $v \in V$ and integrating by parts one can show that

$$(13) \qquad \int_S \langle \sigma(\delta, \nu)du, v \rangle dS = 0 \qquad \text{for any} \quad v \in V.$$

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean metric of $R^n$. Now, we may assume that the normal vector $\nu$ is smoothly extended to a neighbourhood of $S$ in $R^n$. Since $\sigma(\delta, \nu)^2 du = 0$ by virtue of $\delta^2 = 0$, one can choose $v \in V$ which is equal to $\sigma(\delta, \nu)du$ near $S$. This and (13) imply that $\sigma(\delta, \nu)du = 0$ on $S$. Thus we have deduced the boundary value problem (4) from (11).

The $L^p$-theory of elliptic boundary value problems such as developed in [11] enables us to give the following

DEFINITION 3.2. We set

$$(14) \qquad D(B_p) = \{u \in (W^{2,p}(D))^n;\ \sigma(\delta, \nu)u = 0,\ \sigma(\delta, \nu)du = 0 \ \text{on} \ S\},$$

$$(15) \qquad B_p u = -\Delta u \qquad \text{for} \quad u \in D(B_p).$$

Note that $B_p$ is a densely defined closed operator on $(L^p(D))^n$.

The following lemma is of fundamental importance to our purposes.

LEMMA 3.3. *$B_p u \in X_p(D)$ if and only if $u \in D(B_p) \cap X_p(D)$.*

To prove this lemma we need the following

LEMMA 3.4. *Suppose $v \in (W^{2,p}(D))^n$ satisfies $\sigma(\delta, \nu)dv = 0$ on $S$. Then, $\sigma(\delta, \nu)\delta dv = 0$ on $S$.*

Note that, by Theorem 2.1, $\sigma(\delta, \nu)\delta dv$ makes sense as an element of $W^{-1/p,p}(S)$, since $\delta dv \in (L^p(D))^n$ satisfies $\delta(\delta dv) = \delta^2 dv = 0$ in $D$.

PROOF OF LEMMA 3.3. We first assume that $f = B_p u$ is in $X_p(D)$. It suffices to prove that $\delta u = 0$ in $D$. By definition, it is clear that $\delta f = 0$ in $D$, and $\sigma(\delta, \nu)f = 0$ on $S$. From this and Lemma 3.4, we have

$$(16) \qquad -\Delta(\delta u) = -\delta(\Delta u) = \delta f = 0 \qquad \text{in} \quad D,$$

and

$$(17)\qquad \begin{aligned}\sigma(\delta, v)d\delta u &= \sigma(\delta, v)(f-\delta du)\\ &= \sigma(\delta, v)f - \sigma(\delta, v)\delta du\\ &= 0 \qquad \text{on} \quad S.\end{aligned}$$

Since $\sigma(\delta, v)d\delta u = -\partial(\delta u)/\partial v$ in the usual notation, one concludes from (16) and (17) that $\delta u = \text{const.}$ in $D$. Therefore we have

$$(18)\qquad \begin{aligned}0 = (d\delta u, u) &= \int_S \langle \sigma(d, v)\delta u, u\rangle dS + (\delta u, \delta u)\\ &= -\int_S \langle \delta u, \sigma(\delta, v)u\rangle dS + (\delta u, \delta u)\\ &= (\delta u, \delta u).\end{aligned}$$

Thus $\delta u=0$ in $D$, and hence $u$ is in $X_p(D)$.

Conversely, suppose $u$ is in $D(B_p)\cap X_p(D)$, and set $f=B_p u$. It is obvious that

$$(19)\qquad \delta f = -\delta\Delta u = -\Delta(\delta u) = 0.$$

By Lemma 3.4, we have

$$(20)\qquad \begin{aligned}\sigma(\delta, v)f &= \sigma(\delta, v)\delta du + \sigma(\delta, v)d\delta u\\ &= \sigma(\delta, v)d\delta u\\ &= 0 \qquad \text{on} \quad S,\end{aligned}$$

since $\delta u=0$ in $D$. Thus $f$ belongs to $X_p(D)$. The proof is completed.

PROOF OF LEMMA 3.4. For each $q\in C^2(\bar{D})$, we have

$$(21)\qquad \begin{aligned}0 &= (\delta^2 du, q)\\ &= \int_S \langle \sigma(\delta, v)\delta du, q\rangle dS + (\delta du, dq)\\ &= \int_S \langle \sigma(\delta, v)\delta du, q\rangle dS + \int_S \langle \sigma(\delta, v)du, dq\rangle dS + (du, d^2q)\\ &= \int_S \langle \sigma(\delta, v)\delta du, q\rangle dS.\end{aligned}$$

Since $C^2(\bar{D})$ is dense in $W^{1,p'}(D)$, it follows from the surjectivity of the trace operator: $q\to q|_S$ from $W^{1,p'}(D)$ to $W^{1-1/p',p'}(S)=W^{1/p,p'}(S)$ that $\sigma(\delta, v)\delta du=0$ on $S$. This completes the proof.

Lemma 3.3 enables us to define a densely defined closed operator $A_p$ on $X_p(D)$

as follows.

$$\text{(22)} \qquad \begin{cases} D(A_p) = D(B_p) \cap X_p(D), \\ A_p u = B_p u = -\Delta u, \qquad \text{for} \quad u \in D(A_p). \end{cases}$$

THEOREM 3.5. *$P_p$ maps $D(B_p)$ into $D(A_p)$, and $B_pP_p = A_pP_p = P_pB_p$ on $D(B_p)$.*

PROOF. Let $u = w + dq$ be in $D(B_p)$ with $w = P_p u$. Since $u \in (W^{2,p}(D))^n$, the function $q$ is determined by

$$-\Delta q = \delta u \text{ in } D, \quad \text{and} \quad \partial q/\partial \nu = -\sigma(\delta, \nu)u = 0 \text{ on } S.$$

Thus we see $q \in W^{3,p}(D)$, and hence $w = u - dq \in (W^{2,p}(D))^n \cap X_p(D)$. Because $\sigma(\delta, \nu)dw = \sigma(\delta, \nu)du = 0$ on $S$, we have $w \in D(B_p) \cap X_p(D) = D(A_p)$, and

$$B_p u = B_p w + d(-\Delta q) = A_p w + d(-\Delta q),$$

from which we obtain, using Lemma 3.3,

$$P_p B_p u = B_p w = B_p P_p u = A_p w = A_p P_p u.$$

This completes the proof.

