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Growth estimates for the coefficients of generalized formal solutions, and representation of solutions using Laplace integrals and factorial series

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0. Introduction

a) Functions represented by Laplace integrals:

A question of considerable interest in the theory of meromorphic differential equations as well as in other fields of analysis is, given a function $f(z)$ that has an asymptotic expansion as $z \to \infty$ (in some sector), can $f(z)$ be represented by means of a (generalized) Laplace integral such that one can find a convergent expansion of the integrand (in some neighborhood of zero) in terms of the asymptotic expansion of $f(z)$?

In 1918, F. Nevanlinna [17] has given an answer to this question, generalizing some earlier results of G. N. Watson [23]: Suppose that (for fixed reals $a \geq 0$ and $d > 0$) the function $f(z)$ is analytic in the sector

$$S = \{|z| > a, \quad |\arg z| < \pi/(2d)\}$$

(note that throughout this paper the variable z is on the Riemann surface of the Logarithm, hence in case $d < 1/2$ the function $f(z)$ may be multi-valued). Furthermore, assume the existence of a formal power series

$$\sum_1^\infty f_k z^{-k}$$

such that for some positive constant K and every sufficiently large integer j

$$|z^j(f(z) - \sum_1^{j-1} f_k z^{-k})| \leq K^j \Gamma(1 + j/d) \qquad (z \in S). \tag{0.1}$$

Then it is easy to conclude

$$|f_j| \leq K^j \Gamma(1 + j/d) \quad \text{for sufficiently large} \quad j,$$

hence the power series

$$\psi(u) = \sum_1^\infty f_k u^k / \Gamma(1 + k/d) \tag{0.2}$$

converges for $|u| < K^{-1}$. Representing $\psi(u)$ as a generalized inverse Laplace integral over $f(z)$, F. Nevanlinna proved that $\psi(u)$ can be analytically continued into an (explicitly given) region containing the positive real axis, and for every

$\varepsilon>0$ there exists a constant C_ε such that in a slightly smaller region

$$|\psi(u)| \leq C_\varepsilon \exp\{(a+\varepsilon)^d \operatorname{Re} u^d\}. \tag{0.3}$$

This estimate is then used to show

$$f(z) = z^d \int_0^\infty \psi(u^{1/d}) \exp\{-uz^d\} du \tag{0.4}$$

for every z with $\operatorname{Re} z^d > a$, and the integral converges absolutely and uniformly if $\operatorname{Re} z^d \geq a+\varepsilon$ (for every $\varepsilon>0$).

In order to obtain such a representation for $f(z)$ with $\psi(u)$ as in (0.2), (0.3), the assumptions on $f(z)$ are not far from being necessary, since (0.4) can be seen to satisfy (0.1) for $\operatorname{Re} z^d \geq a+\varepsilon$ (with arbitrary $\varepsilon>0$ and K depending upon ε), and the boundary curve $\operatorname{Re} z^d = a+\varepsilon$ has asymptotes parallel to the rays $\arg z = \pm\pi/(2d)$.

Obviously, it is no real restriction that the sector S was taken symmetric to the positive real axis. Making a change of variable $z \mapsto ze^{i\varphi}$ one can treat other sectors as well and obtain a representation (0.4) when integrating along an appropriate ray rather than the positive real axis (cf. Theorem 2).

To see how these results compare to the situation in the theory of meromorphic differential equations, we consider a fixed, but arbitrary equation

$$zx' = A(z)x, \quad A(z) = z^r \textstyle\sum_0^\infty A_k z^{-k}, \tag{0.5}$$

with coefficients of size $n \times n$, such that the series converges for $|z|>a$ (with suitable real $a \geq 0$). It is well known that every such equation has a formal fundamental solution

$$H(z) = F(z) z^L \exp\{Q(z)\} \tag{0.6}$$

with

$$Q(z) = \operatorname{diag}[q_1(z), \ldots, q_n(z)] \tag{0.7}$$

being a diagonal matrix of polynomials in a root of z without constant terms (note that $q_1(z), \ldots, q_n(z)$ need not be distinct), L being a constant matrix, and $F(z)$ being a *formal meromorphic transformation*, i.e. a formal series

$$F(z) = \textstyle\sum_k F_k z^{-k}, \qquad F_{-k} = 0 \quad \text{for large} \quad k \tag{0.8}$$

having an inverse of the same kind (which means that $\det F(z)$ is not the zero series). For the existence of formal solutions, see [9], [22]; for formal solutions of this form, see [2], [3], or [13].

A basic theorem, the Asymptotic Existence Theorem ([22], p. 111) states that to every sector

$$S = S(\alpha, \beta) = \{z;\ |z| > a,\ \alpha < \arg z < \beta\}$$

with sufficiently small opening $\beta - \alpha > 0$, there exists a fundamental solution of the form

$$Y(z; S) = F(z; S) z^L \exp\{Q(z)\} \tag{0.9}$$

such that

$$F(z; S) \cong F(z) \quad \text{in} \quad S$$

in the sense that for every integer j

$$F(z; S) - \sum^{j-1} F_k z^{-k} = O(z^{-j}) \quad \text{in} \quad S \tag{0.10}$$

(note that throughout this paper, if nothing else is said, Landau's symbols O and o are always to be interpreted for $|z| \to \infty$, z in some sector S, such that the O-constant and the convergence to zero are always uniform in every closed subsector of S; if the opening of S happens to be infinite, the phrase "closed subsector" will mean a closed subsector *of finite opening*).

To the author's best knowledge, no direct estimates of the constant in (0.10) (for a closed subsector of S) have been given so far which would make F. Nevanlinna's result applicable. Nonetheless, in the so-called *distinct eigenvalue case*, i.e. when A_0 in (0.5) has n distinct eigenvalues, it is known ([22], [10], [11], [6], [7]) that to every sector S of opening π/r a solution $Y(z; S)$ as in (0.9) exists where $F(z; S)$ can be represented in terms of a Laplace integral, very similar to (0.4). The proofs kind of invert F. Nevanlinna's arguments by showing the convergence of a series analogous to (0.2) (with $d = r$) through a direct estimate upon the coefficients F_k, and from an integral equation an estimate analogous to (0.3) is obtained.

Trjitzinsky [20] and Turrittin [21] have obtained similar results in somewhat more general cases, however a direct analogy cannot hold in general: Consider the different values of

$$d_{jk} = \deg(q_j(z) - q_k(z)), \quad 1 \leq j,\ k \leq n, \tag{0.11}$$

i.e. the rational exponents of the leading term (as $z \to \infty$) of the differences of any two diagonal elements of $Q(z)$; if the difference is identically zero, then $d_{jk} = -\infty$. Ignoring $-\infty$, let (if there are any others)

$$d_1 > d_2 > \cdots > d_t > 0 \tag{0.12}$$

denote the different values in decreasing order (recall that every $q_j(z)$ has zero constant term, hence $d_t > 0$). If all the polynomials are identical (in which case we define $t = 0$), then $z = \infty$ is *an almost regular singular point* of (0.5) in the sense that a scalar-exponential transformation

$$x = \exp\{q(z)I\}\tilde{x}, \quad q(z) = q_1(z) = \cdots = q_n(z)$$

takes (0.5) into an equation having a regular singularity at ∞ (note that in such a case $q(z)$ can be seen to even be a polynomial in z, i.e. not to contain any fractional powers of z). Hence in this case $F(z)$ must converge. Therefore we may assume that $z=\infty$ is an *essentially irregular singularity* of (0.5), i.e. $t \geq 1$ in (0.12). In this case, in [16], [5] one can find an estimate of the coefficients of $F(z)$ in terms of a constant to the k times $\Gamma(1+k/d_t)$, and there is every reason to believe that generally d_t is best-possible in the sense that such an estimate fails (with d in place of d_t) for $d > d_t$. On the other hand, it can be concluded from a characterization of asymptotic sectors (see [5], [13]) that generally only for sectors S of opening π/d_1 there exists a solution $Y(t; S)$ satisfying (0.9), (0.10). Therefore, a series analogous to (0.2) would only converge for $d \leq d_t$, whereas an integral like (0.4) then would have an asymptotic in too large a sector to represent a solution of (0.5), except for $t=1$. (Note that the distinct eigenvalue case is such a case.) In fact, the cases which Turrittin [21] was able to handle, are essentially those where $t=1$ (for representing certain column vector solutions, one can do slightly more general cases, however not for a fundamental solution).

b) Formal solutions with less rapidly growing coefficients:

Since π/d_1 is, generally, the least upper bound upon the opening of sectors S such that a solution satisfying (0.9), (0.10) exists, the only way to generalize the above results to cases $t > 1$ is in allowing generalized formal solutions where the coefficients of $F(z)$ grow at a reduced rate: A formal solution of the usual type is of the kind

$$H(z) = F(z)G(z)$$

where $G(z)$ only involves elementary transcendental functions. If we allow more general functions, then $F(z)$ may have less rapidly growing coefficients: as a trivial example, take $F(z) \equiv I$, and let $G(z)$ be some fundamental solution of (0.5); to make this idea lead to a non-trivial result, we should, however, restrict ourselves to functions $G(z)$ which have a less complicated type of singularity at $z=\infty$ than the solutions of (0.5).

In [1], [2] (see also Section 3 of this paper), the author has introduced *formal solutions of first level*. The main purpose of this paper is to show that the coefficients of first level formal solutions grow at the right order to allow a representation of solutions in terms of Laplace integrals and higher transcendental functions which are simpler (at ∞) than the solutions of (0.5). In a sense being explained later, the degree of complexity of the functions considered is measured by the parameter t, so that for $t=1$ our results will coincide with the classical ones, while for $t>1$ one could iterate our result to finally obtain a representation of solutions as a product of factors, each of which having a Laplace integral representation, times a final factor $z^L \exp\{Q(z)\}$.

When preparing this paper, the author became aware of an expository article of J. P. Ramis [18], which outlines a general theory of those formal series which can be "summed" by either Laplace integrals or generalized factorial series, referring to a number of papers "en préparation" for most of the proofs. As an application to meromorphic differential equations, he proves the possibility of factoring a formal solution of an equation (0.5) into factors that may be individually summed. However, he does not have any result upon the Stokes' phenomenon of the solutions obtained by summing the formal one as described above, nor can he guarantee that using the sum of the first one of the factors as a transformation of the given equation would lead again to a meromorphic equation. Our result can be thought of as one step of a program which by iteration would give a factorization of formal solutions as the one obtained by Ramis, but we do also control the Stokes' phenomenon, and the individual factors obtained are "sectorial transformations", which transform a given meromorphic equation into another one and have (in certain overlapping sectors) a clear asymptotic behavior.

In detail, we proceed as follows: In Section 1 we study sectorial transformations and establish an integral representation for the error term in the asymptotic expansion that is later used to estimate the constant in (0.10). As a side-product, we prove the existence of a meromorphic function that interpolates the coefficients F_k of a formal meromorphic transformation; the integral representation of this meromorphic function may be made a starting point for a detailed study of the behavior of F_k as $k \to \infty$ (see Remark 2.3).

In Section 2 we explain how the formulas obtained so far may be used in the so-called distinct eigenvalue case to reprove formulas representing solutions by means of Laplace integrals. In Section 3 we recall the definition of formal solutions of first level introduced in [1], [2], and in Section 4 we show that the corresponding normal solutions of first level can be represented by means of Laplace integrals in a manner completely analogous to the distinct eigenvalue case. Section 5 then is devoted to a representation using (generalized) factorial series.

