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Relative invariants of prehomogeneous vector spaces and a realization of certain unitary representations I

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1. Introduction

The purpose of this paper is to describe the unitary representations of some real semi-simple Lie groups on the spaces of solutions for certain differential equations.

We are concerned with a Lie group G satisfying the following two conditions:

1. If $\mathfrak{g}$ is the Lie algebra of G, then $\mathfrak{g}$ has a $\boldsymbol{Z}$-graded decomposition $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$.

2. If G_0 is the subgroup of G corresponding to $\mathfrak{g}_0$, then the real prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ possesses a relative invariant.

We take the suitable regular or singular orbits of $(G_0, \mathfrak{g}_1)$ and construct the Hilbert spaces of holomorphic functions on G/K by means of the Fourier-Laplace transform of the functions supported on these orbits. To construct the irreducible and unitary representations of G we use R. A. Kunze's reproducing kernel method [11]. The key of this construction is the Fourier transform of the relative invariant of $(G_0, \mathfrak{g}_1)$, which was also the key in [1], [14], and is studied from a new point of view in [9].

We make some bibliographic comments.

In [4], [5], and [6] Harish-Chandra constructed a certain class of representations of a simply connected real semi-simple Lie group G whose associated symmetric space G/K is hermitian. This class includes the holomorphic discrete series. Rossi and Vergne [12] and Wallach [15], [16] have studied the analytic continuation of the holomorphic discrete series for the scalar case. Furthermore in [12] it is shown that certain of these representations can be realized on the Hardy type Hilbert spaces associated with various boundary orbits in G/K. For the general case similar results were obtained by Inoue [7]. For the groups associated with classical hermitian symmetric spaces of tube type, all these representations were obtained by Gross and Kunze [2], [3] by considering the generalized gamma functions. For the conformal group $SU(2, 2)$ Jakobsen and Vergne constructed the irreducible unitary representations on the solution spaces for wave and Dirac operators [8]. We see these results from the viewpoint of prehomogeneous vector spaces.

For the sake of simplicity we restrict here our attention only to $Sp(n, \boldsymbol{R})$ and $SU(n, n)$.

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2. The situation

Let $G=Sp(n, \boldsymbol{R})$ and let K be its maximal compact subgroup. It is well known that the hermitian symmetric space G/K is realized as an unbounded model:

$$D = \{z = x + iy;\ x, y \in M(n, \boldsymbol{R}),\ {}^t x = x,\ {}^t y = y,\ y \gg 0\}.$$

We shall denote the space of all $n \times n$ real symmetric matrices by $S(n)$, and the cone of all positive definite symmetric matrices by $C(n)$. Then we can write $D=S(n)+iC(n)$. For any $z \in D$ and $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ we put $g \cdot z=(az+b)\cdot(cz+d)^{-1}$.

Let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g}$ has the following decomposition:

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \qquad \text{with} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad (\mathfrak{g}_i=0 \text{ for } |i| \geqq 2)$$

where

$$\mathfrak{g}_{-1} = \left\{\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix};\ x \in S(n)\right\}, \quad \mathfrak{g}_1 = \left\{\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix};\ k \in S(n)\right\},$$

$$\mathfrak{g}_0 = \left\{\begin{pmatrix} A & 0 \\ 0 & -{}^t A \end{pmatrix};\ A \in M(n, \boldsymbol{R})\right\}.$$

Let G_0 be the subgroup of G corresponding to $\mathfrak{g}_0$, i.e.,

$$G_0 = \left\{\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix};\ a \in GL(n, \boldsymbol{R})\right\}.$$

Then, by the adjoint action, the pair $(G_0, \mathfrak{g}_1)$ is a real irreducible prehomogeneous vector space which possesses an irreducible relative invariant f defined by

$$f\left(\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\right) = \det k,$$

and $C(n)$ is one of the regular orbits of this prehomogeneous vector space.

3. Unitary representations (regular case)

We first prove an integral formula which plays an important role in constructing the representations on the spaces of holomorphic functions on D.