COROLLARY 3.6. $(\lambda - B_p)^{-1} P_p = P_p(\lambda - B_p)^{-1} = (\lambda - A_p)^{-1} P_p$, *for any $\lambda$ in the resolvent set of $B_p$.*

PROOF. To show that $(\lambda - A_p)^{-1}$ exists and is bounded on $X_p(D)$ it is sufficient to prove that $\lambda - A_p$ is surjective, since $A_p$ is a restriction of $B_p$. By assumption, for each $f \in X_p(D)$, there exists a unique element $v \in D(B_p)$ such that $f = (\lambda - B_p)v$. Thus we have, by Theorem 3.5,

$$\text{(23)} \qquad f = P_p f = P_p(\lambda - B_p)v = (\lambda - A_p)P_p v,$$

from which follows the existence of $(\lambda - A_p)^{-1}$.

Now, let us fix $f \in (L^p(D))^n$ and choose $v \in D(B_p)$ such that

$$\text{(24)} \qquad (\lambda - B_p)v = f.$$

Applying $P_p$ to both sides of (24) we obtain, by Theorem 3.5,

$$\text{(25)} \qquad P_p(\lambda - B_p)v = (\lambda - A_p)P_p v = (\lambda - B_p)P_p v = P_p f,$$

so that

$$\text{(26)} \qquad P_p v = P_p(\lambda - B_p)^{-1} f = (\lambda - A_p)^{-1} P_p f = (\lambda - B_p)^{-1} P_p f.$$

This completes the proof.

We shall now determine the dual operator $A_p^*$.

LEMMA 3.7. $B_p^* = B_{p'}$.

PROOF. By the regularity theorem for the Neumann problem the spectra of $B_p$ are independent of $p$, and hence are contained in $[0, \infty)$ since $B_2$ is non-negative. Thus, $T_p=(1+B_p)^{-1}$ is a bounded operator on $(L^p(D))^n$ for $1<p<\infty$. By an integration by parts,

$$(27) \qquad (T_p f, g) = a_1(T_p f, T_{p'} g) = (f, T_{p'} g),$$

for any $f \in (L^p(D))^n$ and $g \in (L^{p'}(D))^n$, where $a_1(u, v)=(du, dv)+(\delta u, \delta v)+(u, v)$. Thus, $T_p^*=T_{p'}$ so that $B_p^*=B_{p'}$.

THEOREM 3.8. $A_p^* = A_{p'}$.

PROOF. Let $v$ be in $D(A_{p'})$. Then, an integration by parts yields, for each $u \in D(A_p)$,

$$(28) \qquad \begin{aligned} (A_p u, v) = (\delta du, v) &= \int_S \langle \sigma(\delta, \nu) du, v \rangle dS + (du, dv) \\ &= (du, dv) = \int_S \langle \sigma(d, \nu) u, dv \rangle dS + (u, \delta dv) \\ &= -\int_S \langle u, \sigma(\delta, \nu) dv \rangle dS + (u, A_{p'} v) = (u, A_{p'} v). \end{aligned}$$

Thus we have proved $A_{p'} \subset A_p^*$.

Conversely, suppose that $v$ is in $D(A_p^*)$ and set $f=A_p^* v$. Then, for any $w=u+dq \in D(B_p)$ with $P_p w=u$, we have

$$(29) \qquad (B_p w, v) = (A_p u, v) = (u, f) = (w, f),$$

by Theorems 2.5 and 3.5. From this we see that $v$ is in $D(B_p^*) \cap X_{p'}(D)=D(B_{p'}) \cap X_{p'}(D)=D(A_{p'})$ and $A_p^* v=B_p^* v=B_{p'} v=A_{p'} v$. This completes the proof.

We are now ready to discuss the semigroups and fractional powers generated by $A_p$. In this paragraph $(L^p(D))^n$ is considered as a complex Banach space.

Before stating our results we note the following fact. As is well known (see [6]) there exists a neighbourhood $U$ of $S$ in $R^n$ such that

(30) $U$ and $U \cap D$ are diffeomorphic to $S \times (-\varepsilon, \varepsilon)$ and $S \times (0, \varepsilon)$ respectively, for some $\varepsilon>0$,

and

(31) for each $y' \in S$, the curve $y_n \to (y', y_n) \in S \times (-\varepsilon, \varepsilon)$ represents a straight line in $U$ which is perpendicular to $S$ at $y'$.

Choosing a system of local coordinates $y'=(y_1,\dots, y_{n-1})$ of $S$, one can consider $(y', y_n)=(y_1,\dots, y_{n-1}, y_n)$, $y_n\in(-\varepsilon, \varepsilon)$, as a system of local coordinates on $U$.

In this situation the boundary value problem (4) takes the following form:

$$(4)' \qquad \begin{cases} -\Delta u = f, \quad y_n > 0, \\ u_n(y', 0) = 0, \quad \dfrac{\partial u_k}{\partial y_n}(y', 0) = 0, \quad k = 1,\dots, n-1. \end{cases}$$

Now, let $(ds)^2=\sum_1^{n-1} g_{jk}(y', y_n)dy_j dy_k+(dy_n)^2$ be the representation of the Euclidean metric with respect to the coordinate $(y', y_n)$. Then the following lemma holds, the proof of which is easy and so is omitted.

