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On generalized total curvatures and conformal mappings

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§ 1. Introduction

In this paper, we generalize the concept of total absolute curvatures of submanifolds immersed in a Riemannian manifold and study the properties in relation to conformal mappings. In § 2, generalized total curvatures are defined. We construct certain conformal invariants in § 3, using generalized total curvatures. These invariants contain that of C. C. Hsiung and L. R. Mugridge [4] and T. J. Willmore [7].

§ 2. Generalized total curvatures

Let N be an $(n+q)$-dimensional Riemannian manifold with the metric g and M an n-dimensional submanifold immersed in N. For a normal vector field ξ and a tangent vector field X on M, the second fundamental form A of M is defined to be

$$A^{\xi}(X) := -(\nabla_X \xi)^T,$$

where ∇ is the Levi-Civita connection of N and $(\ )^T$ denotes the tangential component.

Let GL_n be the real general linear group, $\mathfrak{gl}_n$ its Lie algebra and $\mathfrak{s}_n(\subset \mathfrak{gl}_n)$ the subspace which consists of all symmetric matrices. An algebra I is defined to be

$$I := \{\varphi \in C^0(\mathfrak{s}_n) \mid \varphi(gBg^{-1}) = \varphi(B) \quad \text{for any} \quad B \in \mathfrak{s}_n,\ g \in O(n)\},$$

where $C^0(\mathfrak{s}_n)$ is the algebra of all real-valued continuous functions on $\mathfrak{s}_n$ and $O(n)$ is the orthogonal group. For a positive real number r, we define the following subspace

$$I^r := \{\varphi \in I \mid \varphi(bB) = b^r \varphi(B) \quad \text{for any} \quad B \in \mathfrak{s}_n,\ b > 0\}.$$

Let $T_1^{\perp}(M)$ be the normal unit sphere bundle. A linear mapping $\mu_M^N : I^r \to C^0(M)$ is defined to be

$$\mu_M^N(\varphi)(p) := (1/\omega_{q+r-1}) \int_{T_1^{\perp}(M)_p} \varphi(A_p^{\xi}) d\sigma_p(\xi) \qquad \text{for} \quad \varphi \in I^r,\ p \in M,$$

where $d\sigma_p$ is the volume element of the fibre $T_1^{\perp}(M)_p$ and

$$\omega_{q+r-1}: = 2\pi^{(q+r)/2}/\Gamma((q+r)/2),$$

which coincides with the volume of the $(q+r-1)$-dimensional unit sphere when r is an integer.

LEMMA 1. *If N is a totally geodesic submanifold of $\tilde{N}$, then we have*

$$\mu_M^{\tilde{N}}(\varphi) = \mu_M^N(\varphi) \qquad \textit{for} \quad \varphi \in I^r.$$

This fact is due to the factor ω_{q+r-1} in the definition. From now on we will denote μ_M^N by μ_M for simplicity if there will be no ambiguity.

Let $I(GL_n)$ be the algebra of all invariant polynomials on $\mathfrak{gl}_n$. It is clear that elements of $I(GL_n)$ restricted to $\mathfrak{s}_n$ belong to I. The generators $c_k \in I(GL_n)$ $(0 \leqq k \leqq n)$ are defined to be

$$\sum t^k \binom{n}{k} c_k(B): = \det(I_n + tB) \qquad \text{for} \quad B \in \mathfrak{gl}_n.$$

For $p \in M$ and $\xi \in T_1^{\perp}(M)_p$, $c_k(A_p^{\xi})$ is called the k-th mean curvature of M at p with respect to ξ and $K_k^*(p): = \mu_M(|c_k|^{n/k})(p)$ the k-th total absolute curvature at p. The k-th total absolute curvature of M is defined to be

$$TK_k^*(M): = \int_M K_k^*(p) dV_M(p),$$

where dV_M denotes the standard measure on M. Especially $TK_n^*(M)$ is the usual total absolute curvature of M. These curvatures have been studied by many geometers. For example, see [2].

