HIROSHIMA MATH. J.
12 (1982), 627–644

Castelnuovo's regularity of graded rings and modules

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(Received May 4, 1982)

Introduction

In this paper we shall introduce a numerical invariant of a graded module which we call Castelnuovo's regularity, and our aim is to study the structure of graded rings by using this invariant.

Castelnuovo's regularity was first defined by Mumford [12] for coherent sheaves on projective spaces, and on the other hand it is closely related to the a-invariant of a graded ring introduced by Goto and Watanabe ([6], see also Schenzel [19]).

One of our main purposes is to study the structure of minimal free resolutions of graded rings by Castelnuovo's regularity. Such studies were already done by Wahl, Sally and Schenzel in some special cases, and we were stimulated by their work.

After proving a fundamental theorem which claims, in particular, that 0-regular positively graded modules over a homogeneous algebra are generated by their elements of degree zero, we show that flat 0-regular homogeneous algebras are characterized as symmetric algebras.

Then we give a characterization of 0-regular graded modules over 1-regular homogeneous algebras in terms of their minimal free resolutions, and we obtain a generalization of a theorem of Schenzel about minimal free resolutions of certain Cohen-Macaulay algebras (e.g., 1-regular algebras).

For Cohen-Macaulay algebras, there are useful characterizations of regularity by their Hilbert functions and Hilbert series. Using this fact, we examine the structure of minimal free resolutions of certain Cohen-Macaulay algebras which include 2-regular Cohen-Macaulay algebras.

Next, we consider upper bounds of regularity for Cohen-Macaulay and Buchsbaum algebras and study the cases in which given upper bounds are attained.

Finally we remark about a relation between regularity and the degree of defining equations of homogeneous algebras.

I would like to thank my friend Shiroh Itoh for useful discussions and comments in preparing this paper.

Notation and terminology

Throughout this paper, all rings are commutative noetherian rings with unit.

Graded rings are always positively graded, that is, without negative degree part.

For a ring R, a graded ring A is called a graded R-algebra if $A_0=R$, and A is called a homogeneous R-algebra if $A_0=R$ and A is generated by A_1 over R.

For a homogeneous algebra A over a field k, we denote by $\mathrm{e}(A)$ and $\mathrm{emb}(A)$ the multiplicity and the embedding dimension $\dim_k A_1$ of A respectively. If A is Cohen-Macaulay, we denote by $\mathrm{r}(A)$ the Cohen-Macaulay type of A. A is called a hypersurface (of degree e) if A is isomorphic to $k[X_1,\dots, X_v]/(f)$, where $f\neq 0$ is a homogeneous polynomial (of degree e). Similarly, A is called a complete intersection (of type $(e_1,\dots, e_r)$) if A is isomorphic to $k[X_1,\dots, X_v]/(f_1,\dots,f_r)$, where $\{f_1,\dots,f_r\}$ is a homogeneous regular sequence (with $\deg f_i=e_i$).

For a graded module $M=\oplus_{n\in \boldsymbol{Z}} M_n$ over a graded ring A, the i-th local cohomology module $\mathrm{H}^i_P(M)$ of M with support in $P=A_+=\oplus_{n>0}A_n$ is also a graded A-module. We denote its j-th degree part by $[\mathrm{H}^i_P(M)]_j$. Concerning local cohomology theory of graded modules, see Goto and Watanabe [6].

Castelnuovo's regularity

Let $A=\oplus_{n\geq 0}A_n$ be a graded ring, $M=\oplus_{n\in \boldsymbol{Z}} M_n$ a graded A-module and let m be an integer.

DEFINITION 1. We say that M is *m-regular* (in the sense of Castelnuovo) if $[\mathrm{H}^i_P(M)]_j=0$ for every i, j such that $i+j>m$.

The following theorem which is fundamental in this paper is essentially due to Mumford (cf. [12], p. 99), and we prove it for the sake of completeness.

THEOREM 2. *Let A be homogeneous algebra over a ring R, M a finitely generated graded A-module and let m be an integer. Suppose that* $[\mathrm{H}^i_P(M)]_j=0$ *for every i, j such that $i>0$ and $i+j=m+1$. Then* $[\mathrm{H}^i_P(M)]_j=0$ *for every i, j such that $i>0$ and $i+j\geqq m+1$. Morevoer, if* $[\mathrm{H}^0_P(M)]_j=0$ *for every $j\geqq m+1$ (e.g., if* $\mathrm{depth}_P(M)>0$*), then we have $A_iM_j=M_{i+j}$ for every $i\geqq 0$ and $j\geqq m$.*