PROPOSITION 3.1. *If* $\rho>(n-1)/2$, *then*

$$\int_{C(n)} \exp(-\mathrm{Tr}\, ky)(\det k)^{\rho-(n+1)/2}dk$$
$$= \pi^{n(n-1)/4}\prod_{j=1}^{n}\Gamma(\rho-(j-1)/2)(\det y)^{-\rho}$$

for $y\in C(n)$, *where* $dk=\prod_{i\geqq j}dk_{ij}$.

PROOF. Since $y\in C(n)$, there exists $p\in M(n, \boldsymbol{R})$ such that $y={}^tpp$. Then

$$\int_{C(n)} \exp(-\mathrm{Tr}\, ky)(\det k)^{\rho-(n+1)/2}dk$$
$$= \int_{C(n)} \exp(-\mathrm{Tr}\, k^tpp)(\det k)^{\rho-(n+1)/2}dk$$
$$= (\det p)^{-2\rho}\int_{C(n)} \exp(-\mathrm{Tr}\, k)(\det k)^{\rho-(n+1)/2}dk$$
$$= (\det y)^{-\rho}\int_{C(n)} \exp(-\mathrm{Tr}\, k)(\det k)^{\rho-(n+1)/2}dk.$$

We change the integration variable from k_{11} to $\det k$. Let k_1 be the minor determinant given by taking off the first row and the first column from k and let $v=(k_{12},\dots, k_{1n})$. Then

$$(\det k_1)^{-1}(\det k) = k_{11} - {}^tvk_1^{-1}v.$$

Hence we get

$$\int_{C(n)} \exp(-\mathrm{Tr}\, k)(\det k)^{\rho-(n+1)/2}dk$$
$$= \int_{C(n-1)} \exp(-\mathrm{Tr}\, k_1)(\det k_1)^{-1}dk_1\int_0^\infty r^{\rho-(n+1)/2}\exp(-(\det k_1)^{-1}r)dr$$
$$\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(-{}^tvk_1^{-1}v)dk_{12}\cdots dk_{1n},$$

where $C(n-1)$ denotes the cone of all $(n-1)\times(n-1)$ positive definite symmetric matrices and dk_1 denotes the Lebesgue measure on it.

It is well known that

$$\int_0^\infty r^{\rho-(n+1)/2}\exp(-(\det k_1)^{-1}r)dr = \Gamma(\rho-(n-1)/2)(\det k_1)^{\rho-(n-1)/2},$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(-{}^tvk_1^{-1}v)dk_{12}\cdots dk_{1n} = \pi^{(n-1)/2}(\det k_1)^{1/2}.$$

Thus we get

$$\int_{C(n)} \exp(-\mathrm{Tr}\, k)(\det k)^{\rho-(n+1)/2} dk$$
$$= \pi^{(n-1)/2}\, \Gamma(\rho-(n-1)/2) \int_{C(n-1)} \exp(-\mathrm{Tr}\, k_1)(\det k_1)^{\rho-n/2} dk_1.$$

By induction on n we complete the proof.

By analytic continuation and changing the parameter we get the following corollary:

COROLLARY 3.2. *If* $\alpha>-1$, *then*

$$\int_{C(n)} \exp(i\,\mathrm{Tr}\, k(z-w^*))(\det k)^{\alpha} dk$$
$$= 2^{-n\alpha-n(n+1)/2}\pi^{n(n-1)/4} \prod_{j=1}^{n} \Gamma(\alpha+(j+1)/2) \det((z-w^*)/2i)^{-\alpha-(n+1)/2},$$

for $z, w \in D$.

This formula is the Fourier-Laplace transform of the relative invariant and obtained also by the method of micro-local calculus.

Let $\tilde{G}$ be the universal covering group of G. We can define $J_\alpha(\tilde{g}, z) := \det(cz+d)^{\alpha+(n+1)/2}$ for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $z \in D$, $\alpha>-1$ by choosing a branch of the simply connected manifold $\tilde{G}\times D$ such that $J_\alpha(1, z)=1$. Then it is easy to check that J_α satisfies the following conditions:

(3.1) $J_\alpha(\tilde{g}, z)$ is a holomorphic function on D for any fixed $\tilde{g} \in \tilde{G}$,

(3.2) $J_\alpha(1, z)=1$,

(3.3) $J_\alpha(\tilde{g}_1\tilde{g}_2, z)=J_\alpha(\tilde{g}_1, \tilde{g}_2\cdot z)J_\alpha(\tilde{g}_2, z)$.