§ 3. Conformal invariants

It is well-known that $TK_k^*(M)$ is invariant under homotheties of N. Noting that $|c_k|^{n/k} \in I^n$, we can clearly generalize this fact as follows. Let $\bar{N}$ be another Riemannian manifold with the metric $\bar{g}$ and $f: N \to \bar{N}$ a diffeomorphism. If g and $f^*\bar{g}$ are homothetically equivalent, then we have

$$f^*(\mu_{f(M)}^{\bar{N}}(\varphi) dV_{f(M)}) = \mu_M^N(\varphi) dV_M \tag{1}$$

for any $\varphi \in I^n$. In the case where g and $f^*\bar{g}$ are conformally equivalent, the formula (1) does not hold for all of $\varphi \in I^n$ in general. An example of the elements, for which the formula (1) holds, is $(c_1^2 - c_2)^{n/2} \in I^n$:

THEOREM (C. C. Hsiung and L. R. Mugridge [4]). *Let M be a submanifold immersed in a Euclidean space E^{n+q} and f a conformal mapping of E^{n+q}. Then we have*

$$f^*(\mu_{f(M)}((c_1^2-c_2)^{n/2})dV_{f(M)}) = \mu_M((c_1^2-c_2)^{n/2})dV_M,$$

(*in our notation*).

This theorem is due to B.-Y. Chen [1] for $n=2$ and a general q and due to T. J. Willmore [7] for a general ambient space with $q=1$. In the case where M is a surface, we have the following conformal invariant.

THEOREM (B.-Y. Chen [1], J. H. White [6]). *Let M be an orientable closed surface in E^{2+q} and f a conformal mapping of E^{2+q}. Then we have*

$$\int_{f(M)} |\bar{H}|^2 dV_{f(M)} = \int_M |H|^2 dV_M,$$

where H (resp. $\bar{H}$) is the mean curvature vector field of M (resp. $f(M)$) in E^{2+q}.

We will generalize the above theorems as follows.

THEOREM 1. *Let M be a submanifold immersed in a Riemannian manifold N with the metric g and $f: N \to \bar{N}$ a diffeomorphism into a Riemannian manifold $\bar{N}$ with the metric $\bar{g}$. If g and $f^*\bar{g}$ are conformally equivalent, then we have*

$$f^*(\mu^{\bar{N}}_{f(M)}(|\hat{c}_k|^{n/k})dV_{f(M)}) = \mu^N_M(|\hat{c}_k|^{n/k})dV_M \qquad \text{for} \quad k \geqq 2,$$

where $\hat{c}_k \in I^k$ is defined to be

$$\hat{c}_k := \sum_{i=0}^k \binom{k}{i}(-1)^i(c_1)^i c_{k-i}.$$

Note that $\hat{c}_2 = c_2 - c_1^2 (\leqq 0)$. The proof of Theorem 1 will be given in § 4.

REMARK. If the mean curvature vector of M vanishes at $p \in M$, then $\mu_M(|\hat{c}_k|^{n/k})(p) = K_k^*(p)$.

COROLLARY. *In the theorem, let M be an orientable closed surface and N (resp. $\bar{N}$) a space of constant sectional curvature c (resp. $\bar{c}$). Then we have*

$$\int_M |H|^2 dV_M + c\,\mathrm{Vol}(M) = \int_{f(M)} |\bar{H}|^2 dV_{f(M)} + \bar{c}\,\mathrm{Vol}(f(M)),$$

where H (resp. $\bar{H}$) denotes the mean curvature vector field of M (resp. $f(M)$) in N (resp. $\bar{N}$) and Vol( ) *is the volume.*

PROOF. Carry out the integration over the normal unit sphere, and we obtain

$$\mu_M(c_1^2) = \frac{2}{\omega_2}|H|^2 \quad \text{and} \quad \mu_M(c_2) = \frac{2}{\omega_2}(K-c),$$

where K is the Gaussian curvature of M. By applying the Gauss-Bonnet formula to Theorem 1, we obtain the formula.

The formula in the Corollary coincides with that of M. Maeda [5] in the case where N (resp. $\bar{N}$) is the hyperbolic space $H^{2+q}(c)$ (resp. the Euclidean space E^{2+q}) and f is the inclusion mapping from the Poincare disc model into E^{2+q}.

