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Fourier-like transformation and a representation of the Lie algebra $\mathfrak{so}(n+1, 2)$

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1. Introduction

The space M of non-zero cotangent vectors to the unit sphere S^n is an $SO(n+1, 2)$-homogeneous symplectic manifold. The geometry of the $SO(n+1, 2)$-action is studied by several authors. (See Akyildiz [1], Onofri [10], [11], Rawnsley [14], Souriau [19] and Wolf [24], [25].) The present note is motivated by Wolf [24], [25]. We consider the problem of "quantizing" this $SO(n+1, 2)$-action. The standard procedure of geometric quantization does not work because there are no $SO(n+1, 2)$-invariant polarizations. (See Elhadad [2], Ozeki and Wakimoto [12], Wakimoto [22] and Wolf [24].) We will work in the framework of Lie algebras rather than groups. The Lie algebra $\mathfrak{so}(n+1, 2)$ is realized as a Poisson subalgebra $\mathfrak{G}$. By integration of the Hamiltonian vector fields associated with elements of $\mathfrak{G}$, we get the symplectic action of $SO(n+1, 2)$ on M. To construct a representation of $\mathfrak{so}(n+1, 2)$, we use a pair of transversal polarizations: one is the vertical polarization Q and the other is a partially complex polarization P invariant under the geodesic flow. The space $\Gamma_Q(\boldsymbol{L}\otimes L^Q)$ of smooth Q-horizontal sections of a complex line bundle $\boldsymbol{L}\otimes L^Q$ over M is naturally identified with $C^\infty(S^n)$. While there exist no smooth P-horizontal sections in $\Gamma(\boldsymbol{L}\otimes L^P)$ except for zero-section, so we must consider "singular" sections. The supports of singular P-horizontal sections are in a disjoint union of hypersurfaces M_m $(m=0, 1, 2,\dots)$ in M. Each M_m is identified with the Stiefel manifold $SO(n+1)/SO(n-1)$, which is an $SO(2)$-principal bundle over the Grassmann manifold $SO(n+1)/(SO(2)\times SO(n-1))$. The Grassmann manifold is an $SO(n+1)$-homogeneous complex manifold. Let L_m be the $SO(n+1, \mathbf{C})$-homogeneous holomorphic line bundle over the Grassmann manifold given in Kowata and Okamoto [8]. Holomorphic sections of L_m are identified with functions on $SO(n+1)/SO(n-1)$. If we identify M_m with this Stiefel manifold, then holomorphic sections of L_m are identified with functions on M_m. Since $\boldsymbol{L}\otimes L^P$ is a trivial bundle over M, these functions are identified with singular sections of $\boldsymbol{L}\otimes L^P$ with supports in M_m. These sections are P-horizontal. The correspondence: a holomorphic section of $L_m \mapsto$ a P-horizontal section of $\boldsymbol{L}\otimes L^P$ with support in M_m, is bijective. Thus, the consideration of the P-horizontal sections is equivalent to that of all the holomorphic sections of L_m $(m=0, 1, 2,\dots)$

simultaneously. In Section 6, we construct, using the formalism of Gawedzki [3], a Fourier-like transformation (or a pairing) $\mathscr{F}$ from a space of P-horizontal sections to a space of Q-horizontal sections. (Cf. Rawnsley [14].) The restriction of $\mathscr{F}$ to the space of P-horizontal sections with supports in M_m, which is identified with the space of holomorphic sections of L_m, coincides, up to constant multiple, with the "modified Poisson integral" defined in Kowata and Okamoto [8]. By means of this intertwining operator $\mathscr{F}$, we get, after some modifications, an irreducible representation of $\mathfrak{so}(n+1, 2)$ by skew-Hermitian operators on S^n. It seems to the author that the choice of a suitable inner product in the representation space is interesting. (Cf. Takahashi [21].) The quantization obtained here is also the one in the sence of Ōmori [9], that is, the quantization of a function ϕ is a pseudo-differential operator $\hat{\phi}$ (of order one) with principal symbol ϕ. (See also Akyildiz [1], Guillemin and Sternberg [5] and Rawnsley [14].)

For symplectic geometry and geometric quantization, see Gawedzki [3], Guillemin and Sternberg [4], Kostant [7], Simms and Woodhouse [15], Śniatycki [16], Souriau [18], Weinstein [23] and Woodhouse [26].

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2. Preliminaries

Let $\boldsymbol{R}^{n+1}$ and $T^*\boldsymbol{R}^{n+1}$ be the $(n+1)$-space and its cotangent bundle with coordinates $x=(x_1,\dots, x_{n+1})$ and $(x, y)=(x_1,\dots, x_{n+1}, y_1,\dots, y_{n+1})$, respectively. Let us denote $y=(y_1,\dots, y_{n+1})$, $|x|^2=\sum x_j^2$, $|y|^2=\sum y_j^2$, $x\cdot y=\sum x_j y_j$, $X_j=\partial/\partial x_j$ and $Y_j=\partial/\partial y_j$. The bundle of non-zero cotangent vectors to the unit n-sphere $S^n=\{x\in\boldsymbol{R}^{n+1} \mid |x|=1\}$ is written by $M=T^*S^n-\{0\text{-section}\}=\{(x, y)\in T^*\boldsymbol{R}^{n+1} \mid |x|=1,\ x\cdot y=0,\ |y|\neq 0\}$ with the projection $\pi\colon M\to S^n$; $\pi(x, y)=x$. The action form, the symplectic form and the Liouville form on M are given by $\omega=\sum y_j dx_j$, $\Omega=-d\omega=\sum dx_j\wedge dy_j$ and $\Theta=(-1)^{n(n-1)/2}(n!)^{-1}\Omega^n$, respectively. Let $C^\infty(M; \boldsymbol{R})$ be the space of all real-valued smooth functions on M. For each $\phi\in C^\infty(M; \boldsymbol{R})$, a vector field ξ_ϕ on M is defined by $\xi_\phi \lrcorner\, \Omega=d\phi$, which is called the Hamiltonian vector field associated with ϕ. The space $C^\infty(M; \boldsymbol{R})$ is a Lie algebra over $\boldsymbol{R}$ under the Poisson bracket operation given by $\{\phi, \psi\}=\xi_\phi\psi=-\Omega(\xi_\phi, \xi_\psi)$. It is called the Poisson algebra of the symplectic manifold (M, Ω). Let $\phi_{jk}=\phi_{jk}(x, y)$ $(1\leqq j<k\leqq n+3)$ denote the functions on M defined by $\phi_{jk}=x_j y_k-x_k y_j$ $(1\leqq j<k\leqq n+1)$, $\phi_{j,n+2}=y_j$ $(1\leqq j\leqq n+1)$, $\phi_{j,n+3}=|y|x_j$ $(1\leqq j\leqq n+1)$