1. Sectorial transformations

Throughout this section, we consider two fixed polynomial equations

$$zx' = (z^r A_0 + \cdots + A_r)x = A(z)x \tag{1.1}$$

and

$$zx̃' = (z^r \tilde{A}_0 + \cdots + \tilde{A}_r)\tilde{x} = \tilde{A}(z)\tilde{x} \tag{1.2}$$

(where r is some integer ≥ 1), and we assume (1.1) to be *formally meromorphically equivalent to* (1.2) by means of a fixed formal meromorphic transformation $F(z)$ (cf. Introduction), i.e. $F(z)$ satisfies formally

$$zF'(z) = A(z)F(z) - F(z)\tilde{A}(z). \tag{1.3}$$

We will, throughout, refer to the above situation as our *basic situation*. Note that we do not exclude cases where A_0 or $\tilde{A}_0$ (or both) are zero matrices, since for notational convenience we prefer to have the same number r in both equations.

Replacing $F(z)$ in (1.3) by the formal power series $\sum F_k z^{-k}$ and identifying like coefficients, we find

$$(A_r + kI)F_k - F_k\tilde{A}_r + \textstyle\sum_{j=1}^{r} (A_{r-j}F_{k+j} - F_{k+j}\tilde{A}_{r-j}) = 0 \tag{1.4}$$

which may be regarded as a difference equation and is used later.

For some natural N, let reals $\alpha_0,\dots,\alpha_{N-1}$ be given such that

$$0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_{N-1} < 2\pi, \tag{1.5}$$

and define

$$\alpha_{-1} = \alpha_{N-1} - 2\pi,\ \alpha_N = \alpha_0 + 2\pi,\ \alpha_{N+1} = \alpha_1 + 2\pi. \tag{1.6}$$

In our basic situation, every such choice of reals $\alpha_0,\dots,\alpha_{N-1}$ will be called *admissible* iff there are matrices $F_\nu(z)$ (for $\nu=0,\dots,N-1$) which are analytic on the Riemann surface of the Logarithm and satisfy

$$zF_\nu'(z) = A(z)F_\nu(z) - F_\nu(z)\tilde{A}(z), \tag{1.7}$$

$$F_\nu(z) \cong F(z) \qquad \text{in} \quad S(\alpha_{\nu-1}, \alpha_{\nu+1}). \tag{1.8}$$

Defining $F_N(z)=F_0(ze^{-2\pi i})$ (note that $z \in S(\alpha_{N-1}, \alpha_{N+1})$ iff $ze^{-2\pi i} \in S(\alpha_{-1}, \alpha_1)$), we find that (1.7) and (1.8) hold for $\nu=N$ as well. Every such system

$$F_0(z),\dots, F_{N-1}(z);\ F_N(z) = F_0(ze^{-2\pi i})$$

will be called *a system of sectorial transformations corresponding to* $\alpha_0,\dots,\alpha_{N-1}$.

REMARK 1.1. A choice of reals $\alpha_0,\dots,\alpha_{N-1}$ (with implicit definition of $\alpha_{-1}, \alpha_N, \alpha_{N+1}$ as in (1.6)) is certainly admissible, if

$$\delta = \max\{\alpha_{\nu+1} - \alpha_{\nu-1};\ 0 \leq \nu \leq N\}$$

is sufficiently small. To see this, let $H(z)$ and $\tilde{H}(z)$ be formal fundamental solutions of (1.1) and (1.2), resp., such that $H(z)=F(z)\tilde{H}(z)$. Then for sufficiently small δ there exist solutions $Y_\nu(z)$, $\tilde{Y}_\nu(z)$ being asymptotic to $H(z)$, $\tilde{H}(z)$ (resp.) in $S(\alpha_{\nu-1}, \alpha_{\nu+1})$, and $F_\nu(z)=Y_\nu(z)\tilde{Y}_\nu^{-1}(z)$ then satisfies (1.7), (1.8) $(0\leq\nu\leq N-1)$.

REMARK 1.2. Note that (1.7) is a system of homogeneous linear differential equations for the components of $F_\nu(z)$. This system has a singularity of the first kind at $z=0$, hence $F_\nu(z)$ does not grow faster than some fixed power of z as $z\to 0$.

Next we prove a formula for the error term

$$R_{k,\nu}(z) = z^k(F_\nu(z) - \textstyle\sum^{k-1} F_j z^{-j}) \tag{1.9}$$

of sectorial transformations, that will later be used to estimate $R_{k,\nu}(z)$:

PROPOSITION 1. *In our basic situation, let*

$$F_0(z),\dots, F_{N-1}(z);\ F_N(z) = F_0(ze^{-2\pi i})$$

be a system of sectorial transformations corresponding to reals $\alpha_0,\dots,\alpha_{N-1}$.[1] *For an arbitrarily fixed fundamental solution* $G(z)$ *of* (1.2) *and reals* $\gamma_\mu\in(\alpha_{\mu-1}, \alpha_\mu)$, $1\leq\mu\leq N$, $\gamma_0=\gamma_N-2\pi$, *we have*

$$R_{k,\nu}(z) = -(z/2\pi i)\textstyle\sum_{\mu=1}^N \displaystyle\int_0^{\infty(\gamma_\mu)} (\zeta^{k-1}/(\zeta-z))F_\mu(\zeta)G(\zeta)\,[W_\mu-I]G^{-1}(\zeta)d\zeta \tag{1.10}$$

for every sufficiently large integer k and

$$\gamma_\nu < \arg z < \gamma_{\nu+1} \qquad (0\leq\nu\leq N-1); \tag{1.11}$$

herein the matrices W_μ *are defined by*

$$F_{\mu-1}(z)G(z) = F_\mu(z)G(z)W_\mu \qquad (1\leq\mu\leq N). \tag{1.12}$$

PROOF. Since $G(z)$ is a fundamental solution of (1.2) and $F_\nu(z)$ satisfies (1.7), it is easy to see that $F_\nu(z)G(z)$ is a fundamental solution of (1.1), $0\leq\nu\leq N$; hence there are unique constant, invertible matrices W_μ satisfying (1.12). From (1.12) we further see

$$F_\mu(\zeta)G(\zeta)\,[W_\mu-I]G^{-1}(\zeta) = F_{\mu-1}(\zeta) - F_\mu(\zeta), \qquad 1\leq\mu\leq N, \tag{1.13}$$

hence using (1.8) we obtain that the integrands in (1.10) are all asymptotically zero as $z\to\infty$ in $S(\alpha_{\mu-1}, \alpha_\mu)$ $(1\leq\mu\leq N)$. On the other hand, from Remark 1.2 we find that the integrals exist at $\zeta=0$ for sufficiently large k. To evaluate them, we fix ν and k (sufficiently large) and integrate from 0 to $a_\mu=a_\mu(R)$, where

1) Whenever we consider some choice of reals $\alpha_0,\dots,\alpha_{N-1}$, we implicitly assume (1.5) and (1.6).

$$|a_\mu| = R, \quad \arg a_\mu = \gamma_\mu \quad (1 \le \mu \le N).$$

Then we obtain from (1.13)

$$\sum_{\mu=1}^{N} \int_0^{a_\mu} (\zeta^{k-1}/(\zeta - z)) F_\mu(\zeta) G(\zeta) [W_\mu - I] G^{-1}(\zeta) d\zeta$$
$$= \sum_{\mu=1}^{N-1} \left(\int_0^{a_{\mu+1}} - \int_0^{a_\mu} \right) (\zeta^{k-1}/(\zeta - z)) F_\mu(\zeta) d\zeta$$
$$+ \int_0^{a_1} (\zeta^{k-1}/(\zeta - z)) F_0(\zeta) d\zeta - \int_0^{a_N} (\zeta^{k-1}/(\zeta - z)) F_N(\zeta) d\zeta$$
$$= \sum_{\mu=0}^{N-1} \left(\int_0^{a_{\mu+1}} - \int_0^{a_\mu} \right) (\zeta^{k-1}/(\zeta - z)) F_\mu(\zeta) d\zeta;$$

note that $F_N(\zeta) = F_0(\zeta e^{-2\pi i})$ and make a change of variable. Using Cauchy's Theorem, we may for $\mu \neq \nu$ replace the difference of the integrals from 0 to $a_{\mu+1}$, resp. a_μ by a single integral along $|\zeta| = R$ from a_μ to $a_{\mu+1}$. For $\mu = \nu$, since $\gamma_\nu < \arg z < \gamma_{\nu+1}$, we gain a term $-2\pi i\, z^{k-1} F_\nu(z)$ when making the same change in the path of integration (for $R > |z|$) hence the resulting expression is

$$-\,2\pi i\, z^{k-1} F_\nu(z) + \sum_{\mu=0}^{N-1} \int_{a_\mu}^{a_{\mu+1}} (\zeta^{k-1}/(\zeta - z)) F_\mu(\zeta) d\zeta.$$

To evaluate the limit when $R \to \infty$, we replace $F_\mu(\zeta)$ by $\sum^{k-1} F_j \zeta^{-j} + O(\zeta^{-k})$, and since the O-constant is uniform for $\zeta \in S(\gamma_\mu, \gamma_{\mu+1})$, we see that the integral of the error term vanishes as $R \to \infty$, whereas the main term's integral is independent of R as long as $R > |z|$. Therefore we obtain for the right hand side of (1.10)

$$z^k F_\nu(z) - (z/2\pi i) \oint_{|\zeta| = R} (\zeta^{k-1}/(\zeta - z)) \left(\sum^{k-1} F_j \zeta^{-j}\right) d\zeta$$
$$= z^k \left(F_\nu(z) - \sum^{k-1} F_j z^{-j}\right) = R_{k,\nu}(z).$$

Using (1.10), we next obtain a meromorphic function that interpolates the coefficients F_k of $F(z)$ and satisfies the same difference equation as the sequence $\{F_k\}$:

PROPOSITION 2. *Under the same assumptions as in Proposition* 1, *the integral*

$$T(u) = (1/2\pi i) \sum_{\mu=1}^{N} \int_0^{\infty(\gamma_\mu)} \zeta^{u-1} F_\mu(\zeta) G(\zeta) [W_\mu - I] G^{-1}(\zeta) d\zeta \tag{1.14}$$

converges for Re u *sufficiently large and defines a meromorphic function in the u-plane with possible poles at points of the form*

$$u = \tilde{\mu} - \mu, \quad \tilde{\mu} - \mu - 1, \quad \tilde{\mu} - \mu - 2, \ldots \tag{1.15}$$

where μ, $\tilde{\mu}$ *can be any eigenvalue of* A_r, $\tilde{A}_r$, *resp. Moreover, for every* u *not listed in* (1.15)

$$(1.16)\quad (A_r+uI)T(u) - T(u)\tilde{A}_r + \textstyle\sum_{j=1}^{r} (A_{r-j}T(u+j)-T(u+j)\tilde{A}_{r-j}) = 0,$$

and

$$(1.17)\qquad T(k) = F_k \quad \textit{for sufficiently large integer } k.$$

PROOF. The convergence of (1.14) for sufficiently large $\operatorname{Re} u$ follows from the same arguments as the convergence of (1.10). By partial integration

$$uT(u) = -(1/2\pi i)\textstyle\sum_{\mu=1}^{N}\displaystyle\int_0^{\infty(\gamma_\mu)} \zeta^u(d/d\zeta)\{F_\mu(\zeta)G(\zeta)[W_\mu-I]G^{-1}(\zeta)\}d\zeta.$$

