HIROSHIMA MATH. J.
12 (1982), 93–102

Ascendancy in locally finite groups

Shigeaki TÔGÔ

(Received August 26, 1981)

Introduction

In the paper [2], we introduced the notion of weakly ascendant subgroups which is weaker than that of ascendant subgroups, and mainly investigated the relationship of these notions in generalized solvable groups. Recently in the papers [1, 3], ascendancy has been studied in locally solvable, ideally finite Lie algebras.

In this paper, following the line of the papers [1, 3] we shall investigate ascendancy and weak ascendancy in locally finite groups, especially in groups which correspond to locally solvable, ideally finite Lie algebras.

Let G be a group and H be a subgroup of G. In Section 2 we shall show that when $G \in \mathrm{L}(H)\mathfrak{F}$, H is weakly ascendant in G if and only if H is ω-step weakly ascendant in G (Theorem 1). In Section 3 we shall show as the main result of the paper that when $G \in \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$, H is weakly ascendant in G if and only if H is ascendant in G and if and only if H is ω-step ascendant in G (Theorem 3). In Section 4 we shall study the cases where G belongs to $\mathrm{L}(\mathrm{sn})\mathrm{E}\mathfrak{A}$ and $\mathrm{L}(\mathrm{asc})\acute{\mathrm{E}}\mathfrak{A}$. In Section 5 we shall present some characterizations of the class of groups $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ (Theorem 5).

1.

Let G be a group. If X, Y are non-empty subsets of G, we denote by $[X, Y]$ the set of all $[x, y] = x^{-1}y^{-1}xy$ with $x \in X$ and $y \in Y$ and we write $[X, {}_0Y] = X$, $[X, {}_{n+1}Y] = [[X, {}_nY], Y]$ for an integer $n \geq 0$.

If H is respectively an ascendant subgroup, a σ-step ascendant subgroup and a subnormal subgroup of G, we as usual write

$$H \operatorname{asc} G, \quad H \lhd^{\sigma} G \quad \text{and} \quad H \operatorname{sn} G,$$

where σ is an ordinal.

Let $H \leq G$. As in [2], we call H a σ-step weakly ascendant subgroup of G, if there is an ascending series $(S_\alpha)_{\alpha \leq \sigma}$ of subsets of G satisfying the following conditions:

(a) $S_0 = H$ and $S_\sigma = G$.

(b) If α is any ordinal $< \sigma$, then $u^{-1}Hu \subseteq S_\alpha$ for any $u \in S_{\alpha+1}$.

(c) $S_\lambda = \cup_{\alpha<\lambda} S_\alpha$ for any limit ordinal $\lambda \leq \sigma$.
We then write $H \leq^\sigma G$. We call H a weakly ascendant subgroup of G if $H \leq^\sigma G$ for some ordinal σ, and write H wasc G. When $\lambda < \omega$, we call H a weakly subnormal subgroup of G and write H wsn G.

For a class $\mathfrak{X}$ of groups, we write $G \in \mathrm{L}(H)\mathfrak{X}$ if for any finite subset X of G there is an H-invariant $\mathfrak{X}$-subgroup of G containing X. For $\Delta = \lhd$, sn and asc, we write $G \in \mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of G there is an $\mathfrak{X}$-subgroup H of G containing X such that $H \,\Delta\, G$.

As usual, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{A}$, $\mathfrak{N}$, $\mathfrak{E}$ and $\mathrm{E}\mathfrak{A}(=\boldsymbol{P}\mathfrak{A})$ are respectively the classes of finite, finitely generated, abelian, nilpotent, Engel and solvable groups. $\acute{\mathrm{E}}\mathfrak{X}\ (=\acute{\boldsymbol{P}}\mathfrak{X})$ and $\acute{\mathrm{E}}(\lhd)\mathfrak{X}\ (=\acute{\boldsymbol{P}}_n\mathfrak{X})$ (resp. $\acute{\mathrm{E}}_\omega(\lhd)\mathfrak{X}$) are respectively the classes of groups having ascending $\mathfrak{X}$-series of subgroups and ascending $\mathfrak{X}$-series (resp. ω-length ascending $\mathfrak{X}$-series) of normal subgroups. $\mathrm{s}\mathfrak{X}$ is the class of subgroups of $\mathfrak{X}$-groups. $\mathrm{L}\mathfrak{X}$ and $\mathrm{R}\mathfrak{X}$ are respectively the classes of locally and residually $\mathfrak{X}$-groups.