and $\phi_{n+2,n+3}=|y|$. The linear subspace $\mathfrak{G}$ spanned by the functions $\{\phi_{jk}\}$ is a Poisson subalgebra. It is isomorphic to $\mathfrak{so}(n+1, 2)$ under the correspondence: $\phi_{jk}\mapsto E_{jk}-E_{kj}$ $(1\leqq j<k\leqq n+1$ or $n+2\leqq j<k\leqq n+3)$ and $\phi_{jk}\mapsto E_{jk}+E_{kj}$ $(1\leqq j\leqq n+1$ and $n+2\leqq k\leqq n+3)$, where E_{jk} is the $(n+3)\times(n+3)$-matrix which is 1 in the (j, k)-th position and 0 elsewhere. The Hamiltonian vector fields ξ_{jk} associated with ϕ_{jk} are given as follows:

$$\xi_{jk} = \sum\{(\delta_{ki}x_j-\delta_{ij}x_k)X_i + (\delta_{ki}y_i-\delta_{ij}y_k)Y_i\} \quad (1 \leqq j < k \leqq n+1),$$
$$\xi_{j,n+2} = \sum \{(\delta_{ij}-x_ix_j)X_i + (x_jy_i-x_iy_j)Y_i\} \quad (1 \leqq j \leqq n+1),$$
$$\xi_{j,n+3} = \sum (|y|^{-1}x_jy_iX_i-|y|\delta_{ij}Y_i) \quad (1 \leqq j \leqq n+1),$$
$$\xi_{n+2,n+3} = \sum(|y|^{-1}y_iX_i-|y|x_iY_i).$$

Note that $\xi_{n+2,n+3}$ generates the geodesic flow on the unit sphere S^n.

The linear map $\phi_{jk}\mapsto\xi_{jk}$ is a Lie algebra isomorphism of $\mathfrak{G}$ into the Lie algebra of vector fields on M. Since $\{\xi_{jk}\}$ are complete vector fields, they generate, by integration, a symplectic action of $SO(n+1, 2)$ on M. It is well-known that this action preserves no polarizations on M. So, we cannot use the standard method of geometric quantization to construct a representation of the Poisson subalgebra $\mathfrak{G}$. (See Wolf [24], [25].)

In the following sections, we shall employ mainly notions and notations from Gawedzki [3].

3. Polarization P and half-forms

Let U be an open set in M and $u^a=(u^a_1,\dots, u^a_{n+1})$ $(1\leqq a\leqq n)$ be $\boldsymbol{R}^{n+1}$-valued smooth functions on U such that $u^1(x, y)=|y|^{-1}y$ and the matrix ${}^t(x, u^1(x, y), \dots, u^n(x, y))$ is in $SO(n+1)$ for each $(x, y)\in U$. If (V, v^a) is another such a pair, then a map g_{UV}: $U\cap V\to SO(n)$ is defined by $(v^1,\dots, v^n)=(u^1,\dots, u^n)g_{UV}$. For each $(x, y)\in U$, let $P(x, y)$ denote the complex subspace spanned by the tangent vectors $\{\xi_{n+2,n+3}, u^2\cdot Z,\dots, u^n\cdot Z\}$ to M at (x, y), where $u^a\cdot Z=\sum u^a_jZ_j$ with $Z_j=X_j-(-1)^{1/2}|y|Y_j$. Then we have a polarization P on M, which is invariant under the integral flows of ξ_{jk} $(1\leqq j<k\leqq n+1$ or $n+2\leqq j<k\leqq n+3)$, i.e., invariant under the action of $SO(n+1)\times SO(2)$. (See [6].) The frame bundle π_P: $B(M; P)\to M$ of P is a right principal $GL(n, \mathbf{C})$-bundle over M. Coordinate functions are given by φ_U: $U\times GL(n, \mathbf{C})\to\pi_P^{-1}(U)$; $\varphi_U((x, y), g)=(\xi_{n+2,n+3}, u^2\cdot Z,\dots, u^n\cdot Z)g$ together with transition functions g_{UV}. The complex metalinear group is weritten by

$$ML(n, \mathbf{C})=\left\{\tilde{g}=\begin{pmatrix} g & \\ & w\end{pmatrix}\in GL(n+1, \mathbf{C})\,|\,g\in GL(n, \mathbf{C}),\ w\in\mathbf{C}^\times,\ \det g=w^2\right\}$$

with the double covering map σ: $ML(n, \mathbf{C})\to GL(n, \mathbf{C})$; $\sigma(\tilde{g})=g$ and with a holomorphic square root $\chi: ML(n, \mathbf{C}) \to \mathbf{C}^{\times}$; $\chi(\tilde{g})=w$. If we define transition functions $\tilde{g}_{UV}: U \cap V \to ML(n, \mathbf{C})$ by $\tilde{g}_{UV}(x, y)=\begin{pmatrix} g_{UV} & \\ & 1 \end{pmatrix}$, then we have a metalinear frame bundle $\tilde{\pi}_P: \tilde{B}(M; P) \to M$ of P with coordinate functions $\tilde{\varphi}_U: U \times ML(n, \mathbf{C}) \to \tilde{\pi}_P^{-1}(U)$ compatible with φ_U. It is a right principal $ML(n, \mathbf{C})$-bundle over M.

Note that, up to isomorphism, such a bundle is unique for $n \geqq 3$. (See Gawedzki [3, III. 16].)