Since $F_\mu(\zeta)G(\zeta)$, resp. $G(\zeta)$ is a fundamental solution of (1.1), resp. (1.2), we obtain

$$\begin{aligned}&\zeta(d/d\zeta)\{F_\mu(\zeta)G(\zeta)[W_\mu-I]G^{-1}(\zeta)\}\\ &\quad = A(\zeta)F_\mu(\zeta)G(\zeta)[W_\mu-I]G^{-1}(\zeta) - F_\mu(\zeta)G(\zeta)[W_\mu-I]G^{-1}(\zeta)\tilde{A}(\zeta),\end{aligned}$$

and inserting this into the integral for $uT(u)$ proves (1.16) for sufficiently large $\operatorname{Re} u$. Provided A_r+uI and $\tilde{A}_r$ do not have an eigenvalue in common, (1.16) provides a system of linear equations that can be uniquely solved for $T(u)$, and since the determinant of the coefficient matrix is a polynomial in u, we see that $T(u)$ can have at most poles at those points u listed in (1.15). To see (1.17), observe for arbitrary ν, $0\le\nu\le N-1$, using (1.10), that

$$\begin{aligned}F_k &= R_{k,\nu}(z) - z^{-1}R_{k+1,\nu}(z)\\ &= (1/(2\pi i))\textstyle\sum_{\mu=1}^{N}\displaystyle\int_0^{\infty(\gamma_\mu)} \{\zeta^k/(\zeta-z)) - (z\zeta^{k-1}/(\zeta-z))\}\times\\ &\qquad\qquad F_\mu(\zeta)G(\zeta)[W_\mu-I]G^{-1}(\zeta)d\zeta = T(k).\end{aligned}$$

REMARK 1.3. In case $F(z)$ converges, one may take $F_\nu(z)=F(z)$ as sectorial transformations, hence $W_\nu=I$ $(1\le\nu\le N)$. In this case from (1.14), (1.17) we conclude $T(u)\equiv 0$, hence $F_k=0$ for sufficiently large k. The latter also follows directly, since a convergent $F(z)$ satisfying (1.3) must have a regular singularity at $z=0$, hence can only be a polynomial times an integer power of z.

2. The case of distinct eigenvalues

To apply the formulas obtained in Section 1, we first prove, as a main tool, an estimate upon integrals of the kind used in (1.10):

LEMMA 1. *Let a complex number* $\lambda\neq 0$, *reals* α, β *with* $\beta-\alpha>0$ *and a posi-*

tive real d be given, such that

$$\text{(2.1)} \qquad \operatorname{Re}(\lambda z^d) < 0 \qquad \textit{for} \quad \alpha < \arg z < \beta.$$

Furthermore, let $E(z)$ (for $\alpha < \arg z < \beta$) be an $s \times s$ analytic matrix (for some fixed natural s) such that for every sufficiently small $\varepsilon > 0$ there exist positive real constants c_1, c_2 with

$$\text{(2.2)} \qquad \|E(z)\| \leq |z|^{-c_1} c_2 \exp\{|z|^{d-\varepsilon}\} \qquad \textit{for} \quad \alpha < \arg z < \beta.$$

Then for every $\gamma \in (\alpha, \beta)$, the function

$$\text{(2.3)} \qquad R_k(z) = -(z/(2\pi i)) \int_0^{\infty(\gamma)} (\zeta^{k-1}/(\zeta - z)) E(\zeta) \exp(\lambda \zeta^d) d\zeta$$

with integer $k > c_1$ is analytic in the sector $\gamma - 2\pi < \arg z < \gamma$ and can be analytically continued (by altering γ) into the sector

$$\text{(2.4)} \qquad \alpha - 2\pi < \arg z < \beta.$$

Furthermore, for every $\delta > 0$ there exists a constant K (which is independent of k) such that for $k > c_1$

$$\text{(2.5)} \qquad \|R_k(z)\| \leq K^k \Gamma(1 + k/d)$$

in the sector $\alpha + \delta - 2\pi \leq \arg z \leq \beta - \delta$.

PROOF. According to our assumptions, the integral in (2.3) converges absolutely for $k > c_1$, $\gamma - 2\pi < \arg z < \gamma$, hence the analyticity of $R_k(z)$ is obvious, and $R_k(z)$ does not depend upon γ in the sense that for different γ's the integral has the same value for z in the intersection of the corresponding sectors. Therefore $R_k(z)$ can be analytically extended (by altering γ) to the sector (2.4). This only leaves to prove (2.5).

Let $\delta > 0$ be given (without loss in generality, assume $\delta \leq \min\{\beta - \alpha, \pi\}$) and let z with $\alpha + \delta - 2\pi \leq \arg z \leq \beta - \delta$ be arbitrarily fixed. Then take γ such that

$$\alpha + \delta/2 \leq \gamma \leq \beta - \delta/2$$

and

$$\gamma + \delta/2 - 2\pi \leq \arg z \leq \gamma - \delta/2$$

(note that the second formula holds iff $\arg z + \delta/2 \leq \gamma \leq \arg z - \delta/2 + 2\pi$). According to (2.1) there exists a positive constant c_3 (which is independent of γ as long as $\alpha + \delta/2 \leq \gamma \leq \beta - \delta/2$) such that

$$\operatorname{Re}(\lambda \zeta^d) \leq -c_3 |\zeta|^d \qquad \text{for} \quad \arg \zeta = \gamma.$$

Furthermore, since $\delta/2 \leq \arg(\zeta/z) \leq 2\pi - \delta/2$ and $\delta/2 \leq \pi/2$, it follows from an easy geometric argument, that

$$|(\zeta - z)/z| = |1 - \zeta/z| \geq \sin(\delta/2) \qquad \text{for} \quad \arg \zeta = \gamma.$$

Hence, using (2.2) with $0 < \varepsilon < d$ fixed, we obtain

$$\|R_k(z)\| \leq (c_2/2\pi \sin(\delta/2)) \int_0^\infty x^{k-c_1-1} \exp\{(-c_3 + x^{-\varepsilon})x^d\} dx$$

where integration is along $\arg x = 0$ and the power of x is taken to be its principal value. Since $\exp\{(-c_3 + x^{-\varepsilon})x^d\} \exp\{x^d c_3/2\}$ is bounded for $x \geq 0$, there exists a positive constant c_4 such that

$$\exp\{(-c_3 + x^{-\varepsilon})x^d\} \leq c_4 \exp\{-x^d c_3/2\} \qquad \text{for} \quad x \geq 0.$$

Using this estimate and making a change of variable $t = x^d c_3/2$, we find

$$\|R_k(z)\| \leq (c_2 c_4/d 2\pi \sin(\delta/2))(2/c_3)^{(k-c_1)/d} \Gamma((k-c_1)/d) \quad \text{for} \quad k > c_1.$$

Observing that $\Gamma((k-c_1)/d)/\Gamma(1+k/d)$ does not grow faster than a power of k, one now can obtain (2.5).

REMARK 2.1. Note that the integrand in (2.3) is a single-valued function of z, hence $R_k(z)$ might as well be considered in the sector

$$\alpha < \arg z < \beta + 2\pi,$$

and clearly satisfies (2.5) for

$$\alpha + \delta \leq \arg z \leq \beta + 2\pi - \delta$$

(with the same constant $K = K(\delta)$). This will be of importance in the applications.

As a first application of Lemma 1, we consider throughout the remaining part of this section a fixed equation (1.1) for which the coefficient A_0 has n distinct eigenvalues, say $\lambda_1, \ldots, \lambda_n$. Then it is well known that (1.1) has a formal fundamental solution

$$H(z) = F_a(z) z^{\Lambda'} \exp\{Q(z)\}, \tag{2.6}$$

where $Q(z)$ is a diagonal matrix of polynomials in z without constant terms with leading terms $\lambda_1 z^r/r, \ldots, \lambda_n z^r/r$, resp.; Λ' is a diagonal matrix of entries $\lambda'_1, \ldots, \lambda'_n$, say, and $F_a(z)$ is a formal power series in z^{-1} beginning with an invertible matrix. With

$$\tilde{\Lambda}(z) = z(d/dz)Q(z) + \Lambda' = z^r \Lambda_0 + z^{r-1} \Lambda_1 + \cdots + z \Lambda_{r-1} + \Lambda' \tag{2.7}$$

(note that then $\Lambda_0 = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$), we see that $F(z) = F_a(z)$ satisfies (1.3). Hence with these choices of $\tilde{A}$ and F we are in what we call our basic situation, and it appears natural to take

$$G(z) = z^{\Lambda'} \exp\{Q(z)\} \tag{2.8}$$

as the fundamental solution in Proposition 1. In this case, for a system of sectorial transformations corresponding to $\alpha_0, \ldots, \alpha_{N-1}$, if W_μ is defined by (1.12), then

$$\exp\{Q(z)\}[W_\mu - I]\exp\{-Q(z)\} \cong 0 \qquad \text{in} \quad S(\alpha_{\mu-1}, \alpha_\mu) \tag{2.9}$$

($1 \le \mu \le N$). However, (2.9) holds iff W_μ has ones along the diagonal and zeros in all off-diagonal positions (j, k) except for those for which

$$\operatorname{Re}\{(\lambda_j - \lambda_k)z^r/r\} < 0 \qquad \text{for} \quad \alpha_{\mu-1} < \arg z < \alpha_\mu. \tag{2.10}$$

Consequently, for every μ, $1 \le \mu \le N$, the integral

$$-(z/2\pi i)\int_0^{\infty(\gamma_\mu)} (\zeta^{k-1}/(\zeta - z))F_\mu(\zeta)G(\zeta)[W_\mu - I]G^{-1}(\zeta)d\zeta$$

is a finite sum of integrals, to each of which Lemma 1 (with $s=1$ and $d=r$) applies. Hence each term in the sum in (1.10) may be estimated by $K^k\Gamma(1+k/r)$ with a constant $K = K(\delta) > 0$, and the estimate for each term holds in

$$\alpha_{\mu-1} - 2\pi + \delta \le \arg z \le \alpha_\mu - \delta$$

or in (cf. Remark 2.1)

$$\alpha_{\mu-1} + \delta \le \arg z \le \alpha_\mu + 2\pi - \delta.$$

Altogether, we obtain the following

COROLLARY 1. *Under the assumptions made above, for every $\delta > 0$ there exists a constant $K = K(\delta) > 0$ such that for every sufficiently large integer k and $0 \le \nu \le N-1$*

$$\|R_{k,\nu}(z)\| \le K^k\Gamma(1+k/r), \quad \alpha_{\nu-1} + \delta \le \arg z \le \alpha_\nu - \delta. \tag{2.11}$$

In general, the opening of the sector $S(\alpha_{\nu-1}, \alpha_\nu)$ is too small to allow an application of F. Nevanlinna's result described in the introduction. However, for a certain choice of sectorial transformations, the estimate (2.11) will turn out to hold in a large enough sector:

In the distinct eigenvalue case, a number τ is called a Stokes' direction for the equation (1.1), if for any pair of natural numbers (j, k), $1 \le j, k \le n$, and sufficiently small $\delta > 0$ we have

(2.12) $\operatorname{Re}\{(\lambda_j-\lambda_k)z^r\} < 0$, resp. > 0 for $\arg z = \tau-\delta$, resp. $\tau+\delta$.