2.

In this section we study weak ascendancy in locally finite groups. We begin with the following

THEOREM 1. *Let G be a group and $H \leq G$. Assume that $G \in \mathrm{L}(H)\mathfrak{F}$. Then the following conditions are equivalent:*
(1) H wasc G.
(2) $H \leq^\omega G$.

PROOF. Assume that $H \leq^\sigma G$. Then by Lemma 1 in [2] there is a weakly ascending series $(S_\alpha)_{\alpha \leq \sigma}$ from H to G such that $HS_\alpha H = S_\alpha$ for any $\alpha \leq \sigma$. Let $x \in G$. Then x is contained in an H-invariant $\mathfrak{F}$-subgroup $A(x)$ of G. For any $n \in \mathbf{N}$, let $\mu(n)$ be the least ordinal such that

$$[A(x), {}_nH] \subseteq S_{\mu(n)}.$$

Since $[A(x), {}_nH]$ is a finite set, $\mu(n)$ is not a limit ordinal. Observing that $[S_{\alpha+1}, H] \subseteq S_\alpha$ for any $\alpha < \sigma$, we have $\mu(n+1) < \mu(n)$ unless $\mu(n) = 0$. Since the ordinals $\leq \sigma$ are well-ordered, $\mu(n) = 0$ for some $n = n(x) \in \mathbf{N}$. It follows that

$$[A(x), {}_nH] \subseteq S_0 \subseteq H.$$

Especially, $[x, {}_nH] \subseteq H$. We now use Theorem 4 (b) in [2] to conclude that $H \leq^\omega G$.

COROLLARY. *Let G be a group and $H \leq G$. Assume that $G \in \mathrm{L}(H)\mathfrak{F} \cap \mathrm{Min}$. Then the following conditions are equivalent:*

(1) H asc G.
(2) H wasc G.

PROOF. By Theorem 1, if H wasc G then $H \leq^{\omega} G$. By Corollary to Theorem 4 in [2], if $H \leq^{\omega} G$ then H asc G.

THEOREM 2. *Let G be a group and $H \leq G$. Assume that $G \in \mathrm{L}(H)\mathfrak{F}$. Then the following conditions are equivalent*:
(1) $H \leq^{\omega} G$.
(2) $H \leq^{\omega} K$ *for any subgroup K of G containing H.*
(3) $H \leq^{\omega} \langle H, x\rangle$ *for any* $x \in G$.
(4) $H \leq^{\omega} \langle H, [x, H]\rangle$ *for any* $x \in G$.
(5) *For any $x \in G$, there is an $n = n(x) \in \mathbf{N}$ such that* $H \leq^{\omega} \langle H, [x, {}_nH]\rangle$.

PROOF. It is clear that (1)⇒(2)⇒(3)⇒(4)⇒(5). To show that (5)⇒(1), we assume (5). Then there is a weakly ascending series $(S_\alpha(x))_{\alpha\leq\omega}$ from H to $\langle H, [x, {}_nH]\rangle$ such that $HS_\alpha(x)H = S_\alpha(x)$ for any $\alpha \leq \omega$. We put

$$S_{\omega+i}(x) = H(S_\omega(x) \cup [x, {}_{n-1}H] \cup \cdots \cup [x, {}_{n-i}H])H \qquad \text{for } 1 \leq i \leq n,$$
$$S_\beta(x) = S_{\omega+n}(x) \qquad \text{for } \omega+n < \beta < \omega 2.$$

Then for any $\beta < \omega 2$, if $u \in S_{\beta+1}(x)$ then $u^{-1}Hu \subseteq S_\beta(x)$. In fact, if $u \in S_{\omega+i+1}(x)$ ($i < n$), then we may assume that

$$u = h_1 a h_2 \qquad \text{with} \quad h_1, h_2 \in H \quad \text{and} \quad a \in [x, {}_{n-i-1}H].$$

It follows that for any $h \in H$

$$\begin{aligned} u^{-1}hu &= h_2^{-1}a^{-1}h_1^{-1}hh_1ah_2 \\ &= h_2^{-1}[a, h_1^{-1}h^{-1}h_1]h_1^{-1}hh_1h_2 \\ &\in H[x, {}_{n-i}H]H \\ &\subseteq S_{\omega+i}(x). \end{aligned}$$