LEMMA 3.9. *Let $u(t)$ be a solution of the boundary value problem*

$$(32) \qquad \begin{cases} \{\sum_1^{n-1} g^{jk}(y', 0)\xi_j\xi_k - (d/dt)^2\}u(t) = -\lambda e^{i\theta}u(t), \quad t > 0, \\ u_n(0) = 0, \quad (du_j/dt)(0) = 0, \quad 1 \le j \le n-1, \quad u(\infty) = 0, \end{cases}$$

*where $(g^{jk}(y', 0))$ is the inverse matrix of $(g_{jk}(y', 0))$. Then, $u(t)\equiv 0$ whenever $\lambda>0$, $\xi\in R^{n-1}$, $-\pi<\theta<\pi$.*

This lemma together with S. Agmon's trick ([1]) tells us that there exist for each $\omega$, $0<\omega<\pi/2$, constants $C_\omega>0$, $M_\omega>0$, such that each $\lambda$, $|\arg\lambda|\ge\omega$, $|\lambda|\ge M_\omega$ belongs to the resolvent set of $B_p$, and

$$(33) \qquad \|(\lambda - B_p)^{-1}\| < C_\omega/|\lambda|\,.$$

This implies the following

THEOREM 3.10. *$-B_p$ generates a holomorphic semigroup, $e^{-tB_p}$, on $(L^p(D))^n$.*

This theorem and Corollary 3.6, Theorem 3.8 lead us to

COROLLARY 3.11. *$-A_p$ generates a holomorphic semigroup, $e^{-tA_p}$, on $X_p(D)$. Furthermore, we have $P_p e^{-tB_p}=e^{-tB_p}P_p=e^{-tA_p}P_p$, and $(e^{-tA_p})^*=e^{-tA_{p'}}$.*

By the above results we can now discuss the fractional powers of $B_p$ and $A_p$. Without loss of generality we may assume that both $B_p$ and $A_p$ are invertible.

The following theorem plays an important role in the next section.

THEOREM 3.12. (i) $D(A_p^\alpha) = D(B_p^\alpha) \cap X_p(D)$,

(ii) $A_p^\alpha = B_p^\alpha$ *on* $D(A_p^\alpha)$ *for* $0 < \alpha < 1$.

PROOF. Let $u$ be in $D(A_p^\alpha)$ and set $v=A_p^\alpha u \in X_p(D)$. Then we have

$$w \equiv B_p^{-\alpha}v = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda + B_p)^{-1}v\,d\lambda \tag{34}$$
$$= \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda + A_p)^{-1}v\,d\lambda$$
$$= A_p^{-\alpha}v = u,$$

by a well-known formula for fractional powers of operators (see [16]). Thus $w=u \in D(B_p^\alpha) \cap X_p(D)$, and $A_p^\alpha u = B_p^\alpha u$.

Conversely, let $u$ be in $D(B_p^\alpha) \cap X_p(D)$ and set $B_p^\alpha u = v$. Then we have

$$A_p^{-\alpha}P_p v = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda + A_p)^{-1}P_p v\,d\lambda \tag{35}$$
$$= \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}P_p(\lambda + B_p)^{-1}v\,d\lambda$$
$$= P_p B_p^{-\alpha}v = P_p u = u.$$

Thus, $u \in D(A_p^\alpha)$ and hence $A_p^\alpha u = B_p^\alpha u$ by the first part of this proof.

PROPOSITION 3.13. $(A_p^\alpha)^* = A_{p'}^\alpha$ *for* $0 < \alpha < 1$.

PROOF. This is an immediate consequence of $(A_p^{-\alpha})^* = A_{p'}^{-\alpha}$, which follows from

$$\{(\lambda + A_p)^{-1}\}^* = (\lambda + A_p^*)^{-1} = (\lambda + A_{p'})^{-1}, \quad \lambda > 0.$$

## 4. The Navier-Stokes equation with the Neumann condition

In this section we fix $p$, $n<p<\infty$, and consider in $(L^p(D))^n$ the problem (1) and (3). For simplicity, we shall write $A_p = A$, $P_p = P$ and the $L^p$ norm will be denoted by $\|\cdot\|$. All functions considered in this section are assumed to be real.

Now, applying $P$ formally to both sides of (1), we obtain the following evolution equation in the Banach space $X_p = X_p(D)$.

$$\text{(I)} \qquad \begin{cases} \dfrac{du}{dt} + Au = Fu + Pf, \quad t > 0, \\ u(0) = a, \\ \text{where} \quad Fu = -P(u, \text{grad})u. \end{cases}$$

Our aim is to establish for an arbitrary $a \in X_p$ the local existence and uniqueness of strong solutions in the sense of the following definition.

DEFINITION 4.1. Let $Pf$ be in $C((0, T]; X_p)$. We shall say that $u(t)$ is a strong solution of (I) on $[0, T]$ if and only if

( i ) $u(t) \in C([0, T]; X_p) \cap C^1((0, T]; X_p)$, $u(0) = a$,

(ii) $u(t)$ is in $D(A)$ for each $t \in (0, T]$, and $Au(t) \in C((0, T]; X_p)$,

(iii) $\dfrac{du}{dt} + Au = Fu + Pf$ on $(0, T]$.

Let us put $u(t) = e^{\lambda t}v(t)$, $\lambda > 0$. Then $v(t)$ is a solution of

$$(\mathrm{I})' \qquad \begin{cases} \dfrac{dv}{dt} + (\lambda + A)v = e^{\lambda t}Fv + e^{-\lambda t}Pf, & t > 0, \\ v(0) = a. \end{cases}$$

As is shown in the previous section (see the proof of Lemma 3.7) $\lambda + A$ is invertible for any $\lambda > 0$. Therefore we shall assume that, for each $p$, $1 < p < \infty$, both $A_p$ and $B_p$ are invertible and

$$(36) \qquad \|e^{-tA_p}\| \le Ce^{-t}, \quad \|e^{-tB_p}\| \le Ce^{-t} \qquad \text{for some} \quad C = C_p > 0,$$

$$(37) \qquad \|A_p^\alpha e^{-tA_p}\| \le C_\alpha t^{-\alpha}, \quad \|B_p^\alpha e^{-tB_p}\| \le C_\alpha t^{-\alpha}, \qquad \text{for} \quad t > 0,\ 0 < \alpha \le 1,$$
$$\text{with some} \quad C_\alpha = C_{\alpha,p} > 0.$$

The factors $e^{\pm\lambda t}$ on the right hand side of (I)′ are irrelevant since our consideration is local in time.