It is obvious that the set of all Stokes' rays is discrete, hence may be uniquely indexed, using integer indices, such that

(2.13) $$\tau_{-1} < 0 \leq \tau_0,\ \tau_\nu < \tau_{\nu+1} \quad \text{for every integer} \quad \nu.$$

It follows immediately from (2.12) that τ is a Stokes' direction iff so is $\tau+2\pi$, hence the number of Stokes' directions in any half-open interval of length 2π is always the same, say m, and

(2.14) $$\tau_{\nu+m} = \tau_\nu + 2\pi \quad \text{for every integer} \quad \nu.$$

Using purely algebraic arguments, one can show the existence of a *unique* system of sectorial transformations $F_0(z), \ldots, F_{m-1}(z)$; $F_m(z) = F_0(ze^{-2\pi i})$ (corresponding to $F_a(z)$ and $\tau_0, \ldots, \tau_{m-1}$) satisfying (1.7) (with $\tilde{A}(z)$ as in (2.7)) and

(2.15) $$F_\nu(z) \cong F_a(z) \quad \text{in} \quad S(\tau_\nu - \pi/r,\ \tau_{\nu+1}), \quad 0 \leq \nu \leq m$$

(cf. [5]; the matrices $X_\nu(z) = F_\nu(z)z^{\Lambda'} \exp\{Q(z)\}$ are the normal solutions). We will refer to this unique system as *the normalized system of sectorial transformations corresponding to $F_a(z)$ (and* (1.1)).

For the normalized system of sectorial transformations we find for fixed μ, $1 \leq \mu \leq m = N$

$$F_{\mu-1}(\zeta) - F_\mu(\zeta) = F_\mu(\zeta)G(\zeta)[W_\mu - I]G^{-1}(\zeta) \cong 0 \quad \text{in} \quad S(\tau_\mu - \pi/r,\ \tau_\mu),$$

which holds iff $W_\mu - I$ has zeros in all positions (j, k) except for those for which $\operatorname{Re}\{(\lambda_j-\lambda_k)z^r/r\} < 0$ for $\tau_\mu - \pi/r < \arg z < \tau_\mu$. Consequently, the function defined by the integral

$$-(z/2\pi i)\int_0^{\infty(\gamma_\mu)} (\zeta^{k-1}/(\zeta-z))F_\mu(\zeta)G(\zeta)[W_\mu - I]G^{-1}(\zeta)d\zeta$$

can be analytically extended (by altering γ_μ) to the sector (with ν, $0 \leq \nu \leq m-1$, fixed)

$$\tau_\mu - \pi/r - 2\pi < \arg z < \tau_\mu \qquad (\text{if } \mu > \nu),$$

resp.

$$\tau_\mu - \pi/r < \arg z < \tau_\mu + 2\pi \qquad (\text{if } \mu \leq \nu),$$

and using Lemma 1 we see that for every $\delta > 0$ there exists a constant $K_\mu = K_\mu(\delta) > 0$ such that the function can be estimated by $K_\mu^k \Gamma(1+k/r)$ for every sufficiently large k and every z with

$$\tau_\mu - \pi/r - 2\pi + \delta \leq \arg z \leq \tau_\mu - \delta \quad (\text{if } \mu > \nu),$$

resp.

$$\tau_\mu - \pi/r + \delta \le \arg z \le \tau_\mu + 2\pi - \delta \quad (\text{if } \mu \le \nu).$$

Consequently, the sum of these functions (for $\mu = 1, \dots, m$) may be estimated by $K^k \Gamma(1+k/r)$ (with $K = \sum K_\mu$) for every sufficiently large k and z in the intersection of the regions described above, i.e. (cf. (2.14))

$$\tau_\nu - \pi/r + \delta \le \arg z \le \tau_{\nu+1} - \delta.$$

Since (1.10) holds for the analytic continuation of the integrals on the right as well, we therefore obtain

COROLLARY 2. *Given an equation* (1.1) *where* A_0 *has all distinct eigenvalues, then the normalized system of sectorial transformations corresponding to a formal fundamental solution* (2.6) *has the following property*: *Given any* $\delta > 0$, *there exists a positive constant* $K = K(\delta)$ *such that for every sufficiently large integer* k *and* $0 \le \nu \le m-1$

$$\|R_{k,\nu}(z)\| \le K^k \Gamma(1+k/r) \qquad \textit{for} \quad \tau_\nu - \pi/r + \delta \le \arg z \le \tau_{\nu+1} - \delta. \tag{2.16}$$

We see that for the normalized system of sectorial transformations the estimate (2.16) holds in sectors of opening more than π/r (take δ sufficiently small), hence F. Nevanlinna's result described in the introduction shows that each $F_\nu(z)$ may be represented by means of Laplace integrals. Since in the following Section 4 we will generally represent systems of first-level sectorial transformations as Laplace integrals, we need not go into details here.

REMARK 2.2. In this Section, we only gave different proofs for results that are essentially known. In the following Sections we will show that the same arguments used in the distinct eigenvalue case lead to analogous results in the general situation.

REMARK 2.3. It is easy to conclude from either (2.11) or (2.16) that

$$\|F_k\| \le K^k \Gamma(1+k/r) \qquad \text{for sufficiently large} \quad k. \tag{2.17}$$

Such an estimate can also be obtained from (1.14) by an estimate analogous to the one used in the proof of Lemma 1. If $r=1$, and A_0 has distinct eigenvalues, then (1.14) can be used to see that the elements of F_k are linear combinations of sequences having an explicit asymptotic behavior as $k \to \infty$. If one of the sequences grows more rapidly than the others, then the corresponding element of F_k has a clear asymptotic behavior that can be used to calculate one of the elements of the matrices $W_1, \dots, W_m$. This happens if, for some fixed k, there is precisely one j with $|\lambda_k - \lambda_j|$ minimal, and leads to a different proof of a formula obtained by R. Schäfke [19]. (In case $n=2$, $r=1$, one can, with very little effort, reprove the formulas obtained in [14], [15] to calculate the nontrivial elements in W_1, W_2 (note that $m=2$ in this case). We do not however want to go into details about this applications at the moment.) The situation in the case $r>1$ is similar, although technically more complicated; for example, instead of the Gammafunction one has to consider generalized Gammafunctions and their asymptotic behavior [8].

3. Formal solutions of first level

In [1], [2], formal solutions of first level have been introduced. In view of ([2], Proposition 2 and Remark 2.5) together with Remark 3.2 of this paper, these formal solutions may be characterized as follows:

Given a meromorphic differential equation (0.5), then an expression of the form

$$H_1(z) = F(z)G_1(z) \tag{3.1}$$

is called *a first level formal fundamental solution* of (0.5), iff the following two conditions hold:

(i) *The matrix $G_1(z)$ is analytic and invertible for $|z|$ sufficiently large (on the Riemann surface of the Logarithm), and its logarithmic derivative*

$$z^{-1}\tilde{A}(z) = G_1'(z)G_1^{-1}(z) \tag{3.2}$$

is meromorphic at $z=\infty$. Furthermore, there exists a matrix of the form

$$Q_1(z) = p(z)z^{[d]+1}I + d^{-1}z^d\Lambda^{1)} \tag{3.3}$$

where $p(z)$ is a polynomial in z, d a positive rational and

$$\Lambda = \operatorname{diag}[\lambda_1 I_{s_1},\dots, \lambda_l I_{s_l}]^{2)} \tag{3.4}$$

with $l\geq 2$, natural numbers $s_1,\dots, s_l$, $\sum s_k=n$, and distinct complex numbers $\lambda_1,\dots, \lambda_l$, such that for every sufficiently small $\varepsilon>0$

$$\|[G_1(z)\exp\{-Q_1(z)\}]^{\pm 1}\| = o(\exp\{|z|^{d-\varepsilon}\}) \quad in \quad S(-\infty, \infty). \tag{3.5}$$

(ii) *The matrix $F(z)$ is a formal meromorphic transformation from* (0.5) *to*

$$z\tilde{x}' = \tilde{A}(z)\tilde{x} \tag{3.6}$$

(with $\tilde{A}(z)$ as in (3.2)*), i.e. $F(z)$ formally satisfies* (1.3). *Furthermore, for a suitable choice of reals $\alpha_0,\dots, \alpha_{N-1}$ there exist sectorial transformations $F_0(z),\dots,$*

1) By $[d]$ we denote the largest integer not exceeding d.

2) By I_s, for natural s, we always denote the $s\times s$ unit matrix.

$F_{N-1}(z)$; $F_N(z)=F_0(ze^{-2\pi i})$ corresponding to $F(z)$ and $\alpha_0, \ldots, \alpha_{N-1}$[3] *such that for every μ, $1 \leq \mu \leq N$, the constant invertible matrices W_μ defined as in* (1.12) *(with $G_1(z)$ in place of $G(z)$) have I-matrices along the block-diagonal, when blocked according to the block structure of Λ.*

REMARK 3.1. To later avoid minor notational problems, we slightly modified the notation used in [2]: The matrix Λ used in [2] corresponds to $d^{-1}\Lambda$ here (see also Remark 3.2 below).

REMARK 3.2. We are going to show that the rational d in the definition above in fact coincides with the number d_1 defined in the introduction and that, furthermore, the matrix $Q_1(z)$ can (up to the ordering of its diagonal elements) be considered as consisting of those terms of $Q(z)$ (as in (0.7)) that have exponents $\geq d_1 = d$. To do so, let $H(z)$ be a formal fundamental solution of (0.5) (in the usual sense) of the form (0.6) (note, however, that $F(z)$ in (3.1) is to be distinguished from the formal meromorphic transformation in (0.6)). Then the matrix

$$\tilde{H}(z) = F^{-1}(z)H(z), \quad \text{with} \quad F(z) \quad \text{as in (3.1)}$$

is a formal fundamental solution of (3.6) of the form

$$\tilde{H}(z) = \tilde{F}(z)z^L \exp\{Q(z)\},$$

with a formal meromorphic transformation $\tilde{F}(z)$, i.e.

$$H(z) = F(z)\tilde{F}(z)z^L \exp\{Q(z)\}.$$

For every sufficiently small sector S there exists, according to the Asymptotic Existence Theorem, a constant invertible matrix $C=C(S)$ such that

$$G_1(z)C \exp\{-Q(z)\}z^{-L} \cong \tilde{F}(z) \qquad \text{in} \quad S.$$

Using (3.5), we find for sufficiently small $\varepsilon > 0$

$$\text{(3.7)} \qquad \exp\{Q_1(z)\}C \exp\{-Q(z)\} = o(\exp\{|z|^{d-\varepsilon}\}) \qquad \text{in} \quad S.$$

Let $q_j^{(1)}(z) = p(z)z^{[d]+1} + \lambda_j z^d/d$ ($1 \leq j \leq l$) and suppose that for some fixed k ($1 \leq k \leq n$) the degree of $q_j^{(1)}(z) - q_k(z)$, for at least one j, would be larger than d. Then obviously $\deg(q_j^{(1)}(z) - q_k(z)) > d$ for every j, $1 \leq j \leq l$, and there would exist a sector S (of sufficiently small opening) in which $q_j^{(1)}(z) - q_k(z)$ is of exponential order larger than d (for every j, $1 \leq j \leq l$). Consequently, (3.7) would hold only if the k^{th} column of C would be zero, which contradicts to the invertibility of C. Therefore we obtain