Let $L^P(x, y)$ denote the one-dimensional complex vector space of all complex-valued functions f on $\tilde{\pi}_P^{-1}(x, y)$ such that $f(F\tilde{g})=\chi(\tilde{g}^{-1})f(F)$ for any $F \in \tilde{\pi}_P^{-1}(x, y)$ and $\tilde{g} \in ML(n, \mathbf{C})$. Then $L^P = \cup L^P(x, y)$ is called the complex line bundle of half-P-forms on M. It has a non-vanishing section $\nu: M \to L^P$ defined by $\nu(\tilde{\varphi}_U((x, y), e))=1$ for $(x, y) \in U$, where e denotes the unit element in $ML(n, \mathbf{C})$.

Let $\wedge^n(M; P)$ be the bundle of complex n-covectors tangent to M, vanishing after contraction with any vector from P. Then $L^P \otimes L^P$ is naturally isomorphic to $\wedge^n(M; P)$. The isomorphism $L^P \otimes L^P \to \wedge^n(M; P)$ is given by $\nu \otimes \nu \mapsto \underline{\nu \otimes \nu} = (\xi_{n+2,n+3} \lrcorner \Omega) \wedge (u^2 \cdot Z \lrcorner \Omega) \wedge \cdots \wedge (u^n \cdot Z \lrcorner \Omega)$. Let $\wedge^{2n-1}(M; P \cap \bar{P})$ be the bundle of complex $(2n-1)$-covectors tangent to M, vanishing after contraction with any vector from $P \cap \bar{P}$. It is a complex line bundle with a non-vanishing section $\xi_{n+2,n+3} \lrcorner \Theta$. According to Gawedzki [3, (44)], we then have a pairing of $C^\infty(M)$-modules $\langle \cdot, \cdot \rangle_P: \Gamma(L^P) \times \Gamma(L^P) \to \Gamma(|\wedge^{2n-1}(M; P \cap \bar{P})|)$, where $\Gamma(\cdot)$ denotes the space of all smooth sections. For the notation $|\cdot|$, see Gawedzki [3, Ch. II].

Note that

$$\langle f\nu, g\nu \rangle_P = \bar{f}g(2|y|)^{(n-1)/2}|\xi_{n+2,n+3} \lrcorner \Theta|$$

for any $f, g \in C^\infty(M)$.

4. Hilbert space $\mathscr{H}^P$ and its inner product $(\cdot, \cdot)_P$

A "quantum bundle" $\boldsymbol{L}$ for (M, Ω) together with a connection is given in [6]. Since $\boldsymbol{L}$ is a trivial bundle, there is a non-vanishing section $\mathbf{1} \in \Gamma(\boldsymbol{L})$. The connection ∇ and a ∇-invariant Hermitian structure $(\cdot|\cdot)$ on $\boldsymbol{L}$ are given respectively by $\nabla_\xi \mathbf{1} = -(-1)^{1/2}(\xi \lrcorner \omega)\mathbf{1}$ and $(f\mathbf{1}|g\mathbf{1})=\bar{f}g$ for any tangent vector ξ to M and $f, g \in C^\infty(M)$.

For each non-negative integer m, let $r_m = m+(n+1)/2$ and M_m denote a hypersurface of M given by $M_m=\{(x, y) \in M \mid |y|=r_m\}$ together with the inclusion $i_m: M_m \to M$. Let $\chi_m: M \to \boldsymbol{R}$ be the characteristic function of M_m and $\mathscr{H}_m^P$ denote the space of sections of $\boldsymbol{L} \otimes L^P$ spanned by the singular sections

$$\{T_{i_1 \cdots i_m} = \chi_m z_{i_1} \cdots z_{i_m} \mathbf{1} \otimes \nu \mid 1 \leqq i_a \leqq n+1,\ 1 \leqq a \leqq m\},$$

where $z_i = x_i - (-1)^{1/2}|y|^{-1}y_i \in C^\infty(M)$.

Note that $\oplus\sum_{m\geqq 0}\mathscr{H}_m^P$ is the space of all "P-horizontal" sections. (See Gawedzki [3, Ch. III.D] and [6].)

According to Gawedzki [3, (53)], we define a pairing

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_P : \Gamma'(\boldsymbol{L} \otimes L^P) \times \Gamma'(\boldsymbol{L} \otimes L^P) \longrightarrow \Gamma'(|\wedge^{2n-1}(M; P \cap \bar{P})|)$$

by $\langle\!\langle f\mathbf{1}\otimes v, g\mathbf{1}\otimes v \rangle\!\rangle_P = (f\mathbf{1}\,|\,g\mathbf{1})\langle v, v\rangle_P$, where $\Gamma'(\cdot)$ denotes the space of not necessarily continuous sections.

Note that

$$\langle\!\langle T_{i_1\cdots i_m}, T_{j_1\cdots j_m} \rangle\!\rangle_P = (2r_m)^{(n-1)/2}\chi_m \bar{z}_{i_1}\cdots\bar{z}_{i_m} z_{j_1}\cdots z_{j_m} |\xi_{n+2,n+3} \lrcorner \Theta| .$$

Since we consider singular sections of $\boldsymbol{L}\otimes L^P$, whose supports are in M_m, we must modify the pairing as follows: Let $\wedge^{2n-2}(M; \{\xi_{n+2,n+3}, \eta\})$ be the bundle of complex $(2n-2)$-covectors tangent to M, vanishing after contraction with $\xi_{n+2,n+3}$ and $\eta = |y|^{-1}\sum y_j Y_j$. It is a complex line bundle with a non-vanishing section $\eta \lrcorner \xi_{n+2,n+3} \lrcorner \Theta$. Let $\iota: \wedge^{2n-1}(M; P \cap \bar{P}) \to \wedge^{2n-2}(M; \{\xi_{n+2,n+3}, \eta\})$ be a bundle isomorphism given by $\iota(\beta) = \eta \lrcorner \beta$. Then ι induces a bundle isomorphism

$$|\iota| : |\wedge^{2n-1}(M; P \cap \bar{P})| \longrightarrow |\wedge^{2n-2}(M; \{\xi_{n+2,n+3}, \eta\})|$$

defined by $|\iota|(|\beta|) = |\iota(\beta)|$ for any non-zero β. Let $\wedge^{2n-2}(M_m; \{\xi_{n+2,n+3}\})$ be the bundle of complex $(2n-2)$-covectors tangent to M_m, vanishing after contraction with the tangent vector $\xi_{n+2,n+3}$ to M_m. It is a complex line bundle over M_m with a non-vanishing section $\eta \lrcorner \xi_{n+2\,n+3} \lrcorner \Theta$. The pull-back

$$i_m^* : \wedge^{2n-2}(M; \{\xi_{n+2,n+3}, \eta\}) \longrightarrow \wedge^{2n-2}(M_m; \{\xi_{n+2,n+3}\})$$

induces a map $|i_m^*| : |\wedge^{2n-2}(M; \{\xi_{n+2,n+3}, \eta\})| \to |\wedge^{2n-2}(M_m; \{\xi_{n+2,n+3}\})|$.