And let $K_\alpha(z, w) := \det((z-w^*)/2i)^{-\alpha-(n+1)/2}$ for $z, w \in D, \alpha>-1$. Then K_α satisfies the following conditions:

(3.4) $K_\alpha(z, w)$ is holomorphic in $z \in D$ and anti-holomorphic in $w \in D$,

(3.5) $K_\alpha(\tilde{g}\cdot z, \tilde{g}\cdot w)=J_\alpha(\tilde{g}, z)K_\alpha(z, w)\overline{J_\alpha(\tilde{g}, w)}$ for $\tilde{g} \in \tilde{G}$,

(3.6) positivity condition, i.e.,

$$\sum_{i,j=1}^{N} c_i\bar{c}_j K_\alpha(z_j, z_i) \geqq 0 \qquad \text{for any} \quad N \in \boldsymbol{N},\ c_i \in \boldsymbol{C},\ z_i \in D.$$

We prove the last positivity condition.

From Corollary 3.2 we get that

$$K_\alpha(z, w) = 2^{n\alpha+n(n+1)/2}\pi^{-n(n-1)/4} \prod_{j=1}^{n} \Gamma(\alpha+(j+1)/2)^{-1}$$

$$\times \int_{C(n)} \exp(i \operatorname{Tr} k(z-w^*))(\det k)^\alpha dk.$$

Hence

$$\sum_{i,j} c_i \bar{c}_j K_\alpha(z_j, z_i) = PC \int_{C(n)} \sum_{i,j} c_i \bar{c}_j \exp(i \operatorname{Tr} k(z_j - z_i^*))(\det k)^\alpha dk$$
$$\geqq 0. \qquad (PC = \text{positive constant})$$

PROPOSITION 3.3. *Let $\alpha > -1$ and let*

$$L_\alpha := \{\phi\colon C(n) \to \boldsymbol{C};\ \textit{measurable function such that } \|\phi\| < \infty\}$$

where $\|\phi\|^2 := K \sum_{j=1}^{n} \Gamma(\alpha + (j+1)/2)^{-1} \int_{C(n)} |\phi(k)|^2 (\det k)^\alpha dk$,

$K = 2^{n\alpha + n(n+1)/2} \pi^{-n(n-1)/4}$. *For $z \in D$ and $\phi \in L_\alpha$ we put*

$$\check{\phi}(z) := K \prod_{j=1}^{n} \Gamma(\alpha + (j+1)/2)^{-1} \int_{C(n)} \exp(i \operatorname{Tr} kz) \phi(k)(\det k)^\alpha dk.$$

Then the right-hand side is an absolutely convergent integral and defines a holomorphic function $\check{\phi}$ on D. Furthermore

$$H_\alpha := \{\check{\phi}(z);\ \phi \in L_\alpha \textit{ with } \|\check{\phi}\| = \|\phi\|\}$$

is a Hilbert space with the reproducing kernel K_α.

PROOF. If $z = x + iy$, $x \in S(n)$, $y \in C(n)$, then

$$\left| \int_{C(n)} \exp(i \operatorname{Tr} kz) \phi(k)(\det k)^\alpha dk \right| \leqq \int_{C(n)} \exp(-\operatorname{Tr} ky) |\phi(k)| (\det k)^\alpha dk$$
$$\leqq \|\phi\| \left(\int_{C(n)} \exp(-2 \operatorname{Tr} ky)(\det k)^\alpha dk \right)^{1/2} \leqq PC \|\phi\| (\det y)^{-1/2 \cdot (\alpha + (n+1)/2)}.$$

So the integral converges absolutely.

Let $\kappa(k, w) := \exp(-i \operatorname{Tr} kw^*)$. Then $\kappa(\cdot, w) \in L_\alpha$ for any fixed $w \in D$. We put $K(z, w) := \kappa(\cdot, w)^\vee(z)$. We show that $K(z, w)$ is the reproducing kernel in H_α. If $\phi \in L_\alpha$, then

$$\langle \check{\phi}(\cdot), K(\cdot, w) \rangle_{H_\alpha} = \langle \phi(\cdot), \kappa(\cdot, w) \rangle_{L_\alpha}$$
$$= K \prod_{j=1}^{n} \Gamma(\alpha + (j+1)/2)^{-1} \int_{C(n)} \phi(k) \overline{\exp(-i \operatorname{Tr} kw^*)} (\det k)^\alpha dk$$
$$= K \prod_{j=1}^{n} \Gamma(\alpha + (j+1)/2)^{-1} \int_{C(n)} \exp(i \operatorname{Tr} kw) \phi(k)(\det k)^\alpha dk = \check{\phi}(w).$$

Now we recall a theorem of R. A. Kunze.