We first consider the equation (I) in the form of the following integral equation.

$$(\mathrm{II}) \qquad u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}\{Fu(s) + Pf(s)\}ds.$$

We shall give, under some assumptions on $Pf$, a local existence and uniqueness result for (II) and then show that the solution of (II) thus obtained satisfies (I) if $Pf$ is Hölder continuous.

The following lemma is crucial for our purpose.

LEMMA 4.2. *There exists a constant $M > 0$ such that*

(i) $\|P(u, \mathrm{grad})v\| \le M\|A^{1/2}u\| \cdot \|A^{1/2}v\|$,

(ii) $\|A^{-1/4}P(u, \mathrm{grad})v\| \le M\|A^{1/4}u\| \cdot \|A^{1/2}v\|$,

*for any $u, v \in D(A^{1/2})$.*

PROOF. (i) As is shown in the previous section, the following holds with continuous injections.

$$(38) \qquad D(A) \subset D(B) \subset (W^{2,p}(D))^n, \quad D(A^{1/2}) \subset D(B^{1/2}).$$

According to a result of Seeley [15] we have

$$D(B^{\alpha}) = [D(B), (L^p(D))^n]_{1-\alpha}, \quad 0 < \alpha < 1. \tag{39}$$

Thus,

$$\begin{aligned} D(A^{1/2}) \subset D(B^{1/2}) &= [D(B), (L^p(D))^n]_{1/2} \subset [(W^{2,p}(D))^n, (L^p(D))^n]_{1/2} \\ &= (W^{1,p}(D))^n, \end{aligned} \tag{40}$$

with continuous injections. Here and hereafter we denote by $[X, Y]_{\theta}$, $0 \leq \theta \leq 1$, the complex interpolation space (Calderón [2]) of Banach spaces $X$ and $Y$. Furthermore, by the Sobolev imbedding theorem,

$$W^{1,p}(D) \subset C^{\beta}(\bar{D}), \quad \beta = 1 - n/p. \tag{41}$$

Now, let $\psi$ be in $X_{p'}$. Then, by (40), (41) and Hölder's inequality, we obtain

$$\begin{aligned} |(P(u, \mathrm{grad})v, \psi)| &= |((u, \mathrm{grad})v, \psi)| \\ &\leq C \sup_D |u(x)| \int_D |\nabla v(y)|\, |\psi(y)| dy \leq C\|u\|_{1,p}\|v\|_{1,p}\|\psi\|_{0,p'} \\ &\leq C\|A^{1/2}u\| \cdot \|A^{1/2}v\| \cdot \|\psi\|_{0,p'} \quad \text{for} \quad u, v \in D(A^{1/2}), \end{aligned} \tag{42}$$

from which follows (i) since $X_p^* = X_{p'}$ (Theorem 2.5).

(ii) We make use of the following facts:

$$B_p^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-tB_p} dt \qquad \text{for} \quad 0 < \alpha < 1, \quad 1 < p < \infty, \tag{43}$$

$$\int_0^{\infty} t^{\alpha-1} |e^{-tB_p}(x, y)| dt \leq C_{\alpha}/|x-y|^{n-2\alpha} \qquad \text{for} \quad (x, y) \in \bar{D} \times \bar{D}, \quad x \neq y, \quad 1 < p < \infty, \tag{44}$$

where $e^{-tB_p}(x, y)$ denotes the kernel function of $e^{-tB_p}$. (43) is verified easily by (36) and the well-known integral representation of $e^{-tB_p}$ whereas (44) will be proved in Appendix. Thus, for $u, v \in D(A^{1/2})$ and $\psi \in X_{p'}$,

$$\begin{aligned} |(A^{-1/4}P(u, \mathrm{grad})v, \psi)| &= |((u, \mathrm{grad})v, A_{p'}^{-1/4}\psi)| = |((u, \mathrm{grad})v, B_{p'}^{-1/4}\psi)| \\ &\leq C \int_0^{\infty} t^{-3/4} |((u, \mathrm{grad})v, e^{-tB_{p'}}\psi)| dt \\ &\leq C \iint_{D\times D} \frac{|u(x)| \cdot |\nabla v(x)| \cdot |\psi(y)|}{|x-y|^{n-1/2}} dx dy. \end{aligned} \tag{45}$$

Put $w(x) = \int_D \frac{|\psi(y)|}{|x-y|^{n-1/2}} dy$. By the Sobolev inequality we have

$$w \in (L^q(D))^n,\ q^{-1} = 1-(2n)^{-1}-p^{-1}, \quad \text{and} \quad \|w\|_{0,q} \le C\|\psi\|_{0,p'}$$

with a constant $C>0$ independent of $\psi$. Therefore, by Hölder's inequality,

$$(46) \qquad \begin{aligned} |(A^{-1/4}P(u, \operatorname{grad})v, \psi)| &\le C\| \, |u| \cdot |\nabla v| \, \|_{0,q'} \|w\|_{0,q} \\ &\le C\| \, |u| \cdot |\nabla v| \, \|_{0,q'} \|\psi\|_{0,p'} \le C\|u\|_{0,2n}\|v\|_{1,p}\|\psi\|_{0,p'} \\ &\le C\|u\|_{0,2n}\|A^{1/2}v\| \cdot \|\psi\|_{0,p'}, \end{aligned}$$

since $q'^{-1} = 1-q^{-1} = (2n)^{-1}+p^{-1}$. This estimate implies that when $p \ge 2n$,

$$(47) \qquad \|A^{-1/4}P(u, \operatorname{grad})\, v\| \le C\|u\| \cdot \|A^{1/2}v\| \le C\|A^{1/4}u\| \cdot \|A^{1/2}v\|.$$