Now, M_m is S^1-fibered by the orbits of $\xi_{n+2,n+3}$. Let M_m/S^1 denote the orbit space together with the projection $\pi_m: M_m \to M_m/S^1$. Then there exists a unique symplectic structure Ω_m on M_m/S^1 such that $\pi_m^*\Omega_m = i_m^*\Omega$. Let $\Theta_m = (-1)^{(n-1)(n-2)/2}((n-1)!)^{-1}\Omega_m^{n-1}$ be the Liouville form on M_m/S^1. Then the volume of $(M_m/S^1, \Omega_m)$ is given by $|M_m/S^1| = r_m^{n-1}|S^{n-1}|\,|S^n|\,|S^1|^{-1}$, where $|S^d|$ denotes the volume of the unit sphere of dimension d. The bundle $\wedge^{2n-2}(M_m/S^1)$ of complex $(2n-2)$-covectors tangent to M_m/S^1 is a complex line bundle over M_m/S^1 with a non-vanishing section Θ_m. The pull-back $\pi_m^*: \wedge^{2n-2}(M_m/S^1) \to \wedge^{2n-2}(M_m; \{\xi_{n+2,n+3}\})$ induces a map $|\pi_m^*| : |\wedge^{2n-2}(M_m/S^1)| \to |\wedge^{2n-2}(M_m; \{\xi_{n+2,n+3}\})|$.

Note that $|\wedge^{2n-2}(M_m/S^1)|$ is the bundle of densities on M_m/S^1.

LEMMA 1 (cf. Gawedzki [3, Prop. III. 17]). *For any $\mathscr{T}_m, \mathscr{T}'_m \in \mathscr{H}_m^P$, there* *exists a unique smooth density* $\langle\!\langle \mathscr{T}_m, \mathscr{T}'_m \rangle\!\rangle$ *on* M_m/S^1 *such that* $|\pi_m^*| \langle\!\langle \mathscr{T}_m, \mathscr{T}'_m \rangle\!\rangle = |i_m^*| |\iota| \langle\!\langle \mathscr{T}_m, \mathscr{T}'_m \rangle\!\rangle_P$.

For the proof, it is enough to note that the function $(\bar{z}_{i_1} \cdots \bar{z}_{i_m} z_{j_1} \cdots z_{j_m}) \circ i_m$ is constant along the orbits of $\xi_{n+2,n+3}$ and $\mathscr{L}_{\xi_{n+2,n+3}}(\eta \lrcorner \xi_{n+2,n+3} \lrcorner \Theta) = 0$ on M_m, where $\mathscr{L}_\xi$ denotes the Lie derivation with respect to a vector field ξ.

Note that

$$\langle\!\langle T_{i_1 \cdots i_m}, T_{j_1 \cdots j_m} \rangle\!\rangle = (2r_m)^{(n-1)/2} f_{i_1 \cdots i_m j_1 \cdots j_m} |\Theta_m|,$$

where $f_{i_1 \cdots i_m j_1 \cdots j_m} \in C^\infty(M_m/S^1)$ is defined by $f_{i_1 \cdots i_m j_1 \cdots j_m} \circ \pi_m = (\bar{z}_{i_1} \cdots \bar{z}_{i_m} z_{j_1} \cdots z_{j_m}) \circ i_m$.

Similarily as Gawedzki [3, (76)], we define an inner product on $\mathscr{H}_m^P$ by $(\mathscr{T}_m, \mathscr{T}'_m)_P = \varepsilon_m \int_{M_m/S^1} \langle\!\langle \mathscr{T}_m, \mathscr{T}'_m \rangle\!\rangle$, where a positive constant ε_m will be determined in Section 6. We say that a section $\mathscr{T} = \sum_{m \geqq 0} \mathscr{T}_m$, $\mathscr{T}_m \in \mathscr{H}_m^P$, of $\boldsymbol{L} \otimes L^P$ is of finite norm if $(\mathscr{T}, \mathscr{T})_P = \sum (\mathscr{T}_m, \mathscr{T}_m)_P$ is finite. Let $\mathscr{H}^P = \{\mathscr{T} = \sum \mathscr{T}_m \mid$ of finite norm$\}$. Then $\mathscr{H}^P$ is a Hilbert space together with the inner product $(\cdot, \cdot)_P$.

Note that for $m \neq m'$, the subspaces $\mathscr{H}_m^P$ and $\mathscr{H}_{m'}^P$ are orthogonal to each other.

5. Vertical polarization Q

Let (U, u^a) be as in Section 3. The vertical polarization Q is spanned at each point $(x, y) \in U$ by the tangent vectors $\{u^a \cdot Y = \sum u_j^a Y_j \mid 1 \leqq a \leqq n\}$ to M. It is invariant under the intgeral flows of ξ_{jk} $(1 \leqq j < k \leqq n+2)$, i.e., invariant under the action of $SO(n+1, 1)$. Coordinate functions ψ_U and transition functions g_{UV} for the frame bundle $\pi_Q: B(M; Q) \to M$ of Q are given similarily as in Section 3. The metalinear frame bundle $\tilde{\pi}_Q: \tilde{B}(M; Q) \to M$ is defined similarily as in Section 3 together with coordinate functions $\tilde{\psi}_U$ and transition functions $\tilde{g}_{UV}$. Up to isomorphism, such a bundle is unique for $n \geqq 3$. The bundle L^Q of half-Q-forms has a non-vanishing section $\mu: M \to L^Q$ defined by $\mu(\tilde{\psi}_U((x, y), e)) = 1$ for $(x, y) \in U$. $L^Q \otimes L^Q$ is naturally isomorphic to $\wedge^n(M; Q)$. The isomorphism is given by $\mu \otimes \mu \mapsto \mu \underline{\otimes} \mu = (u^1 \cdot Y \lrcorner \Omega) \wedge \cdots \wedge (u^n \cdot Y \lrcorner \Omega) = (-1)^n \pi^* dS^n$, where $dS^n = (\sum x_j X_j) \lrcorner (dx_1 \wedge \cdots \wedge dx_{n+1})$ is the volume form on S^n. According to Gawedzki [3, (44)], we have a pairing $\langle \cdot, \cdot \rangle_Q: \Gamma(L^Q) \times \Gamma(L^Q) \to \Gamma(|\wedge^n(M; Q)|)$.