3) Note that the definition of sectorial transformations immediately is generalized to cases where $A(z)$, $\tilde{A}(z)$ are not polynomials!

$$Q(z) = p(z)z^{[d]+1} + d^{-1}z^d\tilde{\Lambda} + o(z^d) \tag{3.8}$$

with a diagonal matrix $\tilde{\Lambda}$ that may, at present, still have all equal diagonal entries, and we conclude (cf. Introduction for the definition of d_1)

$$d \geq d_1 .$$

From [13], p. 87, we recall that the Asymptotic Existence Theorem is true for every sector of opening π/d_1 (or smaller), hence as above we see that for every such sector there exists a constant, invertible C satisfying (3.7), which is in view of (3.3), (3.8) equivalent to

$$\exp\{d^{-1}\Lambda z^d\}C\exp\{-d^{-1}\tilde{\Lambda}z^d\} = O(1) \quad \text{in} \quad S. \tag{3.9}$$

To discuss (3.9), we may without loss in generality assume the diagonal elements of $\tilde{\Lambda}$ to be ordered in a way that equal ones come consecutively, i.e. we assume

$$\tilde{\Lambda} = \operatorname{diag}[\tilde{\lambda}_1 I_{\tilde{s}_1},\dots, \tilde{\lambda}_{\tilde{l}} I_{\tilde{s}\tilde{l}}]$$

with distinct $\tilde{\lambda}_1,\dots, \tilde{\lambda}_{\tilde{l}}$ and natural $\tilde{s}_k$, $\sum \tilde{s}_k = n$, and we block

$$C = [C_{jk}],\ C_{jk} \text{ of type } s_j \times \tilde{s}_k, \qquad 1 \leq j \leq l,\ 1 \leq k \leq \tilde{l}.$$

We now consider a sector S of opening $\pi/d \leq \pi/d_1$ such that for every pair (j, k), $1 \leq j \leq l$, $1 \leq k \leq \tilde{l}$, either

$$\lambda_j = \tilde{\lambda}_k ,$$

or there exists a ray $\arg z = \gamma$ in S along which

$$\operatorname{Re}(\lambda_j - \tilde{\lambda}_k)z^d > 0$$

(take any $S = S(\alpha, \alpha + \pi/d)$ such that for $\arg z = \alpha$ we have $\operatorname{Re}(\lambda_j - \tilde{\lambda}_k)z^d \neq 0$, for every pair (j, k) with $\lambda_j \neq \tilde{\lambda}_k$). For such a sector S, (3.9) holds iff

$$\text{either} \quad \lambda_j = \tilde{\lambda}_k \quad \text{or} \quad C_{jk} = 0$$

for every pair (j, k), $1 \leq j \leq l$, $1 \leq k \leq \tilde{l}$. For every j, we realize that, due to the invertibility of C, there exists a unique k with $\lambda_j = \tilde{\lambda}_k$ (hence $\tilde{l} = l$) and $\tilde{s}_k \leq s_j$ (otherwise, the columns of C would be linearly dependent). Since $\sum s_\nu = \sum \tilde{s}_\mu = n$, we obtain $\tilde{s}_k = s_j$. Therefore, $\tilde{\Lambda}$ and Λ coincide except for the ordering of their elements, and since Λ contains at least two distinct diagonal elements, we find by definition of d_1

$$d = d_1 .$$

REMARK 3.3. We have shown in [2] that every equation (0.5) having an essentially irregular singular point at ∞ admits a first level formal fundamental solution, and given some fixed formal fundamental solution of first level, we will refer to the block structure of Λ as *the block structure of first level.* It may be seen using Remark 3.2 that this terminology coincides (up to the ordering of the diagonal blocks) with the one used in [1], [4], [13], where the so-called *iterated block structure* has been investigated. If $C=[C_{jk}]$ $(1 \leq j,\ k \leq l)$ is a constant matrix, blocked according to the first level block structure, we define

$$\operatorname{diag}_1 C = \operatorname{diag}[C_{11}, \ldots, C_{ll}].$$

Using this terminology, we see that condition (ii) ensures the existence of sectorial transformations corresponding to $F(z)$ and (suitably chosen) $\alpha_0, \ldots, \alpha_{N-1}$ such that the matrices $W_\mu = G_1^{-1}(z) F_\mu^{-1}(z) F_{\mu-1}(z) G_1(z)$ satisfy

$$\operatorname{diag}_1 W_\mu = I \quad (1 \leq \mu \leq N). \tag{3.10}$$

Every such system of sectorial transformations will be called *of first level*, and the matrices W_μ are called *the corresponding connection matrices* (of first level). It may be recalled from [2] that, given a first level formal solution, there exists a system of first level sectorial transformations corresponding to $F(z)$ and *arbitrarily chosen* $\alpha_0, \ldots, \alpha_{N-1}$ provided that

$$\delta = \max\{\alpha_{\nu+1} - \alpha_{\nu-1};\quad 0 \leq \nu \leq N\}$$

is sufficiently small (with α_{-1}, α_N, α_{N+1} defined as in (1.6)).

REMARK 3.4. Obviously, the factorization of a first level formal fundamental solution $H_1(z)$ into its factors $F(z)$ and $G_1(z)$ is not unique: For any (convergent) meromorphic transformation $T(z)$, the matrices $F(z)T^{-1}(z)$ and $T(z)G_1(z)$ can be easily seen to satisfy (i), (ii) as well. Using this freedom, we may, according to the Birkhoff-Turrittin Reduction Theorem (see [13], p. 15), always assume that the differential equation (3.6) is a polynomial equation (1.2). If equation (0.5) is such that A_0 has n distinct eigenvalues, $\lambda_1, \ldots, \lambda_n$, then one can see that every formal fundamental solution (2.6) satisfies (i), (ii), with

$$F(z) = F_a(z),\ G_1(z) = z^{\Lambda'} \exp\{Q(z)\},\ d = r,\ l = n,$$
$$s_1 = \cdots = s_l = 1,\ p(z) \equiv 0,\ \Lambda = \operatorname{diag}[\lambda_1, \ldots, \lambda_n] = \Lambda_0,$$

(note that in this case (3.10) is trivially satisfied, since from (2.9) we conclude that the diagonal elements of W_μ are all one, $1 \leq \mu \leq N$). Therefore in the distinct eigenvalue case the class of formal solutions of first level contains the formal solutions (2.6), and similarly one can see that whenever the parameter t defined in the introduction (which is the number of different levels in the iterated block structure; see [1], [2], [13]) equals one, then first level formal solutions and formal solutions in the usual sense are almost synonymous.

Whenever $t>1$, the function $G_1(z)$ generally involves higher transcendental functions; however, the following lemma explains that they are always of a simpler kind than the solutions of (0.5):

LEMMA 2. *Let* (0.5) *be an equation having an essentially irregular singular point at* $z=\infty$, *and let*

$$H_1(z) = F(z)G_1(z)$$

be a first level formal fundamental solution of (0.5). *Then*

$$G_1(z) = T(z)\,\mathrm{diag}\,[\hat{G}_1(z),\dots, \hat{G}_l(z)], \tag{3.11}$$

where $T(z)$ *is a meromorphic transformation in the variable* $\tilde{z}=z^{1/p}$ *for a suitable, natural number p, and for* $j=1,\dots, l$, $\hat{G}_j(z)$ *is of size* $s_j\times s_j$ *and a fundamental solution of an equation*

$$z\hat{x}' = \hat{A}_j(z)\hat{x} \tag{3.12}$$

with $\hat{A}_j(z)$ *being meromorphic in* $\tilde{z}$. *Furthermore, equation* (3.12) *is simpler than* (0.5) *in the sense that, if we define* t_j *analogously to the definition of* t *in the introduction, then*

$$t_j \le t-1 \qquad (1\le j\le l). \tag{3.13}$$

PROOF. In [2] we have shown that every $H_1(z)$ can be factored (cf. also Remark 3.4) into a product of $F(z)$ and $G_1(z)$ such that

$$G_1(z) = z^{\tilde{L}}\,\mathrm{diag}\,[G_1^{(1)}(z),\dots, G_l^{(1)}(z)]$$

where $G_j^{(1)}(z)$ is of size $s_j\times s_j$ $(1\le j\le l)$, and $\tilde{L}$ is a constant matrix such that

$$e^{2\pi i\tilde{L}} = DR_1$$

with a constant, invertible, diagonally blocked D (in the first level block structure) and a block-permutation matrix R_1; i.e. R_1 is a permutation matrix, and if we block R_1 according to the first level block structure, then its non-zero blocks are all identity matrices. For every integer μ we find $e^{2\mu\pi i\tilde{L}}=D_\mu R_1^\mu$, where D_μ is again diagonally blocked, and for suitable p we have $R_1^\mu=I$ if p divides μ. Hence $e^{2p\pi i\tilde{L}}$ is diagonally blocked, and since every such invertible matrix has a diagonally blocked logarithm (cf. e.g., [22]), there exists a diagonally blocked $\hat{L}=\mathrm{diag}\,[\hat{L}_1,\dots, \hat{L}_l]$ such that

$$e^{2p\pi i\hat{L}} = e^{2p\pi i\tilde{L}}$$

Defining $T(z)=z^{L}z^{-\tilde{L}}$, we find

$$T(ze^{2p\pi i}) = T(z),$$

hence $T(z)$ is a meromorphic transformation in $\tilde{z}=z^{1/p}$. Obviously, $T^{-1}(z)G_1(z)$ is diagonally blocked, hence (3.11) holds. Moreover,

$$\hat{A}(z) = T^{-1}(z)\tilde{A}(z)T(z) - zT^{-1}(z)T'(z)$$

is also diagonally blocked and $\hat{G}_j(z)$ is a fundamental solution of (3.12), if $\hat{A}_j(z)$ denotes the j^{th} diagonal block of $\hat{A}(z)$. Since $T(z)$ is meromorphic in $\tilde{z}$, we conclude that so is $\hat{A}(z)$. Finally, to obtain (3.13), let (for $j=1,\dots, l$)

$$\hat{H}_j(z) = \Psi_j(z)\exp\{Q_j(z)\}$$

be a formal fundamental solution of (3.12) with a formal logarithmic, invertible matrix $\Psi_j(z)$ (cf. [9]) and a diagonal matrix $Q_j(z)$ of polynomials in a root of z with zero constant terms. Then

$$\tilde{H}(z) = T(z)\operatorname{diag}[\hat{H}_1(z),\dots, \hat{H}_l(z)]$$

is a formal fundamental solution of (3.6) and can be written as (cf. [2])

$$\tilde{H}(z) = \tilde{F}(z)z^{L}\exp\{Q(z)\}$$

with a formal meromorphic transformation $\tilde{F}(z)$, a constant matrix L and $Q(z)=\operatorname{diag}[Q_1(z),\dots, Q_l(z)]$. According to the Main Asymptotic Existence Theorem, for every sufficiently small sector S there exist constant, invertible matrices C_j such that

$$\hat{G}_jC_j \cong \hat{H}_j(z) \quad \text{in} \quad S, \qquad 1 \le j \le l,$$

hence with $C=\operatorname{diag}[C_1,\dots, C_l]$

$$G_1(z)C \cong \tilde{H}(z) \quad \text{in} \quad S.$$

Using (3.5), one can now show in the same manner as in Remark 3.2 that (note $d=d_1$)

$$Q(z) = Q_1(z) + o(z^{d_1}), \text{ with } Q_1(z) \text{ as in (3.3)},$$

i.e.

$$Q_j(z) = p(z)z^{[d_1]+1} + \lambda_j z^{d_1}/d_1 + o(z^{d_1}), \qquad 1 \le j \le l.$$

Since $Q(z)$ is formally meromorphically invariant, we find that the parameter t for the equations (0.5) and (3.6) is the same. Hence, using the above formulas, it follows right from the definition of t and t_j ($1 \le j \le l$) that (3.13) is true.