When $n<p<2n$, we proceed as follows:

As is noted above, we have

$$(48) \qquad D(B^{1/2}) \subset (W^{1,p}(D))^n \subset (L^s(D))^n, \qquad \text{for any} \quad s \ge 1.$$

So, by (39) and the reiteration property of interpolation spaces,

$$(49) \qquad \begin{aligned} D(A^{1/4}) \subset D(B^{1/4}) &= [D(B^{1/2}), (L^p(D))^n]_{1/2} \\ &\subset [(L^s(D))^n, (L^p(D))^n]_{1/2} = (L^r(D))^n, \end{aligned}$$

where $r^{-1} = (2s)^{-1}+(2p)^{-1}$. Now put $s = pn/(p-n)$. Then $r = 2n$, so that by (49),

$$(50) \qquad \|A^{-1/4}P(u, \operatorname{grad})v\| \le C\|u\|_{0,2n}\|A^{1/2}v\| \le C\|A^{1/4}u\| \, \|A^{1/2}v\|.$$

This completes the proof of (ii).

We shall now give the existence result for the integral equation (II) by the use of the following iteration scheme,

$$(51) \qquad u_0(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}Pf(s)ds,$$

$$(52) \qquad u_{m+1}(t) = u_0(t) + \int_0^t e^{-(t-s)A}Fu_m(s)ds, \quad m \ge 0.$$

LEMMA 4.3. *Suppose that* $a \in X_p$, $Pf \in C((0, T]; X_p)$ *and*

$$(53) \qquad \|A^{-1/4}Pf(t)\| = o(t^{-3/4}) \quad \text{as} \quad t \longrightarrow 0.$$

*Then, each* $u_m(t)$ *in* (52) *is well-defined and belongs to* $C([0, T]; X_p) \cap C((0, T]; D(A^\alpha))$, $0<\alpha<3/4$. *Furthermore, there exist constants* $K_{\alpha m}$ *such that*

$$(54) \qquad \|A^\alpha u_m(t)\| \le K_{\alpha m}t^{-\alpha}, \quad 0 \le \alpha < 3/4.$$

PROOF. In view of (53) and (37), we have

$$\begin{aligned}\|A^\alpha u_0(t)\| &\le \|A^\alpha e^{-tA}a\| + \int_0^t \|A^\alpha e^{-(t-s)A}Pf(s)\|ds \\ &\le \|A^\alpha e^{-tA}a\| + \int_0^t \|A^{\alpha+1/4}e^{-(t-s)A}\|\cdot\|A^{-1/4}Pf(s)\|ds \\ &\le \|A^\alpha e^{-tA}a\| + C_\alpha N\int_0^t (t-s)^{-\alpha-1/4}s^{-3/4}ds \\ &\le K_{\alpha 0}t^{-\alpha},\end{aligned}$$

where

$$K_{\alpha 0} = \sup_{0<t\le T} t^\alpha\|A^\alpha e^{-tA}a\| + C_\alpha NB(3/4-\alpha,\ 1/4), \tag{55}$$

$$N = \sup_{0<t\le T} t^{3/4}\|A^{-1/4}Pf(t)\|. \tag{56}$$

Here $B(\cdot,\ \cdot)$ denotes the beta function. Suppose that (54) is valid for $u_0,\dots, u_m$. Then, it follows from Lemma 4.2 that

$$\begin{aligned}\|A^\alpha u_{m+1}(t)\| &\le \|A^\alpha u_0(t)\| + \int_0^t \|A^{\alpha+1/4}e^{-(t-s)A}\|\cdot\|A^{-1/4}Fu_m(s)\|ds \\ &\le K_{\alpha 0}t^{-\alpha} + C_\alpha M\int_0^t (t-s)^{-\alpha-1/4}\|A^{1/4}u_m(s)\|\cdot\|A^{1/2}u_m(s)\|ds \\ &\le \{K_{\alpha 0} + C_\alpha MB(3/4-\alpha,\ 1/4)K_{1/4,m}K_{1/2,m}\}t^{-\alpha}.\end{aligned} \tag{57}$$

Thus we may put

$$K_{\alpha,m+1} = K_{\alpha 0} + C_\alpha MB(3/4-\alpha,\ 1/4)K_{1/4,m}K_{1/2,m}. \tag{58}$$

This completes the proof.

Let us now put $k_m = \max(K_{1/4,m},\ K_{1/2,m})$. From (58) we obtain

$$k_{m+1} \le k_0 + C_1 M\beta k_m^2, \tag{59}$$

where $C_1 = \max(C_{1/4},\ C_{1/2})$ and $\beta = \max\{B(1/4,\ 1/4),\ B(1/2,\ 1/4)\}$. By an elementary calculation it is readily verified that if

$$k_0 < 1/(4C_1 M\beta), \tag{60}$$

then, for each $m>0$,

$$\begin{cases} k_m \le K \equiv \{1-(1-4C_1M\beta k_0)^{1/2}\}/(2C_1M\beta) < 1/(2C_1M\beta), \\ \|A^\gamma u_m(t)\| \le Kt^{-\gamma}, \quad \gamma = 1/2,\ 1/4, \end{cases} \tag{61}$$

so that, by (58),

(62) $$K_{\alpha m} \leq K_\alpha \equiv K_{\alpha 0} + C_\alpha MB(3/4 - \alpha, 1/4)K^2,$$

and

(63) $$\|A^\alpha u_m(t)\| \leq K_\alpha t^{-\alpha}, \qquad \text{for any} \quad m \geq 0,\ 0 \leq \alpha < 3/4.$$

Let us put

(64) $$w_{m+1}(t) \equiv u_{m+1}(t) - u_m(t) = \int_0^t e^{-(t-s)A}\{Fu_m(s) - Fu_{m-1}(s)\}ds,$$
$$(u_{-1}(s) = 0).$$