Note that $\langle \mu, \mu \rangle_Q = |\pi^* dS^n|$.

Let $\Gamma_Q(\boldsymbol{L} \otimes L^Q)$ denote the space of all smooth "Q-horizontal" sections of $\boldsymbol{L} \otimes L^Q$. Then $\Gamma_Q(\boldsymbol{L} \otimes L^Q) = \{f \circ \pi \mathbf{1} \otimes \mu \mid f \in C^\infty(S^n)\}$. (See [6].) According to Gawedzki [3, (76)], an inner product is given by $(f \circ \pi \mathbf{1} \otimes \mu, g \circ \pi \mathbf{1} \otimes \mu)_Q = \int_{S^n} \bar{f} g \, dS^n$. The completion of the pre-Hilbert space $(\Gamma_Q(\boldsymbol{L} \otimes L^Q), (\cdot, \cdot)_Q)$ is denoted by $(\mathscr{H}^Q, (\cdot, \cdot)_Q)$. It may be identified with $L^2(S^n)$ under the correspondence

$f \circ \pi \mathbf{1} \otimes \mu \to f$.

Let $h_{i_1 \cdots i_m}$ be a spherical harmonic of degree m given by $h_{i_1 \cdots i_m} = (-1)^m ((n-1)(n+1)\cdots(2m+n-3))^{-1} X_{i_1} \cdots X_{i_m} (|x|^{1-n})|_{S^n}$, and $\mathscr{H}^Q_m$ the subspace of $\mathscr{H}^Q$ spanned by the sections $\{H_{i_1 \cdots i_m} = h_{i_1 \cdots i_m} \circ \pi \mathbf{1} \otimes \mu \mid 1 \leqq i_a \leqq n+1,\ 1 \leqq a \leqq m\}$.

LEMMA 2. (1) $(H_{i_1 \cdots i_m}, H_{j_1 \cdots j_m})_Q$

$$= (2m+n-1)^{-1} \sum_{a=1}^m \delta_{i_a j_m} (H_{i_1 \cdots \hat{i}_a \cdots i_m}, H_{j_1 \cdots j_{m-1}})_Q$$
$$- ((2m+n-1)(2m+n-3))^{-1} \sum_{a \neq b} \delta_{i_a i_b} (H_{i_1 \cdots \hat{i}_a \cdots \hat{i}_b \cdots i_m j_m}, H_{j_1 \cdots j_{m-1}})_Q .$$

(2) $\sum_{j_1, \cdots, j_m} (H_{i_1 \cdots i_m}, H_{j_1 \cdots j_m})_Q H_{j_1 \cdots j_m}$

$$= ((n+1)(n+3)\cdots(2m+n-1))^{-1} (m!) |S^n| H_{i_1 \cdots i_m} .$$

LEMMA 3. *We have*

$$(T_{i_1 \cdots i_m}, T_{j_1 \cdots j_m})_P = \varepsilon_m \delta_m (H_{i_1 \cdots i_m}, H_{j_1 \cdots j_m})_Q ,$$

where

$$\delta_m = (2r_m)^{(n-1)/2} 2^m (n+1)(n+3)\cdots(2m+n-3)$$
$$(n(n+1)\cdots(m+n-2))^{-1} |M_m/S^1|\, |S^n|^{-1}.$$

PROOF. The actions of $SO(n+1)$ on $\mathscr{H}^P_m$ and on $\mathscr{H}^Q_m$ are naturally defined, which are transitive and leave the inner products $(\cdot, \cdot)_P$ and $(\cdot, \cdot)_Q$ invariant. The isomorphism $\mathscr{H}^m_P \to \mathscr{H}^Q_m$ given by $T_{i_1 \cdots i_m} \mapsto H_{i_1 \cdots i_m}$ is well-defined and commutes with the actions of $SO(n+1)$. It follows that $(T_{i_1 \cdots i_m}, T_{j_1 \cdots j_m})_P = \text{const.}(H_{i_1 \cdots i_m}, H_{j_1 \cdots j_m})_Q$. The constant is determined by calculating $(T_{1 \cdots 1}, T_{1 \cdots 1})_P$ and $(H_{1 \cdots 1}, H_{1 \cdots 1})_Q$.

Since P and Q are transversal, $(L^P \otimes L^Q) \otimes (L^P \otimes L^Q)$ is naturally isomorphic to the bundle $\wedge^{2n}(M)$ of complex $2n$-covectors tangent to M. The isomorphism is given by $(\nu \otimes \mu) \otimes (\nu \otimes \mu) \mapsto (\nu \underline{\otimes} \nu) \wedge (\mu \underline{\otimes} \mu) = \Theta$. We shall choose $\nu \otimes \mu$ as an adjustment of L^P and L^Q. For the adjustment, see Gawedzki [3, Def. IV.4].