We define

$$S_\beta = \bigcup\nolimits_{x\in G} S_\beta(x) \qquad \text{for any } \beta < \omega 2,$$
$$S_{\omega 2} = \bigcup\nolimits_{\beta<\omega 2} S_\beta.$$

Then $S_0 = H$. We have $S_{\omega 2} = G$, since $x \in S_{\omega+n(x)}(x)$. For any $\beta < \omega 2$, if $u \in S_{\beta+1}$ then $u \in S_{\beta+1}(x)$ for some $x \in G$ and therefore

$$u^{-1}Hu \subseteq S_\beta(x) \subseteq S_\beta.$$

Hence we see that $H \leq^{\omega 2} G$. By Theorem 1 we conclude that $H \leq^{\omega} G$.

3.

In this section we consider the case where G belongs to $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})$. First we show

LEMMA 1. (1) $\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A}\leq\acute{\mathrm{E}}(\lhd)\mathfrak{A}$.
(2) $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})\leq\acute{\mathrm{E}}(\lhd)(\mathfrak{A}\cap\mathfrak{F})$.

PROOF. Assume that $G\in\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A}$ (resp. $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})$). We have to construct an $\mathfrak{A}$ (resp. $\mathfrak{A}\cap\mathfrak{F}$)-series $(G_\alpha)_{\alpha\leq\sigma}$ of normal subgroups of G. We put $G_0=1$. Suppose that we have constructed $(G_\beta)_{\beta<\alpha}$ for $\alpha>0$. If α is a limit ordinal, put $G_\alpha=\bigcup_{\beta<\alpha}G_\beta$. If α is not a limit ordinal and $G_{\alpha-1}\neq G$, take $x\in G\setminus G_{\alpha-1}$. Then there is a solvable (resp. solvable finite) normal subgroup K of G containing x. Since $K\not\subseteq G_{\alpha-1}$, we choose the largest integer i such that $K^{(i)}\not\subseteq G_{\alpha-1}$. Here $K^{(i)}$ is as usual the i-th commutator subgroup of K, e.g. $K^{(1)}=\langle[K, K]\rangle$. Put $G_\alpha=G_{\alpha-1}K^{(i)}$. Then $G_\alpha\lhd G$ and

$$G_\alpha/G_{\alpha-1}\cong K^{(i)}/(G_{\alpha-1}\cap K^{(i)})\in\mathfrak{A}\ (\text{resp. }\mathfrak{A}\cap\mathfrak{F}),$$

since $K^{(i+1)}\subseteq G_{\alpha-1}\cap K^{(i)}$. By set-theoretical consideration, we see that there is an ordinal σ such that $G=G_\sigma$.

Lemma 1 tells us that we can now use Theorem 3 (a) in [2] to assert that when $G\in\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A}$, H wasc G if and only if H asc G. For $G\in\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})$ we deduce the following stronger result, which is the main result in this paper.

THEOREM 3. *Let $G\in\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})$ and $H\leq G$. Then the following conditions are equivalent*:
(1) H asc G.
(2) $H\lhd^{\omega}G$.
(3) H wasc G.
(4) $H\leq^{\omega}G$.

PROOF. (2)⇒(1)⇒(3) is clear. Theorem 1 assures that (3)⇒(4). Hence it suffices to show that (4)⇒(2).

Assume (4) and let $(S_\alpha)_{\alpha\leq\omega}$ be a weakly ascending series from H to G. By Lemma 1 in [2] we may assume that $HS_\alpha H=S_\alpha$ and $S_\alpha^{-1}=S_\alpha$ for any $\alpha\leq\omega$. Since $G\in\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})$,

$$G=\bigcup_{\lambda\in\Lambda}A(\lambda)\qquad\text{with}\quad A(\lambda)\lhd G\quad\text{and}\quad A(\lambda)\in\mathrm{E}\mathfrak{A}\cap\mathfrak{F}.$$