4. Representation of solutions as Laplace integrals

a) Normalized first level sectorial transformations:

Consider an arbitrarily fixed meromorphic differential equation (0.5) having an essentially irregular singularity at $z=\infty$, then (0.5) possesses a first level formal fundamental solution (3.1) (cf. [2]).

Quite in the same fashion as in the distinct eigenvalue case (Section 2, in particular (2.12) with d in place of r) we define Stokes' directions corresponding to the numbers $\lambda_1,\dots,\lambda_l$ (occurring in (3.4)), and we call these directions *Stokes' directions of first level* to distinguish from the notation used in the literature so far ([4], [13]). Again, the set of Stokes' directions of first level is discrete and we index them as τ_ν (with integer ν) such that (2.13) holds. Note that in case A_0 has n distinct eigenvalues, then $l=n$ and $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A_0, hence the Stokes' directions of first level coincide with the Stokes' directions introduced in Section 2. More generally, one may see that Stokes' directions of first level are a subset of the set of all Stokes' directions considered in [4], [13], and the inclusion is strict only if t (the number of different levels in the iterated block structure) is larger than one.

Although in [2] we used a slightly different definition of the numbers $\lambda_1,\dots,\lambda_l$ (see Remark 3.1 of this paper), we wish to emphasize that this difference in the notation does not influence the first level Stokes' directions. In particular, we recall from [2] that for suitable $m\geq 1$

$$\tau_{\nu+m} = \tau_\nu + 2\pi \qquad \text{for every} \quad \nu \tag{4.1}$$

(this is so, since one can show that Λ and $\Lambda e^{2\pi i d}$ coincide but for the ordering of the diagonal elements).

In [2], Section 3 we proved a Theorem which we now state, slightly differently formulated:

THEOREM 1. *Let an equation* (0.5) *having an essentially irregular singularity at $z=\infty$ be given, and consider a fixed but arbitrary first level formal fundamental solution* (3.1). *Furthermore, let $e^{2\pi i L_1}$ denote the unique constant, invertible matrix satisfying*

$$G_1(ze^{2\pi i}) = G_1(z)e^{2\pi i L_1}. \tag{4.2}$$

Then there exists a unique system

$$X_0(z),\dots, X_{m-1}(z);\ X_m(z) = X_0(ze^{-2\pi i})e^{2\pi i L_1}$$

of solutions of (0.5) *satisfying*

$$(4.3) \qquad X_\nu(z)G_1^{-1}(z) \cong F(z) \quad \textit{in} \quad S(\tau_\nu - \pi/d, \tau_{\nu+1}), \quad 0 \le \nu \le m$$

and

$$(4.4) \qquad \mathrm{diag}_1\, W_\nu = I, \quad \textit{where} \quad W_\nu = X_\nu^{-1}(z)X_{\nu-1}(z), \quad 1 \le \nu \le m.$$

Unlike in the different eigenvalue case, the condition (4.3) alone is not sufficient to characterize the solutions $X_0(z),\dots, X_{m-1}(z)$ uniquely (see the examples in [2], Section 4). We will however see that the asymptotic behavior of a single $X_\nu(z)$ is sufficient to characterize it uniquely, if, in addition to (4.3), we restrict to solutions for which the error term in the asymptotic does not grow too fast:

Defining

$$(4.5) \qquad F_\nu(z) = X_\nu(z)G_1^{-1}(z), \quad 0 \le \nu \le m,$$

we obtain a system of functions

$$F_0(z),\dots, F_{m-1}(z);\ F_m(z) = F_0(ze^{-2\pi i}),$$

the system of normalized first level sectorial transformations corresponding to $H_1(z)=F(z)G_1(z)$, which is (in view of Theorem 1) uniquely characterized by the conditions

$$(4.6) \qquad F_\nu(z) \cong F(z) \quad \text{in} \quad S(\tau_\nu - \pi/d, \tau_{\nu+1}), \quad 0 \le \nu \le m,$$

$$(4.7) \qquad \mathrm{diag}_1\, W_\nu = I, \quad W_\nu = G_1^{-1}(z)F_\nu^{-1}(z)F_{\nu-1}(z)G_1(z), \quad 1 \le \nu \le m$$

(it follows from (4.6), (4.7) that the matrix

$$A_\nu(z) = zF_\nu'(z)F_\nu^{-1}(z) + F_\nu(z)\tilde{A}(z)F_\nu^{-1}(z)$$

is independent of ν and meromorphic at ∞ with an expansion to be found by replacing $F_\nu(z)$ by the formal transformation $F(z)$, hence $A_\nu(z)=A(z)$ and $F_\nu(z)$ satisfies (1.3) for $\nu=0,\dots, m$).

Using Lemma 1, we next establish an estimate of the error term of the asymptotic expansion for the system of normalized first level sectorial transformations:

PROPOSITION 3. *Let an equation* (0.5) *having an essentially irregular singularity at $z=\infty$ be given, and consider a fixed, but arbitrary first level formal fundamental solution* (3.1) *and the corresponding system of normalized first level sectorial transformations. Then to every $\delta>0$ there exists a positive constant $K=K(\delta)$ such that for every sufficiently large integer k and $0\le\nu\le m-1$ (with $d=d_1$)*

$$(4.8) \qquad \|R_{k,\nu}(z)\| \le K^k\Gamma(1+k/d)$$

for $|z|\geq c$, $\tau_\nu-\pi/d+\delta\leq\arg z\leq\tau_{\nu+1}-\delta$, *with fixed* $c\geq 0$ *such that both* $A(z)$ *and* $\tilde{A}(z)$ *(as in* (3.6)*) are analytic for* $|z|\geq c$.

REMARK 4.1. We have pointed out in Remark 3.4 that a given $H_1(z)$ can always be factored in a way that $\tilde{A}(z)$ is a polynomial equation, and in this case, the constant c in Proposition 3 can be any number larger than a, the radius of convergence of the expansion for $A(z)$. Generally, if $\tilde{A}(z)$ is an arbitrary equation, then $c>\max\{a, \tilde{a}\}$ (with $\tilde{a}$ being the radius of convergence for $\tilde{A}(z)$).

PROOF OF PROPOSITION 3. We first restrict to cases where (0.5) and (3.6) are both polynomial equations, i.e. of the form (1.1) and (1.2). Then, with $F(z)$ as in (3.1), we are in what we called our basic situation, and for fixed ν, $0\leq\nu\leq m-1$, the error term $R_{k,\nu}(z)$ is, according to Proposition 1, the sum (for $\mu=1,\dots, m$) of the integrals

$$-(z/2\pi i)\int_0^{\infty(\gamma_\mu)}(\zeta^{k-1}/(\zeta-z))F_\mu(\zeta)G_1(\zeta)[W_\mu-I]G_1^{-1}(\zeta)d\zeta, \tag{4.9}$$

for $\gamma_\nu<\arg z<\gamma_{\nu+1}$ and every sufficiently large k. Consider some fixed μ, $1\leq\mu\leq m$, and use (1.13) with $G_1(z)$ in place of $G(z)$ together with (4.6) for $\nu=\mu$ and $\nu=\mu-1$, then

$$G_1(z)[W_\mu-I]G_1^{-1}(z)\cong 0 \quad\text{in}\quad S(\tau_\mu-\pi/d,\ \tau_\mu),$$

and according to (3.5) this implies

$$\exp\{d^{-1}\Lambda z^d\}[W_\mu-I]\exp\{-d^{-1}\Lambda z^d\} = o(\exp\{|z|^{d-\varepsilon}\})$$

in $S(\tau_\mu-\pi/d, \tau_\mu)$, for every sufficiently small $\varepsilon>0$. Since $\operatorname{diag}_1 W_\mu=I$, this holds iff for every (j, k), $1\leq j, k\leq l$, the (j, k) block of $W_\mu-I$ is zero, or

$$\operatorname{Re}(\lambda_j-\lambda_k)z^d < 0 \quad\text{in}\quad S(\tau_\mu-\pi/d,\ \tau_\mu).$$

Therefore we see that each block of $F_\mu(\zeta)G_1(\zeta)[W_\mu-I]G_1^{-1}(\zeta)$ is either identically zero or a finite sum of expressions

$$E(\zeta)\exp(\lambda\zeta^d),$$

where $\operatorname{Re}(\lambda\zeta^d)<0$ in $S(\tau_\mu-\pi/d, \tau_\mu)$, and $E(\zeta)$ may be estimated in the form (2.2) (with $\alpha=\tau_\mu-\pi/d$, $\beta=\tau_\mu$). Therefore, if $\mu\geq\nu+1$, Lemma 1 can be applied to show that the function defined by (4.9) can be analytically continued (by altering γ_μ) into the sector

$$\tau_\mu-\pi/d-2\pi < \arg z < \tau_\mu,$$

and for every $\delta>0$ there exists a constant $K=K(\delta, \mu)$ such that for every sufficiently large k the function is bounded by $K^k\Gamma(1+k/d)$ for $\tau_\mu-\pi/d-2\pi+\delta\leq\arg z\leq \tau_\mu-\delta$. If $\mu\leq\nu$, according to Remark 2.1 the same statements are seen to hold for

$$\tau_\mu-\pi/d < \arg z < \tau_\mu+2\pi,$$

resp.

$$\tau_\mu-\pi/d+\delta \leq \arg z \leq \tau_\mu+2\pi-\delta.$$

The sum of these functions (for $\mu=1,\dots, m$) is therefore analytic in the intersection of the open sectors, i.e. for (cf. (4.1))

$$\tau_\nu-\pi/d < \arg z < \tau_{\nu+1},$$

and an estimate of the same kind holds for the sum in the sector

$$\tau_\nu-\pi/d+\delta \leq \arg z \leq \tau_{\nu+1}-\delta.$$

Since for $\arg z\in(\gamma_\nu, \gamma_{\nu+1})\subseteq(\tau_\nu-\pi/d, \tau_{\nu+1})$[1], the sum of the functions (4.9) equals $R_{k,\nu}(z)$, we obtain by means of the uniqueness of analytic functions that $R_{k,\nu}(z)$ is the sum of the analytically extended functions as well. This proves (4.8) in the special case of polynomial equations (where we may take $c=0$).