6. Fourier-like transformation

Let $\boldsymbol{L}^*$ be the dual bundle of $\boldsymbol{L}$ with a dual connection ∇^*. It has a non-vanishing section $\mathbf{1}^* = (\mathbf{1} \mid \cdot)$. Let p_i: $M \times M \to M$, $i=1, 2$, be the projection onto the i-th factor. Let $\boldsymbol{W} = p_1^*(\boldsymbol{L} \otimes L^P) \otimes p_2^*(\boldsymbol{L}^* \otimes L^Q)$. Then $\boldsymbol{W}$ has a non-vanishing section Ξ: $((x, y), (x', y')) \mapsto \mathbf{1}(x, y) \otimes \nu(x, y) \otimes \mathbf{1}^*(x', y') \otimes \mu(x', y')$. For each section $\mathscr{K} = \mathscr{k} \Xi$, $\mathscr{k}$: $M \times M \to \mathbf{C}$, of $\boldsymbol{W}$, and for each $(x, y) \in M$, sections of $\boldsymbol{L} \otimes L^P$ and $\boldsymbol{L}^* \otimes L^Q$ are defined by $\mathscr{K}_P(\cdot, (x, y)) = \mathscr{k}(\cdot, (x, y)) \mathbf{1} \otimes \nu$ and $\mathscr{K}_Q((x, y), \cdot) = \mathscr{k}((x, y), \cdot) \mathbf{1}^* \otimes \mu$, respectively. By $\mathscr{K}_\triangle$ we shall denote a section of $L^P \otimes L^Q$ given by $\mathscr{K}_\triangle(x, y) = \mathscr{k}((x, y), (x, y)) \nu(x, y) \otimes \mu(x, y)$.

DEFINITION. A not necessarily continuous section $\mathscr{K}$ of $\boldsymbol{W}$ will be called a *distinguished kernel for the pair* (P, Q) of polarizations if:

(i) for each $(x, y) \in M$, $\mathscr{K}_P(\cdot, (x, y))$ is P-horizontal,

(ii) for each $(x, y) \in M$, $\mathscr{K}_Q((x, y), \cdot)$ is Q-horizontal, and

(iii) $\mathscr{K}_\triangle = \nu \otimes \mu$ on $\cup M_m$.

(Cf. Gawedzki [3, Def. IV.5].)

From the definition, it follows that the support of a distinguished kernel $\mathscr{K}$ is $(\cup M_m) \times M$.

Note that $\cup M_m$ and M is the "Bohr-Sommerfeld sets" for P and Q, respectively. (See Śniatycki and Toporowski [17, §2].)

LEMMA 4. *There exists a unique distinguished kernel* $\mathscr{K} = k\Xi$ *for* (P, Q). k *is given by*

$$k((x, y), (x', y')) = \sum_{m \geq 0} \sum_{i_1, \dots, i_m} [\chi_m z_{i_1} \cdots z_{i_m}](x, y) h_{i_1 \cdots i_m} \circ \pi(x', y').$$

PROOF. For the existence, it is enough to show that $k_\triangle = 1$ on M_m, where $k_\triangle(x, y) = k((x, y), (x, y))$. For $1 \leqq j < k \leqq n+1$, we have

$$\xi_{jk}(z_{i_1} \cdots z_{i_m}) = \sum_{a=1}^{m} (\delta_{i_a k} z_{i_1} \cdots \hat{z}_{i_a} \cdots z_{i_m} z_j - \delta_{i_a j} z_{i_1} \cdots \hat{z}_{i_a} \cdots z_{i_m} z_k)$$

and

$$\xi_{jk}(h_{i_1 \cdots i_m} \circ \pi) = \sum_{a=1}^{m} (\delta_{i_a k} h_{i_1 \cdots \hat{i}_a \cdots i_m j} - \delta_{i_a j} h_{i_1 \cdots \hat{i}_a \cdots i_m k}) \circ \pi.$$

It follows that $\xi_{jk}(k_\triangle) = 0$. Since $SO(n+1)$ acts on M_m transitively, we have $k_\triangle = \text{const.}$ on M_m. Calculating $k_\triangle(x, y)$ for $x = (1, 0, \dots, 0)$ and $y = (0, -r_m, 0, \dots, 0)$, we have $k_\triangle = 1$ on M_m. The uniqueness follows from the fact that for each fixed $x \in S^n$, $\sum c_{i_1 \cdots i_m} z_{i_1} \cdots z_{i_m} = 0$ for all y such that $(x, y) \in M_m$ implies $c_{i_1 \cdots i_m} = 0$, where $c_{i_1 \cdots i_m} \in \mathbf{C}$ are totally symmetric in all indices and with all pair traces zero.

LEMMA 5. *For each* $(x, y) \in M$, *we have*

$$\begin{aligned}(T_{i_1 \cdots i_m}(\cdot), \mathscr{K}_P(\cdot, (x, y)))_P \\ = \varepsilon_m \delta_m((n+1)(n+3) \cdots (2m+n-1))^{-1}(m!)|S^n| h_{i_1 \cdots i_m} \circ \pi(x, y).\end{aligned}$$

The lemma follows from Lemma 2 and Lemma 3.

Let $\delta: \boldsymbol{L}^* \otimes L^Q \to \boldsymbol{L} \otimes L^Q$ be the bundle anti-isomorphism defined by $\delta(c\mathbf{1}^* \otimes \mu) = \bar{c}\mathbf{1} \otimes \mu$ for $c \in \mathbf{C}$. Now, following Gawedzki [3, (176)], let us define a linear isomorphism $\mathscr{F}_m: \mathscr{H}_m^P \to \mathscr{H}_m^Q$ by

$$\begin{aligned}\mathscr{F}_m(\mathscr{T})(x, y) &= \delta((\mathscr{T}(\cdot), \mathscr{K}_P(\cdot, (x, y)))_P(\mathbf{1}^* \otimes \mu)(x, y)) \\ &= (\mathscr{K}_P(\cdot, (x, y)), \mathscr{T}(\cdot))_P(\mathbf{1} \otimes \mu)(x, y).\end{aligned}$$

LEMMA 6 (cf. Kowata and Okamoto [8]). *$\mathscr{F}_m$ is a unitary transformation if and only if*

$$\varepsilon_m\delta_m = ((n+1)(n+3)\cdots(2m+n-1))^2(m!|S^n|)^{-2}.$$

In this case, $\mathscr{F}_m$ induces a unitary transformation $\mathscr{F}$: $\mathscr{H}^P \to \mathscr{H}^Q$, which gives a unitary equivalence between $\mathscr{H}^P$ and $\mathscr{H}^Q$.

We call $\mathscr{F}$ a *Fourier-like transformation* associated with the transversal polarizations P and Q.