Let

$$1=A(\lambda, 0)\subsetneqq A(\lambda, 1)\subsetneqq\cdots\subsetneqq A(\lambda, n(\lambda))=A(\lambda)$$

be a sequence of normal subgroups of G such that each $A(\lambda, i)/A(\lambda, i-1)$ is a simple subgroup of $G/A(\lambda, i-1)$. Since $A(\lambda, 1)$ and $A(\mu, 1)$ are minimal normal subgroups of G, if $[A(\lambda, 1), A(\mu, 1)]\neq 1$ then $\langle[A(\lambda, 1), A(\mu, 1)]\rangle = A(\lambda, 1) = A(\mu, 1)$, which contradicts the solvability of $A(\lambda, 1)$. Hence

$$[A(\lambda, 1), A(\mu, 1)] = 1.$$

We put $T = A(\lambda, j-1)A(\mu, k-1)$. Then $T \lhd G$. Since $A(\lambda, j)T/T$ and $A(\mu, k)T/T$ are minimal or trivial in G/T, it follows that $[A(\lambda, j)T, A(\mu, k)T] \subseteq T$. Hence

$$[A(\lambda, j), A(\mu, k)] \subseteq A(\lambda, j-1)A(\mu, k-1).$$

We define $\Omega(0) = 1$ and construct $\Omega(1)$ as follows. In the case where $A(\lambda) \not\subseteq H$, let $r(\lambda)$ be the least integer such that $A(\lambda, r(\lambda)) \not\subseteq H$ and define

$$W(\lambda, 1) = \{z \in A(\lambda, r(\lambda)) \mid z^{-1}Hz = H\}.$$

Then $W(\lambda, 1) \leq G$. We assert that $W(\lambda, 1) \not\subseteq H$. In fact, since $A(\lambda, r(\lambda)) \in \mathfrak{F}$, there is some $n \in \mathbf{N}$ such that $A(\lambda, r(\lambda)) \subseteq S_n$. Hence $HA(\lambda, r(\lambda)) \subseteq S_n$. It follows that $H \leq^n HA(\lambda, r(\lambda))$ with a weakly ascending series

$$(S_i \cap HA(\lambda, r(\lambda)))_{i\leq n}.$$

Take the least i such that $S_i \cap HA(\lambda, r(\lambda)) \not\subseteq H$. Then $S_i \cap A(\lambda, r(\lambda)) \not\subseteq H$. Hence we can choose $z \in (S_i \cap A(\lambda, r(\lambda))) \setminus H$. Then z^{-1} belongs to the same set. It is immediate that

$$z^{-1}Hz,\ zHz^{-1} \subseteq S_{i-1} \cap HA(\lambda, r(\lambda)) \subseteq H.$$

Hence $z^{-1}Hz = H$ and therefore $z \in W(\lambda, 1) \setminus H$.

In the other case where $A(\lambda) \subseteq H$, let $r(\lambda) = n(\lambda)$ and define $W(\lambda, 1)$ as above. In any case, $W(\lambda, 1) \subseteq A(\lambda, r(\lambda))$ and $A(\lambda, r(\lambda)-1) \subseteq H$. Hence

$$\begin{aligned}[W(\lambda, 1), W(\mu, 1)] &\subseteq [A(\lambda, r(\lambda)), A(\mu, r(\mu))] \\ &\subseteq A(\lambda, r(\lambda)-1)A(\mu, r(\mu)-1) \\ &\subseteq H.\end{aligned}$$

Now define

$$\Omega(1) = \langle W(\lambda, 1) \mid \lambda \in \Lambda\rangle.$$

Since each element of $\Omega(1)$ normalizes H, we have $H\Omega(1) \leq G$ and $H \lhd H\Omega(1)$. Furthermore $[\Omega(1), \Omega(1)] \subseteq H$, which can be seen by observing that for $z_\lambda \in W(\lambda, 1)$, $z_\mu \in W(\mu, 1)$ and $z_\nu \in W(\nu, 1)$

$$[z_\lambda z_\mu, z_\nu] = [z_\lambda, z_\nu]^{z_\mu}[z_\mu, z_\nu] \in H.$$

It follows that

$$[H\Omega(1), H\Omega(1)] \subseteq H.$$

Next let $k \geq 1$ and assume that we have constructed the ascending series $(\Omega(i))_{i \leq k}$ of subgroups of G such that for $i \leq k$

$$H\Omega(i) \leq G \quad \text{and} \quad [H\Omega(i), H\Omega(i)] \subseteq H\Omega(i-1).$$

We shall construct $\Omega(k+1)$ as follows. In the case where $A(\lambda) \not\subseteq H\Omega(k)$, let $r(\lambda)$ be the least integer such that $A(\lambda, r(\lambda)) \not\subseteq H\Omega(k)$ and define

$$W(\lambda, k+1) = \{z \in A(\lambda, r(\lambda)) \mid z^{-1}Hz, zHz^{-1} \subseteq H\Omega(k)\}.$$