THEOREM 3.4. (Kunze [11], see also [8].) *Let G be a group of holomorphic transitive transformations of a complex domain D. Let H be a Hilbert space of holomorphic functions on D having a reproducing kernel $K(z, w)$. Let $J(g, z)$ be a continuous function on $G \times D$ satisfying the conditions* (3.1), (3.2), (3.3), *and for $e_0 \in D$ the representation $g \mapsto J(g, e_0)$ is unitary on G^{e_0}, the stabilizer of e_0 in G. If $K(z, w)$ satisfies the conditions* (3.4), (3.5), (3.6), *and if $(T(g)f)(z) = J(g^{-1}, z)^{-1} f(g^{-1} \cdot z)$, then T is an irreducible unitary representation of G on H.*

By the Kunze's theorem we conclude

THEOREM 3.5. *For $\alpha > -1$, the representation $(T_\alpha(\tilde{g})\check{\phi})(z) := J_\alpha(\tilde{g}^{-1}, z)^{-1} \check{\phi}(\tilde{g}^{-1} \cdot z)$ is an irreducible unitary representation of $\tilde{G}$ on H_α. If α is a non-negative integer or half-integer, this is a representation of $G_2 = Mp(n, \boldsymbol{R})$, the metaplectic group. Furthermore if $m = \alpha + (n+1)/2$ is an integer, this is a representation of $G = Sp(n, \boldsymbol{R})$ itself, given by*

$$(T_\alpha(g)\check{\phi})(z) = \det(cz+d)^{-m} \check{\phi}((az+b)(cz+d)^{-1}) \quad \text{for} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

4. Unitary representations (singular case)

We have constructed the Hilbert spaces of holomorphic functions on D by the Fourier-Laplace transform of the square-integrable functions supported on the regular orbit $C(n)$. From now on, we will consider the singular orbits of the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$.

Let $b(C(n)) := \{k \in \overline{C(n)};\ \det k = 0\} = \cup_{j=0}^{n-1} O_j$ where $O_j = \{k \in b(C(n));\ \operatorname{rank} k = j\}$.

One can easily see that

$$\lim_{\alpha \to -(n-j+1)/2} (\det k)^\alpha \Gamma(\alpha+1)^{-1} \cdots \Gamma(\alpha + (n-j+1)/2)^{-1} dk$$

defines a semi-invariant measure $d\mu_j(k)$ on O_j.

From Proposition 3.1 we conclude

COROLLARY 4.1. *For $j = 1, \dots, n-1$,*

$$\int_{O_j} \exp(i \operatorname{Tr} k(z - w^*)) d\mu_j(k) = 2^{-nj/2} \pi^{n(n-1)/4} \prod_{m=1}^{j} \Gamma(m/2) \det((z - w^*)/2i)^{-j/2}.$$

We put

$$\begin{aligned} K_{(j)}(z, w) :&= \det((z-w^*)/2i)^{-j/2} \\ &= 2^{nj/2} \pi^{-n(n-1)/4} \prod_{m=1}^{j} \Gamma(m/2)^{-1} \int_{O_j} \exp(i \operatorname{Tr} k(z-w^*)) d\mu_j(k). \end{aligned}$$

In parallel with Proposition 3.3 and Theorem 3.5 we conclude

PROPOSITION 4.2. *Let $j=1,\dots,n-1$ and let*

$$L_{(j)}:=\{\phi\colon O_j\to \boldsymbol{C};\ \textit{measurable function such that } \|\phi\|<\infty\}$$

where $\|\phi\|^2:=K\prod_{m=1}^{j}\Gamma(m/2)^{-1}\int_{O_j}|\phi(k)|^2 d\mu_j(k)$, $K=2^{nj/2}\pi^{n(n-1)/4}$.