Because of the inequality (easily derived from Lemma 4.2)

(65) $$\|A^{-1/4}(Fu - Fv)\|$$
$$\leq M\{\|A^{1/2}u\| \cdot \|A^{1/4}(u - v)\| + \|A^{1/2}(u - v)\| \cdot \|A^{1/4}v\|\},$$

one can see that, for $0<\alpha<3/4$,

(66)

$$\|A^\alpha w_{m+1}(t)\|$$
$$\leq C_\alpha \int_0^t (t-s)^{-\alpha-1/4} M\{\|A^{1/2}u_m(s)\| \cdot \|A^{1/4}w_m(s)\| + \|A^{1/4}u_{m-1}(s)\| \cdot \|A^{1/2}w_m(s)\|\}ds$$
$$\leq C_\alpha MK \int_0^t (t-s)^{-\alpha-1/4}\{s^{-1/2}\|A^{1/4}w_m(s)\| + s^{-1/4}\|A^{1/2}w_m(s)\|\}ds.$$

From this we obtain, by induction on $m$,

(67) $$\|A^\gamma w_m(t)\| \leq (2C_1 MK\beta)^m K t^{-\gamma}, \quad \gamma = 1/2,\ 1/4,$$

so that

(68) $$\|A^\alpha w_{m+1}(t)\| \leq 2C_\alpha MK^2(2C_1MK\beta)^m B(3/4 - \alpha, 1/4)t^{-\alpha}, \quad 0 \leq \alpha < 3/4.$$

Since $2C_1MK\beta<1$ by (61), we see from (68) that $\sum_m \|A^\alpha w_m(t)\|$ converges uniformly on compact subsets of $(0, T]$ and so does $\sum_m \|w_m(t)\|$ uniformly on $[0, T]$. Thus, there exist $u$ in $C([0, T]; X_p)$ and $v_\alpha$ in $C((0, T]; X_p)$ such that

(69) $$u_m(t) \longrightarrow u(t) \quad \text{in} \quad X_p \quad \text{uniformly on} \quad [0, T],$$

and

(70) $A^\alpha u_m(t) \longrightarrow v_\alpha(t)$ in $X_p$ uniformly on compact subsets of $(0, T]$, $0<\alpha<3/4$.

Since $A^\alpha$ is a closed operator we see that $v_\alpha(t) = A^\alpha u(t)$, and hence $A^\alpha u \in C((0, T]; X_p)$, $0<\alpha<3/4$. Moreover, by (63) and (61), we have

(71) $\|A^{\gamma}u(t)\| \leq Kt^{-\gamma}$, $\gamma = 1/4, 1/2$, $\|A^{\alpha}u(t)\| \leq K_{\alpha}t^{-\alpha}$, $0 \leq \alpha < 3/4$.

Combining (63) and (71) with the inequality (easily derived from Lemma 4.2),

$$\|Fu - Fv\| \leq M(\|A^{1/2}u\| + \|A^{1/2}v\|)\,\|A^{1/2}(u-v)\|, \tag{72}$$

and taking Lemma 4.2 into account, one obtains

$$\begin{cases} Fu_m(t) \longrightarrow Fu(t) \quad \text{on} \quad (0, T], \\ \|A^{-1/4}Fu_m(t)\| \leq Ct^{-3/4} \quad \text{with} \quad C > 0 \quad \text{independent of} \quad m. \end{cases} \tag{73}$$

Applying the dominated convergence theorem to the scheme (52) one concludes that

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}\{Fu(s) + Pf(s)\}ds \quad \text{on} \quad [0, T]. \tag{74}$$

Thus we have constructed a solution of (II) under the assumption (60). In view of (53), (55), (56) and the fact that

$$t^{\alpha}\|A^{\alpha}e^{-tA}a\| \longrightarrow 0, \quad \text{as} \quad t \longrightarrow 0, \qquad \text{for each} \quad a \in X_p, \tag{75}$$

we have only to choose $T>0$ sufficiently small in order for (60) to be valid. Thus we have proved the following

THEOREM 4.4. *For any $a \in X_p$ and any $Pf \in C((0, \infty); X_p)$ satisfying* (53), *there exist a $T>0$ and a solution $u(t)$ of* (II) *belonging to $C([0, T]; X_p) \cap C((0, T]; D(A^{\alpha}))$ for any $\alpha$, $0<\alpha<3/4$.*

The solution $u(t)$ constructed above satisfies

$$\|A^{1/2}u(t)\| = o(t^{-1/2}), \quad \|A^{1/4}u(t)\| = o(t^{-1/4}), \quad t \longrightarrow 0, \tag{76}$$

as will be seen from the fact that we can make $k_0$ in (60), and hence $K$ in (61), arbitrarily small by choosing $T>0$ small. This observation leads us to the following definition of the function space $S[0, T]$, in which the uniqueness result is to be shown.

DEFINITION 4.5. Let $u(t)$ be continuous on $[0, T]$ with values in $X_p$. We say that $u(t)$ is in $S[0, T]$ if and only if $A^{1/2}u(t)$ is continuous on $(0, T]$ and satisfies $\|A^{1/2}u(t)\| = o(t^{-1/2})$, $t \to 0$.