Then $W(\lambda, k+1) \not\subseteq H\Omega(k)$. In fact, since $A(\lambda, r(\lambda)) \in \mathfrak{F}$, $A(\lambda, r(\lambda)) \subseteq S_m$ and therefore $HA(\lambda, r(\lambda)) \subseteq S_m$ for some $m \in \mathbf{N}$. It follows that $H \leq^m HA(\lambda, r(\lambda))$ with a weakly ascending series

$$(S_i \cap HA(\lambda, r(\lambda)))_{i \leq m}.$$

Take the least i such that $S_i \cap HA(\lambda, r(\lambda)) \not\subseteq H\Omega(k)$. Then $S_i \cap A(\lambda, r(\lambda)) \not\subseteq H\Omega(k)$. Hence we can choose $z \in (S_i \cap A(\lambda, r(\lambda))) \smallsetminus H\Omega(k)$. Then z^{-1} belongs to the same set and

$$z^{-1}Hz, zHz^{-1} \subseteq S_{i-1} \cap HA(\lambda, r(\lambda)) \subseteq H\Omega(k).$$

Hence $z \in W(\lambda, k+1) \smallsetminus H\Omega(k)$.

In the other case where $A(\lambda) \subseteq H\Omega(k)$, let $r(\lambda) = n(\lambda)$ and define $W(\lambda, k+1)$ as above.

In any case, $W(\lambda, k) \subseteq W(\lambda, k+1)$. In fact, since $H\Omega(k-1) \subseteq H\Omega(k)$, denoting $r(\lambda)$ defined above by $r_k(\lambda)$ we have $r_{k-1}(\lambda) \leq r_k(\lambda)$. Hence $A(\lambda, r_{k-1}(\lambda)) \not\subseteq A(\lambda, r_k(\lambda))$, from which it follows that $W(\lambda, k) \subseteq W(\lambda, k+1)$. Especially in the first case $W(\lambda, k+1) \not\subseteq W(\lambda, k)$ and therefore $W(\lambda, k) \subsetneqq W(\lambda, k+1)$.

Since $W(\lambda, k+1) \subseteq A(\lambda, r(\lambda))$ and $A(\lambda, r(\lambda)-1) \subseteq H\Omega(k)$, we have

$$\begin{aligned}[W(\lambda, k+1), W(\mu, k+1)] &\subseteq [A(\lambda, r(\lambda)), A(\mu, r(\mu))] \\ &\subseteq A(\lambda, r(\lambda)-1)A(\mu, r(\mu)-1) \\ &\subseteq H\Omega(k).\end{aligned}$$

Especially $[W(\lambda, k+1), W(\mu, k)] \subseteq H\Omega(k)$ and therefore

$$[W(\lambda, k+1), \Omega(k)] \subseteq H\Omega(k).$$

We see that $W(\lambda, k+1) \leq G$. In fact, for $z_1, z_2 \in W(\lambda, k+1)$ and $h \in H$ we have

$$z_1^{-1}hz_1 = h_1u \qquad \text{with} \quad h_1 \in H \quad \text{and} \quad u \in \Omega(k),$$

and therefore

$$\begin{aligned}(z_1z_2)^{-1}h(z_1z_2) &= z_2^{-1}(h_1u)z_2\\ &= (z_2^{-1}h_1z_2)[z_2, u^{-1}]u\\ &\in H\Omega(k).\end{aligned}$$

Similarly $(z_1z_2)h(z_1z_2)^{-1}\in H\Omega(k)$. Hence $z_1z_2\in W(\lambda, k+1)$. Since $z_1^{-1}\in W(\lambda, k+1)$, $W(\lambda, k+1)\leq G$. Now we define

$$\Omega(k+1) = \langle W(\lambda, k+1) \mid \lambda\in\Lambda\rangle.$$

Each element of $\Omega(k+1)$ transforms both H and $\Omega(k)$ into $H\Omega(k)$ and therefore normalizes $H\Omega(k)$. Hence $H\Omega(k+1)=H\Omega(k)\Omega(k+1)\leq G$ and $H\Omega(k)\lhd H\Omega(k+1)$. Furthermore $[\Omega(k+1), \Omega(k+1)]\subseteq H\Omega(k)$. It follows that

$$[H\Omega(k+1), H\Omega(k+1)] \subseteq H\Omega(k).$$

Since $W(\lambda, 1)\subsetneqq W(\lambda, 2)\subsetneqq\cdots\subsetneqq W(\lambda, k)\subsetneqq W(\lambda, k+1)$ if $A(\lambda)\not\subseteq H\Omega(k)$, we have

$$A(\lambda)\subseteq H\Omega(i) \qquad \text{for } i=|A(\lambda)|.$$

Hence $G=\bigcup_{i=1}^{\infty} H\Omega(i)$. Thus we conclude that $H\lhd^{\omega} G$.