For $z\in D$ and $\phi\in L_{(j)}$ we put

$$\check{\phi}(z):=K\prod_{m=1}^{j}\Gamma(m/2)^{-1}\int_{O_j}\exp(i\operatorname{Tr}kz)\phi(k)d\mu_j(k).$$

Then the right-hand side is an absolutely convergent integral and defines a holomorphic function $\check{\phi}$ on D. Furthermore

$$H_{(j)}:=\{\check{\phi}(z);\ \phi\in L_{(j)}\ \textit{with}\ \|\check{\phi}\|=\|\phi\|\}$$

is a Hilbert space with the reproducing kernel $K_{(j)}$.

THEOREM 4.3. *For $j=1,\dots,n-1$, the representation*

$$\begin{aligned}(T_{(j)}(\tilde{g})\check{\phi})(z)&:=J_{(j)}(\tilde{g}^{-1},z)^{-1}\check{\phi}(\tilde{g}^{-1}\cdot z)\\ &=\det(cz+d)^{-j/2}\check{\phi}((az+b)(cz+d)^{-1}),\end{aligned}$$

for $g^{-1}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G$, is an irreducible unitary representation of G_2 on $H_{(j)}$.

5. Results for representation spaces

In the previous section we have obtained the holomorphic functions by means of the Fourier-Laplace transform. If $\check{\phi}\in H_{(j)}$ $(j=1,\dots,n-1)$, we can take the hyperfunction boundary value to $S(n)$. (For the terminology of the theory of hyperfunctions, see, for example, [13].) We denote it by $\check{\phi}(x+iC0)$ or simply by $\check{\phi}(x)$.

PROPOSITION 5.1. *If $\check{\phi}\in H_{(j)}$, then $\check{\phi}(x)$ is in fact a tempered distribution on $S(n)$.*

PROOF. Since $\check{\phi}\in H_{(j)}$,

$$|\check{\phi}(z)|=|\langle\check{\phi}(\cdot),K(\cdot,z)\rangle|\leqq\|\check{\phi}\|\cdot\|K(\cdot,z)\|$$

by Schwarz' inequality. On the other hand $K(\cdot,z)=\exp(-i\operatorname{Tr}kz^*)^{\vee}(\cdot)$. Hence

$$
\begin{aligned}
\|K(\cdot, z)\|^2 &= |\langle K(\cdot, z), K(\cdot, z)\rangle| \\
&\leqq PC\int_{O_j} |\exp(-i \operatorname{Tr} kz^*) \exp(i \operatorname{Tr} kz)| d\mu_j(k) \\
&\leqq PC\int_{O_j} \exp(-2\operatorname{Tr} ky) d\mu_j(k) \leqq PC\int_{O_j} \exp(-\operatorname{Tr} k) d\mu_j(k)\cdot(\det y)^{-2j}.
\end{aligned}
$$

$$(z=x+iy)$$

Hence $\check{\phi}(x)$ is a tempered distribution on $S(n)$.

We put

$$D_{(j)} := \{\check{\phi}(x+iC0);\ \phi\in L_{(j)} \text{ with } \|\check{\phi}\| = \|\phi\|\}.$$

Then $D_{(j)}$ is a Hilbert space of distributions on which G_2 acts irreducibly and unitarily.

We shall see that $D_{(j)}$ is the solution space for certain hyperbolic differential equation. Recall that $(G_0, \mathfrak{g}_1)$ is a real prehomogeneous vector space with an irreducible relative invariant

$$f\left(\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\right) = \det k.$$

$\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are non-singularly paired by the Killing form and we identify $\mathfrak{g}_{-1}$ with the dual space $\mathfrak{g}_1^*$. Then $(G_0, \mathfrak{g}_{-1})$ is the dual prehomogeneous vector space with an irreducible relative invariant

$$f^*\left(\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}\right) = \det x.$$

Let $f(\operatorname{grad}_x)$ be a hyperbolic differential operator on $\mathfrak{g}_{-1}$ with constant coefficients defined by

$$f(\operatorname{grad}_x)\exp(\operatorname{Tr} kx) = f(k)\exp(\operatorname{Tr} kx).$$

(Notice that the bilinear form $\operatorname{Tr} kx$ is proportional to the Killing form.)