§4. Proof of Theorem 1

Let $\sigma: \mathfrak{s}_n \to \mathfrak{s}_n$ be a homomorphism defined to be

$$\sigma(B) := B - c_1(B)I_n \qquad \text{for} \quad B \in \mathfrak{s}_n,$$

and $\sigma^*: I \to I$ the induced homomorphism.

A straightforward calculation gives

Lemma 2. $\sigma^* c_k = \hat{c}_k$.

Therefore, in order to get the formula in Theorem 1, it is sufficient to prove

Theorem 2. *If g and $f^*\bar{g}$ are conformally equivalent, then we have*

$$f^*(\mu^{\bar{N}}_{f(M)}(\varphi)dV_{f(M)}) = \mu^N_M(\varphi)dV_M \qquad \textit{for} \quad \varphi \in \sigma^*(I^n).$$

At first we prove the following lemmas. Let ρ be a smooth function on N such as $f^*\bar{g} = e^{2\rho}g$.

Lemma 3. *For $p \in M$, $\xi \in T^\perp(M)_p$ and $X \in T(M)_p$, we have*

$$(f^*\bar{A})^\xi_p(X) = A^\xi_p(X) - (\xi\rho)X,$$

where $\bar{A}$ is the second fundamental form of $f(M)$ in $\bar{N}$.

Proof. It is clear that $f^*\bar{A}$ is the second fundamental form of M relative to the induced metric $f^*\bar{g}$. Then we get

$$(f^*\bar{A})^\xi_p(X) = -((f^*\bar{\nabla})_X\xi)^T,$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{N}$ and $f^*\bar{\nabla}$ is the induced connection. Since g and $f^*\bar{g}$ are conformally equivalent, we have

$$(f^*\bar{\nabla})_X Y - \nabla_X Y = (X\rho)Y + (Y\rho)X - g(X, Y)\operatorname{grad}\rho$$

for vector fields X, Y on N. This formula implies

$$(f^*\bar{\nabla})_X\xi - \nabla_X\xi = (X\rho)\xi + (\xi\rho)X \tag{2}$$

for $X \in T(M)_p$, $\xi \in T^\perp(M)_p$. By taking the tangential parts of the both sides of (2), we have the lemma.

REMARK. Let ω_α^β (resp. $\bar{\omega}_\alpha^\beta$) be the normal connection form relative to a local orthonormal frame field ξ_α (resp. $e^{-\rho}f_*\xi_\alpha$). By taking the normal part of the both sides of the formula (2), we see that $f^*\bar{\omega}_\alpha^\beta=\omega_\alpha^\beta$. Thus we find that transgression forms with respect to the normal connection are invariant under changes of metrics on the ambient space (cf. [3]).

LEMMA 4. *For $\varphi\in I^r$ and $p\in M$, we have*

$$\mu_{f(M)}(\varphi)(f(p)) = e^{-r\rho}\int_{T_1^\perp(M)_p}\varphi(A_p^\xi-(\xi\rho)I_p)d\sigma_p(\xi),$$

where I_p is the identity transformation of $T(M)_p$.

PROOF. For $\xi\in T_1^\perp(M)_p$, take $\bar{\xi}\in T_1^\perp(f(M))_{f(p)}$ such that $\bar{\xi}=e^{-\rho(p)}f_*\xi$. From Lemma 3, we see

$$\varphi(\bar{A}_{f(p)}^{\bar{\xi}}) = \varphi((f^*\bar{A})_p^{e^{-\rho(p)}\xi}) = e^{-r\rho(p)}\varphi((f^*\bar{A})_p^\xi) = e^{-r\rho(p)}\varphi(A_p^\xi-(\xi\rho)I_p).$$

Since the corresponding $\xi\to\bar{\xi}$ is an isometry, we get the required formula.

LEMMA 5. $f^*(\mu_{f(M)}(\varphi))=e^{r\rho}\mu_M(\varphi)$ *for* $\varphi\in\sigma^*(I^n)$.

Since $\sigma(A_p^\xi-(\xi\rho)I_p)=\sigma(A_p^\xi)$, Lemma 5 follows from Lemma 4.

Now, Theorem 2 is an immediate consequence of Lemma 5, since

$$f^*(dV_{f(M)}) = e^{n\rho}dV_M.$$

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