As a consequence of Theorem 3, we have

THEOREM 4. *Let $G\in \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})$ and $H\leq G$. Then the following conditions are equivalent*:

(1) $H\lhd^{\omega} G$.
(2) $H\lhd^{\omega} K$ *for any subgroup K of G containing H.*
(3) $H\lhd^{\omega}\langle H, x\rangle$ *for any $x\in G$.*
(4) $H\lhd^{\omega}\langle H, [x, H]\rangle$ *for any $x\in G$.*
(5) *For any $x\in G$, there is an $n=n(x)\in\mathbf{N}$ such that $H\lhd^{\omega}\langle H, [x, {}_nH]\rangle$.*

PROOF. Since $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})$ is S-closed, by Theorem 3 ω-step weak ascendancy can be replaced by ω-step ascendancy in Theorem 2.

As another consequence of Theorem 3, we have

PROPOSITION 1. *Let G be a group and $H\leq G$. Assume that $G\in \mathrm{E}\mathfrak{A}(\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F}))$ and $H\in\mathrm{E}\mathfrak{A}$. If H wsn G, then $H\lhd^{\omega} G$.*

PROOF. There is a solvable normal subgroup K of G such that $G/K\in \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A}\cap\mathfrak{F})$. If H wsn G, then H wsn HK. Since $HK\in\mathrm{E}\mathfrak{A}$, H sn HK by Theorem 3 (c) in [2]. On the other hand, HK/K wsn G/K. By Theorem 3, $HK/K\lhd^{\omega} G/K$ and therefore $HK\lhd^{\omega} G$. Thus $H\lhd^{\omega} G$.

4.

In this section we consider the cases where G belongs to $\mathrm{L(sn)}\mathfrak{F}$, $\mathrm{L(sn)E}\mathfrak{A}$ and $\mathrm{L(asc)}\acute{\mathrm{E}}\mathfrak{A}$.

PROPOSITION 2. *Let G be a group and $H \leq G$. Assume that $G \in \mathrm{L(sn)}\mathfrak{F}$ and $H \in \mathfrak{G}$. Then the following conditions are equivalent*:

(1) $H \operatorname{asc} G$.
(2) $H \operatorname{sn} G$.
(3) $H \operatorname{wasc} G$.
(4) $H \operatorname{wsn} G$.

PROOF. It suffices to show that (3)⇒(2). By assumption, there is a subgroup K of G such that

$$H \leq K \operatorname{sn} G \quad \text{and} \quad K \in \mathfrak{F}.$$

If $H \operatorname{wasc} G$, then $H \operatorname{wasc} K$. Since $K \in \mathfrak{F}$, it follows that $H \operatorname{wsn} K$. Hence $H \operatorname{sn} K$ by Corollary to Theorem 4 in [2]. Thus $H \operatorname{sn} G$.

PROPOSITION 3. *Let G be a group and $H \leq G$. Assume that $G \in \mathrm{L(sn)E}\mathfrak{A}$ and $H \in \mathfrak{G}$. Then the following conditions are equivalent*:

(1) $H \operatorname{sn} G$.
(2) $H \operatorname{wsn} G$.

PROOF. By assumption, there is a subgroup K of G such that

$$H \leq K \operatorname{sn} G \quad \text{and} \quad K \in \mathrm{E}\mathfrak{A}.$$

If H wsn G, then H wsn K and hence H sn K by Theorem 3 (c) in [2]. Therefore H sn G.

PROPOSITION 4. *Let G be a group and $H \leq G$. Assume that one of the following conditions is satisfied*:

(a) $G \in \mathrm{L(asc)}(\acute{\mathrm{E}}(\lhd)\mathfrak{A})$ *and* $H \in \mathfrak{G}$.
(b) $G \in \mathrm{L(asc)}\acute{\mathrm{E}}\mathfrak{A}$ *and* $H \in \mathfrak{F}$.