THEOREM 5.2. *For $j=1,\dots, n-1$, the elements of $D_{(j)}$ satisfy the differential equation $f(\operatorname{grad}_x)u=0$.*

PROOF. We have only to show that any element of $H_{(j)}$ satisfies $f(\operatorname{grad}_z)u=0$ in the complex domain. If $\check{\phi}\in H_{(j)}$, then

$$\check{\phi}(z) = PC\prod_{m=1}^{j}\Gamma(m/2)^{-1}\int_{O_j}\exp(i\operatorname{Tr} kz)\phi(k)d\mu_j(k)$$

for some $\phi\in L_{(j)}$. By differentiating under the integral sign,

$$f(\mathrm{grad}_z)\check{\phi}(z) = PC\prod_{m=1}^{j}\Gamma(m/2)^{-1}\int_{O_j} f(\mathrm{grad}_z)\exp(i\,\mathrm{Tr}\,kz)\phi(k)d\mu_j(k) = 0.$$

Finally we will mention the singularities for the elements of $D_{(j)}$. Let $C^*(n)$ be the dual cone of $C(n)$, i.e.,

$$C^*(n) = \{\xi;\ \mathrm{Tr}\,\xi k \geqq 0 \text{ for any } k \in C(n)\}.$$

It is easy to see that $C^*(n) = \overline{C(n)}$.

From Theorem 5.2 we conclude that for any $\check{\phi} \in D_{(j)}$,

$$S.S.\check{\phi} \subset S(n) \times b(C^*(n))$$

where $S.S.$ means the singularity spectrum of a hyperfunction.

Moreover we can conclude

THEOREM 5.3. *Let $\check{\phi} \in D_{(j)}$. Then*

$$S.S.\check{\phi} \subset \{(x, \xi);\ x \in S(n),\ \xi \in b(C^*(n)),\ \mathrm{rank}\,\xi \leqq j\}.$$

PROOF. We consider the minor determinants $f\begin{pmatrix} i_1\cdots i_m \\ j_1\cdots j_m \end{pmatrix}$ of degree $n-m$ and the differential operators $f\begin{pmatrix} i_1\cdots i_m \\ j_1\cdots j_m \end{pmatrix}(\mathrm{grad}_x)$ defined by

$$f\begin{pmatrix} i_1\cdots i_m \\ j_1\cdots j_m \end{pmatrix}(\mathrm{grad}_x)\exp(\mathrm{Tr}\,kx) = f\begin{pmatrix} i_1\cdots i_m \\ j_1\cdots j_m \end{pmatrix}(k)\exp(\mathrm{Tr}\,kx).$$

If $\check{\phi} \in D_{(j)}$ and $n-m \geqq j$, then, parallel to the proof of Theorem 5.2, we get $f\begin{pmatrix} i_1\cdots i_m \\ j_1\cdots j_m \end{pmatrix}(\mathrm{grad}_x)\check{\phi} = 0$. Thus the theorem is proved.

6. The case of $SU(n, n)$

Let $G = SU(n, n)$. The maximal compact subgroup K is isomorphic to $S(U(n) \times U(n))$, and the hermitian symmetric space G/K is realized as

$$D := \{z = x + iy;\ x^* = x,\ y^* = y,\ y \gg 0\}.$$

We denote now the space of all $n \times n$ hermitian matrices by $H(n)$, and the cone of all positive definite hermitian matrices by $C(n)$. Then $D = H(n) + iC(n)$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in D$, we put $g \cdot z = (az+b)(cz+d)^{-1}$.

Let

$$\mathfrak{g}_{-1} = \left\{\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix};\ x \in H(n)\right\},\quad \mathfrak{g}_1 = \left\{\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix};\ k \in H(n)\right\},$$

$$\mathfrak{g}_0 = \left\{\begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}; A \in M(n, \boldsymbol{C}), \operatorname{Tr} A \in \boldsymbol{R}\right\},$$

Then $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$, the $\boldsymbol{Z}$-graded decomposition.

If G_0 is the subgroup corresponding to $\mathfrak{g}_0$, the pair $(G_0, \mathfrak{g}_1)$ is a real prehomogeneous vector space by the adjoint action and it possesses an irreducible relative invariant f defined by

$$f\left(\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}\right) = \det k.$$

$C(n)$ is one of the regular orbits of this prehomogeneous vector space.