Note that $\|A^{1/4}u(t)\| = o(t^{-1/4})$ holds for any $u(t)$ in $S[0, T]$ by the well-known moment inequality (see [16]),

$$\|A^{1/4}v\| \leq C\|A^{1/2}v\|^{1/2} \cdot \|v\|^{1/2}. \tag{77}$$

THEOREM 4.6. *The solution of* (II) *is unique within the class $S[0, T]$.*

PROOF. Let $u, v \in S[0, T]$ be two solutions of (II) corresponding to the same data. Then, since

$$w(t) \equiv u(t) - v(t) = \int_0^t e^{-(t-s)A}\{Fu(s) - Fv(s)\}ds, \tag{78}$$

we have, by (65),

$$\|A^\gamma w(t)\| \le C_1 M K_0 \int_0^t (t-s)^{-\gamma-1/4}(s^{-1/4}\|A^{1/2}w(s)\| + s^{-1/2}\|A^{1/4}w(s)\|)ds \tag{79}$$

for $\gamma=1/2, 1/4$. Here we have chosen $K_0$ so that $\|A^\gamma u(t)\| \le K_0 t^{-\gamma}$, $\|A^\gamma v(t)\| \le K_0 t^{-\gamma}$ for $\gamma=1/2, 1/4$. Since both $u$ and $v$ are in $S[0, T]$, we can choose $T_0>0$ so that (79) holds on $(0, T_0]$ with $K_0$ sufficiently small. By induction on $m$ it is easily shown from (79) that

$$\|A^\gamma w(t)\| \le (2C_1 M\beta K_0)^m 2K_0 t^{-\gamma} \quad \text{on} \quad (0, T_0]. \tag{80}$$

Since we may assume $2C_1 M\beta K_0 < 1$, it follows from (80) that $w(t)=0$ on $[0, T_0]$. Repeating the above argument for $t \ge T_0$, we can choose $T_j$, $j \ge 1$, such that $w(t)=0$ on $[\sum_{k=0}^{j} T_k, \sum_{k=0}^{j+1} T_k]$, $j \ge 0$. Since $A^{1/2}u(t)$ and $A^{1/2}v(t)$ are continuous on $(0, T]$, we see easily that $\{T_j\}$ is bounded away from 0. Thus we may conclude that $w(t)=0$ on $[0, T]$.

Finally we shall prove the following theorem concerning the equation (I).

THEOREM 4.7. *If, in addition to the assumptions of Theorem* 4.4, *Pf is Hölder continuous on* $[\varepsilon, T]$ *for any* $\varepsilon>0$, *then the solution* $u(t)$ *of* (II) *is the solution of* (I) *in the sense of Definition* 4.1.

PROOF. As is easily verified $u(t)$ satisfies for each $\varepsilon>0$,

$$u(t) = e^{-(t-\varepsilon)A}u(\varepsilon) + \int_\varepsilon^t e^{-(t-s)A}\{Fu(s)+Pf(s)\}ds \quad \text{on} \quad [\varepsilon, T]. \tag{81}$$

So, our assertion is true if we show the Hölder continuity of $Fu(t)$ on $[\varepsilon, T]$, which, in view of (72), follows if $A^{1/2}u(t)$ is Hölder continuous. Therefore, in the following, we prove the Hölder continuity of $A^{1/2}u(t)$ on $[\varepsilon, T]$.

Put $u(t)=u_0(t)+w(t)$, where

$$u_0(t) = e^{-tA}a + \int_0^t e^{-(t-s)A}Pf(s)ds, \tag{82}$$

$$w(t) = \int_0^t e^{-(t-s)A}Fu(s)ds. \tag{83}$$

The first term on the right hand side of (82) is estimated as follows.

(84)

$$\|A^{1/2}(e^{-(t+h)A}-e^{-tA})a\| = \|(e^{-hA}-I)A^{1/2}e^{-tA}a\|$$

$$\leq \|(e^{-hA}-I)A^{-\alpha}\|\cdot\|A^{\alpha+1/2}e^{-tA}a\| \leq C_\alpha h^\alpha \varepsilon^{-\alpha-1/2} \quad \text{for} \quad \varepsilon\leq t\leq t+h,\ 0<\alpha\leq 1/2.$$

Note that here (and hereafter) we use $\|(e^{-hA}-I)A^{-\alpha}\|\leq C_\alpha h^\alpha$, which is easily checked by an elementary calculation.

For the second term of (82) we have

$$(85)\quad A^{1/2}\int_0^{t+h} e^{-(t+h-s)A}Pf(s)ds - A^{1/2}\int_0^t e^{-(t-s)A}Pf(s)ds$$

$$= \int_0^t A^{1/2}\{e^{-(t+h-s)A} - e^{-(t-s)A}\}Pf(s)ds + \int_t^{t+h} A^{1/2}e^{-(t+h-s)A}Pf(s)ds$$

$$\equiv I_1 + I_2,$$

so that

$$(86)\quad \|I_1\| \leq \int_0^t \|A^{1/2}(e^{-hA}-I)e^{-(t-s)A}Pf(s)\|ds$$

$$\leq \|(e^{-hA}-I)A^{-\alpha}\|\int_0^t \|A^{\alpha+3/4}e^{-(t-s)A}\|\cdot\|A^{-1/4}Pf(s)\|ds$$

$$\leq C_\alpha h^\alpha N\int_0^t (t-s)^{-\alpha-3/4}s^{-3/4}ds$$

$$\leq C_\alpha h^\alpha t^{-\alpha-1/2}B(1/4-\alpha,\ 1/4)$$

$$\leq C_\alpha h^\alpha \varepsilon^{-\alpha-1/2}B(1/4-\alpha,\ 1/4) \quad \text{for} \quad \varepsilon\leq t\leq t+h, \quad 0<\alpha<1/4,$$

and

$$(87)\qquad \|I_2\| \leq \int_t^{t+h} \|A^{1/2}e^{-(t+h-s)A}Pf(s)\|ds$$

$$\leq \int_t^{t+h} \|A^{3/4}e^{-(t+h-s)A}\|\ \|A^{-1/4}Pf(s)\|ds$$

$$\leq C\int_t^{t+h} (t+h-s)^{-3/4}ds(\sup_{\varepsilon<t<T}\|A^{-1/4}Pf(t)\|)$$

$$\leq C_\varepsilon h^{1/4} \quad \text{for} \quad \varepsilon\leq t\leq t+h.$$

Combining (86) and (87) with (84) we see that $A^{1/2}u_0(t)$ is Hölder continuous on $[\varepsilon, T]$ with exponent $\alpha$, $0<\alpha<1/4$.