Note that

$$\mathscr{F}(T_{i_1\cdots i_m}) = (n+1)(n+3)\cdots(2m+n-1)(m!|S^n|)^{-1}H_{i_1\cdots i_m}.$$

7. Representation of the Poisson subalgebra $\mathfrak{G}$

By means of the polarization P, any function in the Poisson subalgebra spanned by $\{\phi_{jk} \mid 1 \leqq j<k \leqq n+1$ or $n+2 \leqq j<k \leqq n+3\}$ is geometrically quantized. (See [6].) The Hermitian operator $\hat{\phi}^P_{jk}$ on $\mathscr{H}^P$ corresponding to ϕ_{jk} is given as follows:

$$\hat{\phi}^P_{jk}(T_{i_1\cdots i_m}) = -(-1)^{1/2} \textstyle\sum_{a=1}^m (\delta_{i_a k} T_{i_1\cdots \hat{i}_a \cdots i_m j} - \delta_{i_a j} T_{i_1\cdots \hat{i}_a \cdots i_m k})$$

for $1 \leqq j<k \leqq n+1$ and

$$\hat{\phi}^P_{n+2,n+3}(T_{i_1\cdots i_m}) = (m+(n-1)/2)T_{i_1\cdots i_m}.$$

On the other hand, by means of the polarization Q, any function in the Poisson subalgebra spanned by $\{\phi_{jk} \mid 1 \leqq j<k \leqq n+2\}$ is geometrically quantized as follows: For any vector field ξ on M, whose integral flow preserves Q, a ξ-derivation $\mathscr{L}^{1/2}_\xi$ on $\Gamma(L^Q)$ is defined by $2(\mathscr{L}^{1/2}_\xi \mu)\otimes\mu = L_\xi(\mu\otimes\mu)$. (See Gawedzki [3, Prop. II. 6].)

LEMMA 7. *We have $\mathscr{L}^{1/2}_{\xi_{jk}}\mu = 0$ for $1 \leqq j<k \leqq n+1$, and $\mathscr{L}^{1/2}_{\xi_{j,n+2}}\mu = -(n/2)x_j\mu$ for $1 \leqq j \leqq n+1$.*

Now, according to the usual method of geometric quantization, the Hermitian operators $\hat{\phi}^Q_{jk}$ $(1 \leqq j<k \leqq n+2)$ on $\mathscr{H}^Q$ corresponding to ϕ_{jk} are given by $\hat{\phi}^Q_{jk} = -(-1)^{1/2}\{(\nabla_{\xi_{jk}} + (-1)^{1/2}\phi_{jk})\otimes\mathscr{L}^{1/2}_{\xi_{jk}}\}$. They span a Lie algebra isomorphic to $\mathfrak{so}(n+1, 1)$.

Note that for any $f \in C^\infty(S^n)$,

$$\hat{\phi}^Q_{jk}(f\circ\pi\mathbf{1}\otimes\mu) = -(-1)^{1/2}\{(x_jX_k - x_kX_j)f\}\circ\pi\mathbf{1}\otimes\mu$$

for $1 \leqq j<k \leqq n+1$, and

$$\hat{\phi}^Q_{j,n+2}(f\circ\pi\mathbf{1}\otimes\mu) = -(-1)^{1/2}\{(\textstyle\sum_{i=1}^{n+1}(\delta_{ij} - x_ix_j)X_i - (n/2)x_j)f\}\circ\pi\mathbf{1}\otimes\mu$$

for $1 \leqq j \leqq n+1$. (See, for example, Śniatycki [16, (7.82)].)

LEMMA 8. *We have* $\mathscr{F} \circ \hat{\phi}^P_{jk} \circ \mathscr{F}^{-1} = \hat{\phi}^Q_{jk}$ *for* $1 \leqq j < k \leqq n+1$.

In the following, $\hat{\phi}^Q_{jk}$ is written simply by $\hat{\phi}_{jk}$ for $1 \leqq j < k \leqq n+1$. Now, let us define

$$\hat{\phi}_{n+2,n+3} = \mathscr{F} \circ \hat{\phi}^P_{n+2,n+3} \circ \mathscr{F}^{-1}.$$

Then we have

$$\hat{\phi}_{n+2,n+3}(f \circ \pi \mathbf{1} \otimes \mu) = ((\Delta + (n-1)^2/4)^{1/2} f) \circ \pi \mathbf{1} \otimes \mu,$$

where Δ is the Laplace-Beltrami operator on the unit sphere S^n. (Cf. Rawnsley [14].) $\hat{\phi}_{n+2,n+3}$ is a Hermitian, pseudo-differential operator of order one with principal symbol $\phi_{n+2,n+3}$. Since $\mathfrak{G}$ is generated by $\{\phi_{jk} \mid 1 \leqq j < k \leqq n+2$ or $n+2 \leqq j < k \leqq n+3\}$, we expect that the Lie algebra generated by $\{(-1)^{1/2}\hat{\phi}_{jk} \mid 1 \leqq j < k \leqq n+1$ or $n+2 \leqq j < k \leqq n+3\} \cup \{(-1)^{1/2}\hat{\phi}^Q_{j,n+2} \mid 1 \leqq j \leqq n+1\}$ is naturally isomorphic to $\mathfrak{G}$. But we have the following:

PROPOSITION 9. *For each fixed* $\lambda \in \mathbf{C}$ *and* $1 \leqq j \leqq n+1$, *define*

$$D^\lambda_{j,n+2}(f \circ \pi \mathbf{1} \otimes \mu) = -(-1)^{1/2}\{(\textstyle\sum_{i=1}^{n+1} (\delta_{ij} - x_i x_j) X_i + \lambda x_j) f\} \circ \pi \mathbf{1} \otimes \mu$$

for any $f \in C^\infty(S^n)$, *and*

$$D^\lambda_{j,n+3} = (-1)^{1/2} [D^\lambda_{j,n+2}, \hat{\phi}_{n+2,n+3}].$$

Then we have

$$(-1)^{1/2} [D^\lambda_{j,n+2}, D^\lambda_{k,n+3}] = \delta_{jk} \hat{\phi}_{n+2,n+3}$$

if and only if $\lambda = -(n \pm 1)/2$.