Since all proofs are parallel to the case of $Sp(n, \boldsymbol{R})$, they are omitted.

PROPOSITION 6.1. *Let $\rho > n-1$. Then*

$$\int_{C(n)} \exp(-\operatorname{Tr} ky)(\det k)^{\rho-n} dk = \pi^{n(n-1)/2} \prod_{j=1}^{n} \Gamma(\rho-j+1)(\det y)^{-\rho}$$

for $y \in C(n)$, where dk is the Lebesgue measure on $C(n)(\cong \boldsymbol{R}^{n^2})$.

COROLLARY 6.2. *Let $\alpha > -1$. Then*

$$\int_{C(n)} \exp(i \operatorname{Tr} k(z-w^*))(\det k)^{\alpha} dk$$
$$= 2^{-n\alpha-n^2} \pi^{n(n-1)/2} \prod_{j=1}^{n} \Gamma(\alpha+j) \det((z-w^*)/2i)^{-\alpha-n}$$

for $z, w \in D$.

Let $\tilde{G}$ be the universal covering group of G. We can define $J_\alpha(\tilde{g}, z) := \det(cz+d)^{\alpha+n}$ for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $z \in D$, $\alpha > -1$ by choosing a branch such that $J_\alpha(1, z)=1$. Let $K_\alpha(z, w) := \det((z-w^*)/2i)^{-\alpha-n}$ for $z, w \in D$, $\alpha > -1$. Then J_α and K_α satisfy the conditions (3.1), (3.2), (3.3) and (3.4), (3.5), (3.6) respectively.

PROPOSITION 6.3. *Let $\alpha > -1$ and let*

$$L_\alpha := \{\phi : C(n) \to \boldsymbol{C};\ \textit{measurable function such that } \|\phi\| < \infty\}$$

where $\|\phi\|^2 := K \prod_{j=1}^{n} \Gamma(\alpha+j)^{-1} \int_{C(n)} |\phi(k)|^2 (\det k)^{\alpha} dk, \quad K = 2^{n\alpha+n^2} \pi^{-n(n-1)/2}.$

For $z \in D$ and $\phi \in L_\alpha$ we put

$$\check{\phi}(z) := K \prod_{j=1}^{n} \Gamma(\alpha+j)^{-1} \int_{C(n)} \exp(i \operatorname{Tr} kz) \phi(k)(\det k)^{\alpha} dk.$$

Then the right-hand side is an absolutely convergent integral and defines a holomorphic function $\check{\phi}$ on D. Furthermore

$$H_\alpha := \{\check{\phi}(z);\ \phi \in L_\alpha \textit{ with } \|\check{\phi}\| = \|\phi\|\}$$

is a Hilbert space with the reproducing kernel K_α.

THEOREM 6.4. *For* $\alpha > -1$, *the representation* $(T_\alpha(\tilde{g})\check{\phi})(z) := J_\alpha(\tilde{g}^{-1}, z)^{-1} \cdot \check{\phi}(\tilde{g}^{-1}\cdot z)$ *is an irreducible unitary representation of* $\tilde{G}$ *on* H_α. *If* α *is a non-negative integer, this is a representation of* $G = SU(n, n)$ *itself, given by*

$$(T_\alpha(g)\check{\phi})(z) = \det(cz+d)^{-\alpha-n}\check{\phi}((az+b)(cz+d)^{-1}) \quad \textit{for} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $b(C(n)) := \{k \in \overline{C(n)};\ \det k = 0\} = \cup_{j=0}^{n-1} O_j$ where $O_j = \{k \in b(C(n));\ \text{rank } k = j\}$.

It is easy to see that

$$\lim_{\alpha\to -n+j} (\det k)^\alpha \Gamma(\alpha+1)^{-1} \cdots \Gamma(\alpha+n-j)^{-1} dk$$

defines a semi-invariant measure $d\mu_j(k)$ on O_j.