The estimation of (83) is carried out as follows. Put

$$(88)\qquad A^{1/2}w(t+h) - A^{1/2}w(t) = \int_0^t A^{1/2}\{e^{-(t+h-s)A} - e^{-(t-s)A}\}Fu(s)ds$$
$$+ \int_t^{t+h} A^{1/2}e^{-(t+h-s)A}Fu(s)ds$$
$$\equiv I_3 + I_4.$$

Then, by virtue of (76), we have, with some $K>0$,

$$(89)\quad \|I_3\| \leq \|(e^{-hA}-I)A^{-\alpha}\| \int_0^t \|A^{\alpha+3/4}e^{-(t-s)A}\| \cdot \|A^{-1/4}Fu(s)\| ds$$
$$\leq C_\alpha h^\alpha \int_0^t (t-s)^{-\alpha-3/4}MK^2 s^{-3/4} ds$$
$$\leq C_\alpha MK^2 \varepsilon^{-\alpha-1/2} B(1/4-\alpha,\, 1/4)h^\alpha, \quad \text{for} \quad \varepsilon \leq t \leq t+h, \quad 0<\alpha<1/4,$$

and

$$(90)\qquad \|I_4\| \leq \int_t^{t+h} \|A^{3/4}e^{-(t+h-s)A}\| \cdot \|A^{-1/4}Fu(s)\| ds$$
$$\leq C\int_t^{t+h} (t+h-s)^{-3/4}K^2 M s^{-3/4} ds$$
$$\leq CK^2 M t^{-3/4} \int_t^{t+h} (t+h-s)^{-3/4} ds$$
$$\leq CK^2 M \varepsilon^{-3/4} h^{1/4} \qquad \text{for} \quad \varepsilon \leq t \leq t+h.$$

Thus $A^{1/2}w(t)$ is also Hölder continuous on $[\varepsilon, T]$ with exponent $\alpha$, $0<\alpha<1/4$. This completes the proof.

REMARK 4.8. Kato and Fujita [8] treated problem (1) and (2) in the form of the equation (I) in $X_2$, $n=3$, with $A$ denoting the Stokes operator. They proved the local existence and uniqueness under the assumption $a \in D(A^{1/4})$. This assumption is needed because, in their case, one must require $u \in D(A^{3/4})$ in order to obtain a good estimate for the nonlinear term $Fu$ whereas we have only to require $u \in D(A^{1/2})$ in our case (see Lemma 4.2 in this note).

**Appendix**

Here we prove the inequality (44), i.e.,

$$(44)\quad \int_0^\infty t^{\alpha-1}|U(t,x,y)|dt \leq C_\alpha/|x-y|^{n-2\alpha}, \quad (x,y)\in \bar{D}\times\bar{D}, \quad x\neq y,\ 0<\alpha<1,$$

where $U(t, x, y)$ denotes the kernel function of $e^{-tB_p}$, $1<p<\infty$.

For $0<t\leq 1$, the following estimate is known (see [14]).

$$|U(t, x, y)| \leq Ct^{-n/2}\exp\{-|x-y|^2/ct\}. \tag{91}$$

Therefore, denoting $r=|x-y|\neq 0$, we obtain

$$\begin{aligned}\int_0^1 t^{\alpha-1}|U(t, x, y)|dt &\leq C\int_0^1 t^{\alpha-1-n/2}\exp(-r^2/ct)dt \\ &\leq C\int_0^\infty t^{\alpha-1-n/2}\exp(-r^2/ct)dt \leq C_\alpha r^{2\alpha-n}.\end{aligned} \tag{92}$$

For $t>1$, we proceed as follows. Let $0<\lambda_1\leq\lambda_2\leq\cdots$ be the eigenvalues of $B_2$ and $\{\phi_m\}$ be the corresponding orthonormal system of eigenforms. (Note that $B_p$ is assumed to be invertible by adding a positive constant to the Laplacian.) As is well known $U(t, x, y)$ admits the following expansion,

$$U(t, x, y) = \textstyle\sum_{m=1}^\infty \exp(-\lambda_m t)\phi_m(x)\otimes\phi_m(y), \tag{93}$$

so that we obtain for $t>1$,

(94)

$$\begin{aligned}&|U(t, x, y)| \\ &\leq \textstyle\sum_{m=1}^\infty \exp(-\lambda_m t)|\phi_m(x)|\cdot|\phi_m(y)| \\ &\leq \{\textstyle\sum_{m=1}^\infty \exp(-\lambda_m t)|\phi_m(x)|^2\}^{1/2}\{\sum_{m=1}^\infty \exp(-\lambda_m t)|\phi_m(y)|^2\}^{1/2} \\ &\leq \exp(-\lambda_1(t-1))\{\textstyle\sum_{m=1}^\infty \exp(-\lambda_m)|\phi_m(x)|^2\}^{1/2}\times\{\sum_{m=1}^\infty \exp(-\lambda_m)|\phi_m(y)|^2\}^{1/2} \\ &= \exp(-\lambda_1(t-1))\{\mathrm{Tr}\, U(1, x, x)\}^{1/2}\{\mathrm{Tr}\, U(1, y, y)\}^{1/2} \\ &\leq C\exp(-\lambda_1 t),\end{aligned}$$

since $D$ is bounded and $U(t, x, y)$ is smooth on $(0, \infty)\times\bar{D}\times\bar{D}$. Here $\mathrm{Tr}\, S$ denotes the trace of a matrix $S$. Thus,

$$\int_1^\infty t^{\alpha-1}|U(t, x, y)|dt \leq C\int_0^\infty t^{\alpha-1}\exp(-\lambda_1 t)dt = C_\alpha\Gamma(\alpha). \tag{95}$$

Combining (92) with (95) we therefore obtain

$$\begin{aligned}\int_0^\infty t^{\alpha-1}|U(t, x, y)|dt &\leq C_1/|x-y|^{n-2\alpha} + C_2 \\ &\leq C_3/|x-y|^{n-2\alpha},\end{aligned} \tag{96}$$

since $D$ is bounded. This completes the proof of (44).

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