So, we shall modify $\hat{\phi}^Q_{j,n+2}$ $(1 \leqq j \leqq n+1)$ to define an operator $\hat{\phi}_{j,n+2}$ on $\Gamma_Q(\boldsymbol{L} \otimes L^Q)$ by

$$\hat{\phi}_{j,n+2}(f \circ \pi \mathbf{1} \otimes \mu) = -(-1)^{1/2}\{(\textstyle\sum_{i=1}^{n+1} (\delta_{ij} - x_i x_j) X_i - ((n-1)/2) x_j) f\} \circ \pi \mathbf{1} \otimes \mu$$

for any $f \in C^\infty(S^n)$. Then, by analogy with $\phi_{j,n+3} = \{\phi_{j,n+2}, \phi_{n+2,n+3}\}$, we shall define $\hat{\phi}_{j,n+3} = (-1)^{1/2} [\hat{\phi}_{j,n+2}, \hat{\phi}_{n+2,n+3}]$ for $1 \leqq j \leqq n+1$. It is a pseudo-differential operator of order one with principal symbol $\phi_{j,n+3}$.

LEMMA 10. *For* $1 \leqq j \leqq n+1$, *we have*

$$\begin{aligned} \hat{\phi}_{j,n+2}(H_{i_1 \cdots i_m}) = (-1)^{1/2}\{&(m + (n-1)/2) H_{i_1 \cdots i_m j} - 2^{-1} \textstyle\sum_{a=1}^m \delta_{i_a j} H_{i_1 \cdots \hat{i}_a \cdots i_m} \\ &+ (2m+n-3)^{-1} \textstyle\sum_{1 \leqq a < b \leqq m} \delta_{i_a i_b} H_{i_1 \cdots \hat{i}_a \cdots \hat{i}_b \cdots i_m j}\} \end{aligned}$$

and

$$\hat{\phi}_{j,n+3}(H_{i_1\cdots i_m}) = (m+(n-1)/2)H_{i_1\cdots i_m j}+2^{-1}\sum_{a=1}^{m}\delta_{i_a j}H_{i_1\cdots\hat{i}_a\cdots i_m}$$
$$-(2m+n-3)^{-1}\sum_{1\leqq a<b\leqq m}\delta_{i_a i_b}H_{i_1\cdots\hat{i}_a\cdots\hat{i}_b\cdots i_m j}\,.$$

Let $\hat{\mathfrak{G}}$ (resp. $\tilde{\mathfrak{G}}$) denote the linear space over $\boldsymbol{R}$ spanned by the operators $\hat{\phi}_{jk}$ (resp. $(-1)^{1/2}\hat{\phi}_{jk}$) $(1\leqq j<k\leqq n+3)$, and $\rho\colon \mathfrak{G}\to\tilde{\mathfrak{G}}$ be the linear map given by $\phi_{jk}\mapsto(-1)^{1/2}\hat{\phi}_{jk}$.

LEMMA 11. *$\tilde{\mathfrak{G}}$ is a Lie algebra under the bracket operation. ρ is an isomorphism of $\mathfrak{G}$ onto $\tilde{\mathfrak{G}}$.*

As operators on the Hilbert space $\mathscr{H}^Q$, $\hat{\phi}_{jk}(1\leqq j\leqq n+1$ and $n+2\leqq k\leqq n+3)$ are not Hermitian. To make them Hermitian, we shall modify $(\mathscr{H}^Q, (\cdot, \cdot)_Q)$ as follows: Let $\langle\cdot, \cdot\rangle$ denote the inner product on $\Gamma_Q(\boldsymbol{L}\otimes L^Q)$ defined by

$$\langle f\circ\pi\mathbf{1}\otimes\mu,\ g\circ\pi\mathbf{1}\otimes\mu\rangle = (f\circ\pi\mathbf{1}\otimes\mu,\ \hat{\phi}_{n+2,n+3}(g\circ\pi\mathbf{1}\otimes\mu))_Q = \int_{S^n}\bar{f}(\Lambda g)dS^n,$$

where $\Lambda=(\Delta+(n-1)^2/4)^{1/2}$. We assume here $n\geqq 2$. Note that

$$\langle H_{i_1\cdots i_m}, H_{j_1\cdots j_m}\rangle = (m+(n-1)/2)(H_{i_1\cdots i_m}, H_{j_1\cdots j_m})_Q\,.$$

Let $H_{1/2}(S^n)$ be the Sobolev space on S^n with the inner product $\langle\cdot, \cdot\rangle$ given by

$$\langle f, g\rangle = \int_{S^n}\bar{f}(\Lambda g)dS^n.$$

Then the completion of the pre-Hilbert space $(\Gamma_Q(\boldsymbol{L}\otimes L^Q), \langle\cdot, \cdot\rangle)$ is identified with $(H_{1/2}(S^n), \langle\cdot, \cdot\rangle)$ under the correspondence $f\circ\pi\mathbf{1}\otimes\mu\to f$.

LEMMA 12. *Each element of $\hat{\mathfrak{G}}$ is a Hermitian operator on $H_{1/2}(S^n)$.*

The lemma follows easily from Lemma 2.

THEOREM. *$\rho\colon\mathfrak{G}\to\tilde{\mathfrak{G}}$ provides an irreducible representation of the Lie algebra $\mathfrak{so}(n+1, 2)$ on the Sobolev space $H_{1/2}(S^n)$ by skew-Hermitian, pseudo-differential operators of order one.*

The irreducibility follows from the fact that the restriction of ρ to a subalgebra isomorphic to $\mathfrak{so}(n+1, 1)$ is irreducible. (See Akyildiz [1] and Takahashi [21, §5].)

By integration, ρ gives rise to a "Fourier integral representation" of $SO(n+1, 2)$ or its covering group. (Cf. Guillemin and Sternberg [5].) Note that the period of the geodesic flow generated by $\xi_{n+2,n+3}$ is 2π, while the period of the one-parameter group of unitary transformations generated by $(-1)^{1/2}\hat{\phi}_{n+2,n+3}$ is 2π for odd n and 4π for even n. (Compare with Souriau [20, §10].)

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