Then the following conditions are equivalent:

(1) $H \operatorname{asc} G$.
(2) $H \operatorname{wasc} G$.

PROOF. Assume (a) (resp. (b)). Then there is a subgroup K of G such that

$$H \leq K \operatorname{asc} G \quad \text{and} \quad K \in \acute{\mathrm{E}}(\lhd)\mathfrak{A} \text{ (resp. } \acute{\mathrm{E}}\mathfrak{A}).$$

If H wasc G, then H wasc K and hence H asc K by Theorem 3 (a) (resp. (b)) in [2]. Therefore H asc G.

5.

In this section we shall study the class $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ and present some of its characterizations.

LEMMA 2. $\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \leq \acute{\mathrm{E}}_{\omega}(\lhd)\mathfrak{A}$.

PROOF. In the proof of Theorem 3, we put $H=1$. Then we see that

$$\Omega(i) \lhd G \quad \text{and} \quad \Omega(i+1)/\Omega(i) \in \mathfrak{A} \qquad \text{for any } i < \omega,$$
$$G = \bigcup_{i<\omega} \Omega(i).$$

Hence $G \in \acute{\mathrm{E}}_{\omega}(\lhd)\mathfrak{A}$.

We denote by $\mathfrak{X}_0$ the class of groups in which every non-trivial finite subgroup H satisfies the condition $H \neq H^{(1)}$ $(=\langle [H, H] \rangle)$.

LEMMA 3. (1) $\{\mathrm{S}, \mathrm{L}, \acute{\mathrm{E}}, \mathrm{R}\}\mathfrak{X}_0 = \mathfrak{X}_0$.
(2) $\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \leq \mathfrak{X}_0$ *and* $\acute{\mathrm{E}}_{\omega}(\lhd)\mathfrak{A} \leq \mathfrak{X}_0$.

PROOF. (1) Assume that $G \in \acute{\mathrm{E}}\mathfrak{X}_0$ and let $(G_\alpha)_{\alpha \leq \sigma}$ be an $\mathfrak{X}_0$-series of G. For any non-trivial finite subgroup H of G, let α be the least ordinal such that $H \leq G_\alpha$. Then α is not a limit ordinal. Since $G_\alpha/G_{\alpha-1} \in \mathfrak{X}_0$ and $HG_{\alpha-1}/G_{\alpha-1}$ is a non-trivial finite subgroup of $G_\alpha/G_{\alpha-1}$, we have $H^{(1)}G_{\alpha-1}/G_{\alpha-1} \neq HG_{\alpha-1}/G_{\alpha-1}$, whence $H^{(1)} \neq H$. Therefore $G \in \mathfrak{X}_0$. Thus we have $\acute{\mathrm{E}}\mathfrak{X}_0 = \mathfrak{X}_0$.

Assume that $N_\lambda \lhd G$ and $G/N_\lambda \in \mathfrak{X}_0$ for any $\lambda \in \Lambda$. Let $H/\bigcap_\lambda N_\lambda$ be a finite non-trivial subgroup of $G/\bigcap_\lambda N_\lambda$. If $(H/\bigcap_\lambda N_\lambda)^{(1)} = H/\bigcap_\lambda N_\lambda$, then $H^{(1)}(\bigcap_\lambda N_\lambda) = H$. Choose λ such that $H \not\leq N_\lambda$. Then $H^{(1)}N_\lambda = HN_\lambda$ from which it follows that $(HN_\lambda/N_\lambda)^{(1)} = HN_\lambda/N_\lambda$. This contradicts the assumption that $G/N_\lambda \in \mathfrak{X}_0$. Hence $(H/\bigcap_\lambda N_\lambda)^{(1)} \neq H/\bigcap_\lambda N_\lambda$ and $G/\bigcap_\lambda N_\lambda \in \mathfrak{X}_0$. Thus we have $\mathrm{R}\mathfrak{X}_0 = \mathfrak{X}_0$.

It is immediate that $\mathrm{S}\mathfrak{X}_0 = \mathfrak{X}_0$ and $\mathrm{L}\mathfrak{X}_0 = \mathfrak{X}_0$.