In order to complete the proof, we recall that, according to the Birkhoff-Turrittin Reduction Theorem, for an arbitrary equation (0.5) there exists a (convergent) meromorphic transformation $T(z)$ such that the equation

$$z\hat{x}' = \hat{A}(z)\hat{x},\ \hat{A}(z) = T^{-1}(z)A(z)T(z) - zT^{-1}(z)T'(z) \tag{4.10}$$

is a polynomial equation. If $H_1(z)=F(z)G_1(z)$ is an arbitrary first level formal fundamental solution of (0.5), then $\hat{H}_1(z)=T^{-1}(z)H_1(z)$ is a first level formal fundamental solution of (4.10). Furthermore, according to Remark 3.4 there exists another meromorphic transformation $\tilde{T}(z)$ such that $\tilde{G}_1(z)=\tilde{T}(z)G_1(z)$ is a fundamental solution of a polynomial equation. Hence factorizing $\hat{H}_1(z)=\hat{F}(z)\tilde{G}_1(z)$, Proposition 3 is proved for the system $\hat{F}_0(z),\dots, \hat{F}_{m-1}(z)$; $\hat{F}_m(z)=\hat{F}_0(ze^{-2\pi i})$ of normalized first level sectorial transformations corresponding to $\hat{H}_1(z)=\hat{F}(z)\tilde{G}_1(z)$. Checking (4.6), (4.7) for

$$F_\nu(z) = T(z)\hat{F}_\nu(z)\tilde{T}(z) \qquad (0\leq\nu\leq m),$$

we find that $F_0(z),\dots, F_{m-1}(z)$; $F_m(z)=F_0(ze^{-2\pi i})$ is the unique system of normalized first level sectorial transformations corresponding to $H_1(z)=F(z)G_1(z)$. Hence a two-fold application of Lemma 3 (below) completes the proof.

LEMMA 3. *Let $F(z; S)$ be an $n\times n$ matrix of functions analytic in*

$$S = \{z;\ |z|\geq a,\ \alpha<\arg z<\beta\}$$

1) Recall from the definition that $\tau_\nu-\pi/d$ is also a Stokes' direction, i.e. $\tau_\nu-\pi/d\leq\tau_{\nu-1}$.

(with reals $\alpha<\beta$ and $a\geq 0$), and let $\sum F_j z^{-j}$ be a formal meromorphic series such that to every $\delta>0$ there exists a positive constant $K=K(\delta)$ with

$$\|z^k(F(z;\,S) - \textstyle\sum^{k-1} F_j z^{-j})\| \leq K^k \Gamma(1+k/d)$$

for every sufficiently large integer k and

$$|z| \geq a,\ \alpha+\delta \leq \arg z \leq \beta-\delta$$

(with some fixed real $d>0$, independent of k and δ). Furthermore, let $T(z)=\sum T_j z^{-j}$ be a meromorphic transformation, converging for $|z|\geq\rho$ ($\rho>0$), and define

$$\tilde{F}(z;\,S) = T(z)F(z;\,S),\quad \tilde{F}_j = \textstyle\sum_\nu T_\nu F_{j-\nu}$$

(for every integer j). Then to every $\delta>0$ there exists a constant $\tilde{K}=\tilde{K}(\delta)$ such that

$$\|z^k(\tilde{F}(z;\,S) - \textstyle\sum^{k-1} \tilde{F}_j z^{-j})\| \leq \tilde{K}(\delta)\Gamma(1+k/d)$$

for every sufficiently large k and

$$|z| \geq \max(a,\,\rho),\ \alpha+\delta \leq \arg z \leq \beta-\delta.$$

Proof. For technical convenience, first note that if the lemma is proved for some $F(z;\,S)$, then it holds for $F(z;\,S)+P(z)$ as well, where $P(z)$ may be any matrix polynomial (if we replace $\sum F_j z^{-j}$ by $\sum F_j z^{-j}+P(z)$). This allows us to assume without loss in generality

$$F_j = 0 \qquad \text{if}\ \ j<0.$$

Furthermore, if $T(z)=z^g$ with some integer g, then

$$z^k(\tilde{F}(z;\,S) - \textstyle\sum^{k-1} \tilde{F}_j z^{-j}) = z^{k+g}(F(z;\,S) - \textstyle\sum^{k+g-1} F_j z^{-j}),$$

hence the lemma is correct in this case (note that $\Gamma(1+(k+g)/d)/\Gamma(1+k/d)$ is not growing faster than a suitable power of k). Hence we also may assume

$$T_j = 0 \qquad \text{for}\ \ j<0.$$

In this case,

$$z^k(\tilde{F}(z;\,S) - \textstyle\sum_{j=0}^{k-1} \tilde{F}_j z^{-j}) = z^k T(z)(F(z;\,S) - \textstyle\sum_{j=0}^{k-1} F_j z^{-j}) + R_k(z),$$
$$R_k(z) = z^k \textstyle\sum_{j=k}^{\infty} z^{-j} \sum_{\nu=0}^{k-1} T_{j-\nu} F_\nu.$$

Since $T(z)$ stays analytic at $z=\infty$, it is sufficient to estimate $R_k(z)$. It follows from our assumptions on $F(z;\,S)$ that

$$\|F_k\| \leq K^k\Gamma(1+k/d) \quad \text{for sufficiently large} \quad k,$$

and (by possibly enlarging K) we may assume that (for every large k)

$$\|F_j\| \leq K^k \Gamma(1+k/d) \qquad \text{for} \quad j \leq k-1.$$

For sufficiently small $\varepsilon > 0$

$$\|T_j\| \leq c\,(\rho-\varepsilon)^j \qquad \text{for every} \quad j.$$

Hence

$$\|R_k(z)\| \leq K^k \Gamma(1+k/d) \sum_{j=k}^{\infty} |z|^{k-j} c \sum_{\nu=0}^{k-1} (\rho-\varepsilon)^{j-\nu} \leq \hat{K}^k \Gamma(1+k/d)$$

for $|z| \geq \rho$ with suitable $\hat{K} = \hat{K}(\delta, \varepsilon) > 0$.

b) Normal solutions of first level:

While generally there are infinitely many functions having a given formal power series as their asymptotic expansion in a given sector S, F. Nevanlinna [17] proved that there is only one satisfying (0.1), if the sector S has opening at least π/d. In view of this uniqueness result and Proposition 3, we may reformulate Theorem 1 as follows:

THEOREM 1′. *Under the same assumptions as in Theorem* 1, *for every fixed* ν, $0 \leq \nu \leq m-1$, *there exists a unique fundamental solution* $X_\nu(z) = F_\nu(z) G_1(z)$, *such that for every* $\delta > 0$ *there exists a constant* $K = K(\delta) > 0$ *with*

$$\|z^k(F_\nu(z) - \textstyle\sum^{k-1} F_j z^{-j})\| \leq K^k \Gamma(1+k/d) \tag{4.11}$$

for every sufficiently large integer k *and*

$$|z| \geq c, \;\; \tau_\nu - \pi/d + \delta \leq \arg z \leq \tau_{\nu+1} - \delta$$

(*with* c *as in Proposition* 3). *The system*

$$X_0(z), \ldots, X_{m-1}(z);\; X_m(z) = X_0(ze^{-2\pi i}) e^{2\pi i L_1}$$

then satisfies (4.3), (4.4).

REMARK 4.2. The system of solutions

$$X_0(z), \ldots, X_{m-1}(z);\; X_m(z) = X_0(ze^{-2\pi i}) e^{2\pi i L_1}$$

will be called *the system of first level normal solutions* of (0.5) corresponding to $H_1(z)$. We wish to emphasize that, according to Theorem 1′, a single $X_\nu(z)$ is uniquely characterized by (4.11), while (4.3), (4.4) cannot directly be used to identify a single fundamental solution as one of the normal solutions of first level.

For every integer $\mu \neq 0$, define

$$F_{\nu+\mu m}(z) = F_\nu(ze^{-2\mu\pi i}) \qquad (0 \le \nu \le m-1),$$

then, since $\tau_{\nu+\mu m} - \pi/d + \delta \le \arg z \le \tau_{\nu+\mu m+1} - \delta$ iff $\tau_\nu - \pi/d + \delta \le \arg z - 2\mu\pi \le \tau_{\nu+1} - \delta$ (cf. (4.1)), the functions $F_\nu(z)$ satisfy (4.11) *for every integer* ν. The functions

$$X_\nu(z) = F_\nu(z) G_1(z) \qquad \text{for every integer} \quad \nu$$

then satisfy (cf. (4.2))

$$X_{\nu+m}(z) = X_\nu(ze^{-2\pi i}) e^{2\pi i L_1} \qquad \text{for every} \quad \nu$$

and we call $X_\nu(z)$ *the* ν^{th} *first level normal solution of* (0.5) *corresponding to* $H_1(z)$. Note that, unlike the matrix $F_\nu(z)$, the function $X_\nu(z)$ does not depend upon a particular factorization of $H_1(z)$ into $F(z)$ and $G_1(z)$.

Using F. Nevanlinna's result described in the introduction, we will now represent the first level normal solutions in terms of Laplace integrals:

THEOREM 2. *For an arbitrarily given equation* (0.5) *having an essentially irregular singularity at* ∞, *consider a first level formal fundamental solution* $H_1(z) = F(z) G_1(z)$. *Then the function* $\Psi(u)$, *locally given by*

$$\Psi(u) = \textstyle\sum_1^\infty F_j u^j / \Gamma(1 + j/d), \tag{4.12}$$

may be analytically continued (for every fixed integer ν*) into the sector*

$$\pi/2d - \tau_{\nu+1} < \arg u < \pi/2d - \tau_\nu,$$

and for every $\gamma \in (\pi/2 - d\tau_{\nu+1},\ \pi/2 - d\tau_\nu)$ *and every* $\varepsilon > 0$ *there exists a constant* $c_{\varepsilon,\gamma}$ *such that*

$$\|\Psi(u)\| \le c_{\varepsilon,\gamma} \exp\{|u|^d (\rho + \varepsilon)^d\} \qquad \text{for} \quad \arg u = d^{-1}\gamma \tag{4.13}$$

(with $\rho = \max(a, \tilde{a})$ *where* a, *resp.* $\tilde{a}$, *is the radius of convergence of* $A(z)$, *resp.* $\tilde{A}(z)$*). Furthermore, for the* ν^{th} *first level normal solution* $X_\nu(z) = F_\nu(z) G_1(z)$ *corresponding to* $H_1(z)$ *we have (with* γ *as above)*

$$F_\nu(z) = \textstyle\sum_{j \le 0} F_j z^{-j} + z^d \displaystyle\int_0^{\infty(\gamma)} \Psi(u^{1/d}) \exp\{-uz^d\} du \tag{4.14}$$

for every $z \in S(\tau_\nu - \pi/d,\ \tau_{\nu+1})$ *with*

$$\operatorname{Re} z^d e^{i\gamma} > \rho^d \tag{4.15}$$

(and the integral converges absolutely).

PROOF. Consider some fixed, but arbitrary integer ν, and let $\gamma \in (\pi/2 - d\tau_{\nu+1},\ \pi/2 - d\tau_\nu)$ be arbitrarily fixed. Define

$$F_\nu^{(\gamma)}(z) = F_\nu(ze^{-i\gamma/d}), \quad F_j^{(\gamma)} = (e^{i\gamma/d})^j F_j$$

(for every integer j), then for every $\delta > 0$ there exists a $K(\delta) = K > 0$ such that (cf. (4.11))

$$\|z^k(F_\nu^{(\gamma)}(z) - \sum^{k-1} F_j^{(\gamma)} z^{-j})\| \leq K^k \Gamma(1+k/d) \tag{4.16}$$

for every sufficiently large k and

$$|z| \geq \rho + \varepsilon/2, \; \tau_\nu - \pi/d + \gamma/d + \delta \leq \arg z \leq \tau_{\nu+1} + \gamma/d - \delta,$$

with arbitrarily fixed $\varepsilon > 0$ (independent of δ).