COROLLARY 6.5. *For* $j = 1, \dots, n-1$,

$$\int_{O_j} \exp(i \operatorname{Tr} k(z-w^*) d\mu_j(k) = 2^{-nj}\pi^{n(n-1)/2} \textstyle\prod_{m=1}^{j} \Gamma(m) \det((z-w^*)/2i)^{-j}.$$

We put

$$\begin{aligned} K_{(j)}(z, w) &:= \det((z-w^*)/2i)^{-j} \\ &= 2^{nj}\pi^{-n(n-1)/2} \textstyle\prod_{m=1}^{j} \Gamma(m)^{-1} \displaystyle\int_{O_j} \exp(i \operatorname{Tr} k(z-w^*)) d\mu_j(k). \end{aligned}$$

PROPOSITION 6.6. *Let* $j = 1, \dots, n-1$ *and let*

$$L_{(j)} := \{\phi : O_j \to \boldsymbol{C};\ \textit{measurable function such that } \|\phi\| < \infty\}$$

where $\|\phi\|^2 := K \prod_{m=1}^{j} \Gamma(m)^{-1} \int_{O_j} |\phi(k)|^2 d\mu_j(k)$, $K = 2^{nj}\pi^{-n(n-1)/2}$. *For* $z \in D$ *and* $\phi \in L_{(j)}$ *we put*

$$\check{\phi}(z) := K \textstyle\prod_{m=1}^{j} \Gamma(m)^{-1} \displaystyle\int_{O_j} \exp(i \operatorname{Tr} kz) \phi(k) d\mu_j(k).$$

Then the right-hand side is an absolutely convergent integral and defines a holomorphic function $\check{\phi}$ *on D. Furthermore*

$$H_{(j)} := \{\check{\phi}(z);\ \phi \in L_{(j)} \textit{ with } \|\check{\phi}\| = \|\phi\|\}$$

is a Hilbert space with the reproducing kernel $K_{(j)}$.

THEOREM 6.7. *For* $j = 1, \dots, n-1$, *the representation*

$$\begin{aligned} (T_{(j)}(g)\check{\phi})(z) &:= J_{(j)}(g^{-1}, z)^{-1}\check{\phi}(g^{-1}\cdot z) \\ &= \det(cz+d)^{-j}\check{\phi}((az+b)(cz+d)^{-1}) \end{aligned}$$

for $g^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *is an irreducible unitary representation of* G *on* $H_{(j)}$.

If $\check{\phi}\in H_{(j)}$, we can take the tempered distribution boundary value to $H(n)$. We denote it by $\check{\phi}(x+iC0)$ or simply by $\check{\phi}(x)$.

We put

$$D_{(j)}:=\{\check{\phi}(x+iC0);\ \phi\in L_{(j)} \text{ with } \|\phi\|=\|\check{\phi}\|\}.$$

Then $D_{(j)}$ is a Hilbert space on which G acts irreducibly and unitarily.

Let $f(\mathrm{grad}_x)$ be the hyperbolic differential operator on $\mathfrak{g}_{-1}$ defined by

$$f(\mathrm{grad}_x)\exp(\mathrm{Tr}\,kx)=f(k)\exp(\mathrm{Tr}\,kx).$$

THEOREM 6.8. *For* $j=1,\dots,n-1$, *the elements of* $D_{(j)}$ *satisfy the differential equation* $f(\mathrm{grad}_x)u=0$. *Furthermore, by considering the minor determinants,*

$$S.S.\check{\phi}\subset\{(x,\xi);\ x\in H(n),\ \xi\in b(C^*(n)),\ \mathrm{rank}\,\xi\leqq j\}$$

for $\check{\phi}\in D_{(j)}$.

For example, if $n=2$, then we can identify $H(2)$ with $\boldsymbol{R}^4$ by

$$\begin{pmatrix} x_0+x_1 & x_2-ix_3 \\ x_2+ix_3 & x_0-x_1 \end{pmatrix}=\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and $f(\mathrm{grad}_x)=\Box=(\partial/\partial x_0)^2-(\partial/\partial x_1)^2-(\partial/\partial x_2)^2-(\partial/\partial x_3)^2$, the wave operator. The elements of $D_{(1)}$ satisfy the differential equation $\Box u=0$ [8].

We have considered the real prehomogeneous vector space $(G_0, \mathfrak{g}_1)$ and its regular orbit $C(n)$. It is an interesting problem to develop the representation theory for the general orbits of this prehomogeneous vector space.

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