(2) Both $\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F})$ and $\acute{\mathrm{E}}_{\omega}(\lhd)\mathfrak{A}$ are contained in $\acute{\mathrm{E}}\mathfrak{A}$. But by (1) we have

$$\acute{\mathrm{E}}\mathfrak{A} \leq \acute{\mathrm{E}}\mathfrak{X}_0 = \mathfrak{X}_0.$$

THEOREM 5. *For any class $\mathfrak{X}$ of groups such that*

$$\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_{\omega}(\lhd)\mathfrak{A} \leq \mathfrak{X} \leq \mathfrak{X}_0,$$

we have

$$\mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}).$$

PROOF. Assume that $G \in \mathfrak{X}_0 \cap \mathrm{L}(\lhd)\mathfrak{F}$ and let X be any finite subset of G. Then X is contained in a finite normal subgroup H of G. We use induction on $n=|H|$ to show that $H \in \mathrm{E}\mathfrak{A}$. It clearly holds for $n=1$. Let $n \geq 2$ and assume the case $n-1$. Since $G \in \mathfrak{X}_0$, $H \neq H^{(1)}$. By induction hypothesis we have $H^{(1)} \in \mathrm{E}\mathfrak{A}$. Hence $H \in \mathrm{E}\mathfrak{A}$. It follows that $G \in \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$. Therefore

$$\mathfrak{X}_0 \cap \mathrm{L}(\lhd)\mathfrak{F} \leq \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}).$$

Now let $\mathfrak{X}$ be any class of groups such that

$$\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_{\omega}(\lhd)\mathfrak{A} \leq \mathfrak{X} \leq \mathfrak{X}_0.$$

Then by Lemmas 1 and 2 we have

$$\begin{aligned} \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) &\leq \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_{\omega}(\lhd)\mathfrak{A} \cap \mathrm{L}(\lhd)\mathfrak{F} \\ &\leq \mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} \\ &\leq \mathfrak{X}_0 \cap \mathrm{L}(\lhd)\mathfrak{F} \\ &\leq \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}). \end{aligned}$$

Therefore we have $\mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$.

COROLLARY. *If $\mathfrak{X}$ is one of the classes*

$$\acute{\mathrm{E}}\mathfrak{A},\ \acute{\mathrm{E}}(\lhd)\mathfrak{A},\ \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}),\ \acute{\mathrm{E}}\mathrm{LE}\mathfrak{A},\ \mathrm{RLE}\mathfrak{A},\ \mathrm{R}\acute{\mathrm{E}}\mathfrak{E} \text{ and } \acute{\mathrm{E}}\mathfrak{E},$$

then $\mathfrak{X} \cap \mathrm{L}(\lhd)\mathfrak{F} = \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$.

PROOF. Since $\mathrm{LE}\mathfrak{A} \leq \mathfrak{X}_0$, by Lemma 3 we see that

$$\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \leq \acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \acute{\mathrm{E}}\mathfrak{A} \leq \acute{\mathrm{E}}\mathrm{LE}\mathfrak{A} \leq \acute{\mathrm{E}}\mathfrak{X}_0 = \mathfrak{X}_0 \quad \text{and}$$
$$\acute{\mathrm{E}}_{\omega}(\lhd)\mathfrak{A} \leq \mathrm{LE}\mathfrak{A} \leq \mathrm{RLE}\mathfrak{A} \leq \mathrm{R}\mathfrak{X}_0 = \mathfrak{X}_0.$$

Since $\mathfrak{E} \cap \mathfrak{F} \leq \mathfrak{N}$, we have $\mathfrak{E} \leq \mathfrak{X}_0$. Hence by Lemma 3 we see that

$$\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \leq \acute{\mathrm{E}}\mathfrak{E} \leq \mathrm{R}\acute{\mathrm{E}}\mathfrak{E} \leq \mathrm{R}\acute{\mathrm{E}}\mathfrak{X}_0 = \mathfrak{X}_0.$$

Thus by Theorem 5 we have the assertion.

References

[1] E. L. Stitzinger: Ascendancy in locally solvable, ideally finite Lie algebras, Hiroshima Math. J., **11** (1981), 1–4.

[2] S. Tôgô: Subnormality and ascendancy in groups, Hiroshima Math. J., **10** (1980), 597–605.

[3] S. Tôgô, M. Honda and T. Sakamoto: Ideally finite Lie algebras, Hiroshima Math. J., **11** (1981), 299–315.

Department of Mathematics,
Faculty of Science,
Hiroshima University