If we take δ so small that

$$\gamma/d + \delta \leq \pi/2d - \tau_\nu, \quad \gamma/d - \delta \geq \pi/2d - \tau_{\nu+1},$$

then the region where (4.16) is valid contains the sector

$$|z| \geq \rho + \varepsilon/2, \quad -\pi/2d \leq \arg z \leq \pi/2d.$$

Therefore we conclude from F. Nevanlinna [17] (compare the introduction) that

$$\Psi^{(\gamma)}(u) = \sum_1^\infty F_k^{(\gamma)} u^k / \Gamma(1+k/d)$$

converges for $|u| \leq K^{-1}$ and may be analytically continued along the positive real axis up to ∞, such that there exists a constant $c_{\varepsilon,\gamma} > 0$ with

$$\|\Psi^{(\gamma)}(u)\| \leq c_{\varepsilon,\gamma} \exp\{(\rho+\varepsilon)^d u^d\} \quad \text{for positive real} \quad u,$$

and

$$F_\nu^{(\gamma)}(z) = \sum_{j \leq 0} F_j^{(\gamma)} z^{-j} + z^d \int_0^\infty \Psi^{(\gamma)}(u^{1/d}) \exp\{-uz^d\} du$$

for every z with $\operatorname{Re} z^d > \rho + \varepsilon/2$ and $-\pi/2d \leq \arg z \leq \pi/2d$. Defining $\Psi(u) = \Psi^{(\gamma)}(ue^{-i\gamma/d})$, we see that (4.12) is satisfied, due to the definition of $F_j^{(\gamma)}$, and $\Psi(u)$ can be analytically continued along the ray $\arg u = \gamma/d$. Since γ has been taken arbitrarily from the interval $(\pi/2 - d\tau_{\nu+1}, \pi/2 - d\tau_\nu)$, this proves the analyticity of $\Psi(u)$ in

$$\pi/2d - \tau_{\nu+1} < \arg u < \pi/2d - \tau_\nu,$$

and (4.13) follows from the estimate for $\Psi^{(\gamma)}(u)$. Finally, (4.14) may be proved using the corresponding formula for $F_\nu^{(\gamma)}(z)$ and making a change of variable.

5. Representation of solutions as factorial series

A large number of papers (e.g. [6], [7], [10], [11], [12], [17], [18], [19],

[20], [22]) are concerned with the "summation" of a formal solution in form of a convergent (generalized) factorial series. All the papers listed above succeed in summing the whole formal fundamental solution only in the different eigenvalue case or (which is almost the same) in cases where the number of different levels in the iterated block structure is one. Using Theorem 2, we are now able to treat the general case, considering first level formal fundamental solutions.

PROPOSITION 4. *Under the assumptions made in Theorem* 2, *let* p, q *be relatively prime naturals such that*

$$d = d_1 = p/q,$$

and define

$$\Phi_j(u) = \textstyle\sum_{k=1}^{\infty} F_{kp-j} u^{qk-1}/\Gamma(qk), \quad 0 \leq j \leq p-1. \tag{5.1}$$

Then for every fixed ν, *the functions* $\Phi_j(u)$ *are analytic for*

$$\pi/2 - d\tau_{\nu+1} < \arg u < \pi/2 - d\tau_\nu, \tag{5.2}$$

and for every $\varepsilon > 0$ *and every* $\gamma \in (\pi/2 - d\tau_{\nu+1}, \pi/2 - d\tau_\nu)$ *there exists a constant* $c_{\varepsilon,\gamma}$ *such that for* $j = 0, \dots, p-1$ *and* ρ *as in Theorem* 2

$$\|\Phi_j(u)\| \leq c_{\varepsilon,\gamma} \exp\{(\rho+\varepsilon)^d |u|\} \qquad \textit{for} \quad \arg u = \gamma. \tag{5.3}$$

Furthermore, for every $z \in S(\tau_\nu - \pi/d, \tau_{\nu+1})$ *with* (4.15)

$$F_\nu(z) = \textstyle\sum_{j \leq 0} F_j z^{-j} + \sum_{j=0}^{p-1} z^j \int_0^{\infty(\gamma)} \Phi_j(u) \exp\{-uz^d\} du. \tag{5.4}$$

PROOF. If $H_1(z) = F(z)G_1(z)$ is a first level formal fundamental solution of (0.5), then $\hat{H}_1(z) = \hat{F}(z)G_1(z)$, $\hat{F}(z) = z^p F(z)$ is a first level formal fundamental solution of $z\hat{x}' = (A(z) - pI)\hat{x}$, and if we define (analogously to (4.12))

$$\hat{\Psi}(u) = \textstyle\sum_1^{\infty} \hat{F}_j u^j / \Gamma(1 + j/d),$$

then (note that $\hat{F}_j = F_{j+p}$ for every j)

$$\hat{\Psi}(u^{1/d}) = \textstyle\sum_{p+1}^{\infty} F_j u^{j/d-q} / \Gamma(1 + j/d - q).$$

Since for $\alpha, \beta > 0$

$$u^{\alpha+\beta}/\Gamma(1+\alpha+\beta) = (1/\Gamma(\alpha)) \int_0^u (u-t)^{\alpha-1} t^\beta / \Gamma(1+\beta) dt$$

if we integrate along a straight line and define the powers according to any fixed selection of $\arg u (= \arg t = \arg(u-t))$, one proves by termwise integration of the power series (for sufficiently small $|u|$)

(5.5) $\Phi_j(u) - F_{p-j}u^{q-1}/\Gamma(q)$

$$= (1/p) \sum_{k=0}^{p-1} e^{2kj\pi i/d}(1/\Gamma(j/d+q-1)) \int_0^u (u-t)^{j/d+q-2} \hat{\Psi}((te^{2k\pi i})^{1/d})dt$$

$j=0,\ldots, p-1$. (If $q=1$, then (5.5) should, for $j=0$, be read $\Phi_0(u)-F_1=(1/p) \sum_{k=0}^{p-1} \hat{\Psi}((te^{2k\pi i})^{1/d})$. It follows from the definition of first level Stokes' directions that, if μ denotes the number of Stokes' directions in $[0, \pi/d)$, we have (cf. [2])

$$\tau_{\nu+\mu} = \tau_\nu + \pi/d \qquad \text{(for every integer } \nu).$$

Hence for every fixed ν and $\gamma \in (\pi/2 - d\tau_{\nu+1}, \pi/2 - d\tau_\nu)$ we have $\arg u = \gamma$ iff $\arg(ue^{2k\pi i}) = \gamma + 2k\pi \in (\pi/2 - d\tau_{\nu-2k\mu+1}, \pi/2 - d\tau_{\nu-2k\mu})$ (for $k=0,\ldots, p-1$). Therefore we conclude from Theorem 2 that to every $\varepsilon > 0$ (and γ and ν as above) there exists a constant $\hat{c}_{\varepsilon,\gamma}$ such that

$$\|\hat{\Psi}((ue^{2k\pi i}))^{1/d}\| \le \hat{c}_{\varepsilon,\gamma} \exp\{|u|(\rho+\varepsilon)^d\} \qquad \text{for} \quad \arg u = \gamma.$$

Using this estimate and (5.5), one can now easily obtain the analyticity of $\Phi_j(u)$ for u as in (5.2) plus the estimate (5.3) (by directly estimating (5.5)).

To see (5.4), we prove (by termwise integration)

$$(5.6) \qquad (d/du)^{1-q} \Psi(u^{1/d}) = \sum_{j=0}^{p-1} (1/\Gamma((p-j)/d)) \int_0^u (u-t)^{(p-j)/d-1} \Phi_j(t)dt$$

(if the left hand side is interpreted as the $(q-1)$-fold iterated integral from 0 to u). By means of $(q-1)$-fold partial integration we then obtain

$$z^d \int_0^{\infty(\gamma)} \Psi(u^{1/d}) \exp\{-uz^d\}du$$
$$= z^p \int_0^{\infty(\gamma)} \{\sum_{j=0}^{p-1} (1/\Gamma((p-j)/d)) \int_0^u (u-t)^{(p-j)/d-1} \Phi_j(t)dt\} \exp\{-uz^d\}du,$$

and interchanging the order of integration we obtain (5.4), using

$$\int_0^{\infty(\gamma)} u^{(p-j)/d-1} \exp\{-uz^d\}du = z^{j-p}\Gamma((p-j)/d), \quad \arg z^d \in (\gamma - \pi/2, \gamma + \pi/2).$$

Since $\Phi_j(u)$ satisfies (5.3) and is regular in a strip containing the ray $\arg u = \gamma$ (with γ as in Proposition 4), one can now show in a standard manner (cf. [22]) that the integrals in (5.4) can be expressed as convergent generalized factorial series, completely analogous to the distinct eigenvalue case. This then shows:

THEOREM 3. *Under the assumptions made in Theorem* 2, *let* p, q *be relatively prime naturals such that*

$$d = d_1 = p/q.$$

Then for every fixed integer ν, the ν^{th} first level normal solution $X_\nu(z)$ corresponding to $H_1(z)$ can be represented as

$$(5.7)\qquad X_\nu(z)=\left\{\sum_{j\le 0} F_j z^{-j}+\sum_{k=0}^{p-1} z^k \sum_{j=0}^{\infty} \frac{B_j^{(k)}(\omega)j!}{z^d(z^d+\omega)\cdots(z^d+j\omega)}\right\} G_1(z)$$

for every $z\in S(\tau_\nu-\pi/d,\ \tau_{\nu+1})$ with $\operatorname{Re} z^d e^{i\gamma}>\rho^d$ (where ρ,γ are as in Theorem 2) and $\omega=|\omega|e^{(-\gamma-\pi)i}$ with sufficiently large $|\omega|$; the coefficients $B_j^{(k)}(\omega)$ are polynomials in ω of degree at most j and can be found explicitly in terms of the coefficients of $F(z)$.

Since a proof of Theorem 3 can be given in just the same way as in [22], pp. 323–332, we omit it here. We remark that an explicit bound upon the possible values of $|\omega|$ can be given in terms of the values $\lambda_1,\dots,\lambda_l$, but we do not want to consider this question now.

As a final remark, we recall that first level formal solutions have been proved to be the objects that naturally occur in connection with representations of solutions by either Laplace integrals or factorial series. So far, the definition of first level formal solutions is such that one hardly may hope to calculate them explicitly (via recursion formulas for the coefficients of $F(z)$), since one does not know how to find the matrix $G_1(z)$. However, the author hopes to prove in a separate article that formal solutions of first level can also be characterized by conditions (i) plus a restriction upon the growth of the coefficients of $F(z)$. Assuming this to be done, it might still be difficult *but would at least theoretically be possible to calculate first level formal solutions.* For example, when $d=d_1$ is an integer, one would have to find a formal meromorphic transformation $F(z)=\sum F_j z^{-j}$ with $\|F_j\|\le K^j\Gamma(1+j/d)$ for sufficiently large j such that $\tilde{A}(z)=F^{-1}(z)A(z)F(z)-zF^{-1}(z)F'(z)$ would be a diagonally blocked meromorphic function (in the block structure of first level) and each block has a highest order term with just one eigenvalue; then $z\tilde{x}'=\tilde{A}(z)\tilde{x}$ would have a fundamental solution $G_1(z)$ satisfying (i). To find $F(z)$ might still be hard, but could be considered an algebraic problem.

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