PROOF. First, we may assume that R is a local ring with infinite residue field by localizing and by considering the base change $R\to R(T)$. Next, we may assume that $A=R[X_1,\dots, X_n]$ with $\deg X_i=1$. We prove the assertions by induction on n. If $n=0$ or $\mathrm{Ass}_+(M)=\{\mathfrak{p}\in \mathrm{Ass}(M);\ \mathfrak{p}\not\supset P\}=\emptyset$, the assertions are trivial since $\mathrm{H}^0_P(M)=M$ and $\mathrm{H}^i_P(M)=0$ for every $i>0$. Suppose that $n>0$ and $\mathrm{Ass}_+(M)=\{\mathfrak{p}_1,\dots, \mathfrak{p}_r\}$ ($\mathfrak{p}_i$ are homogeneous ideals). Then since $P\not\subset \mathfrak{p}_1\cup\cdots\cup \mathfrak{p}_r$, we can find an element $a\in A_1-\mathfrak{m}A_1\cup \mathfrak{p}_1\cup\cdots\cup\mathfrak{p}_r$, where $\mathfrak{m}$ is the maximal ideal of R (cf. [14], Th. 2.3). Hence we can take a as a part of a free basis of A_1, and by changing coordinates we may assume that $a=X_n$. Put $X=X_n$ and consider the following exact sequence:

$$0 \longrightarrow N \longrightarrow M \xrightarrow{X} XM(1) \longrightarrow 0.$$

Then since $\mathrm{Supp}(N) \subset \mathrm{V}(P)$, we get $\mathrm{H}^i_P(M) \cong \mathrm{H}^i_P(XM(1))$ for every $i>0$. Put $\bar{A}=A/XA \cong R[X_1,\dots, X_{n-1}]$, $\bar{P}=\bar{A}_+$ and $\bar{M}=M/XM$. If $i+j=m+1$ and $i>0$, then from the exact sequence $0 \to XM \to M \to \bar{M} \to 0$, we have the exact sequence

$$0 = [\mathrm{H}^i_P(M)]_j \longrightarrow [\mathrm{H}^i_P(\bar{M})]_j \longrightarrow [\mathrm{H}^{i+1}_P(XM)]_j = [\mathrm{H}^{i+1}_P(M)(-1)]_j = 0,$$

so that $[\mathrm{H}^i_{\bar{P}}(\bar{M}))]_j=0$. By induction hypothesis we have $[\mathrm{H}^i_{\bar{P}}(\bar{M})]_j=0$ for every i, j such that $i>0$ and $i+j \geqq m+1$. If $i>0$ and $i+j=m+2$, then

$$0 = [\mathrm{H}^i_P(M)(-1)]_j \cong [\mathrm{H}^i_P(XM)]_j \longrightarrow [\mathrm{H}^i_P(M)]_j \longrightarrow [\mathrm{H}^i_P(\bar{M})]_j = 0$$

is exact, so that $[\mathrm{H}^i_P(M)]_j=0$. Repeating this argument, we get the first assertion. For the second assertion, we have $\bar{A}_i\bar{M}_j=\bar{M}_{i+j}$ for every $i \geqq 0$ and $j \geqq m$ by induction hypothesis (note that $[\mathrm{H}^0_{\bar{P}}(\bar{M})]_j=0$ for every $j \geqq m+1$). Thus we have $A_iM_j+XM_{i+j-1}=M_{i+j}$. By induction on i, we get $M_{i+j-1}=A_{i-1}M_j$. Hence $M_{i+j}=A_iM_j+XM_{i+j-1}=A_iM_j+XA_{i-1}M_j=A_iM_j$. Q.E.D.

Given a homogeneous R-algebra A, put $X=\mathrm{Proj}(A)$ and let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. If we put $M=\bigoplus_{n \in \boldsymbol{Z}} \mathrm{H}^0(X, \mathscr{F}(n))$, then we have $\mathscr{F}=\tilde{M}$ (the $\mathcal{O}_X$-module associated to M), $\mathrm{H}^0_P(M)=\mathrm{H}^1_P(M)=0$ and $\mathrm{H}^{i+1}_P(M)=\bigoplus_{n \in \boldsymbol{Z}}\mathrm{H}^i(X, \mathscr{F}(n))$ for every $i>0$ (cf. EGA [8], II, Th. 2.7.5). Therefore, by Th. 2, M is m-regular if and only if $\mathrm{H}^i(X, \mathscr{F}(j))=0$ for every i, j such that $i>0$ and $i+j=m$, i.e., $\mathscr{F}$ is m-regular in the sense of Mumford [12].

DEFINITION 3. We define the (Castelnuovo's) *regularity* $\mathrm{reg}(M)$ of M by $\mathrm{reg}(M)=\inf\{m \in \boldsymbol{Z};\ M \text{ is } m\text{-regular}\}$.

The following proposition follows easily from the long exact sequence of local cohomology, so we omit the proof.

PROPOSITION 4. *Let A be a graded ring and consider the following exact sequence of graded A-modules*:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

Then: (1) *If* $\mathrm{reg}(M) > \mathrm{reg}(M'')$, *then* $\mathrm{reg}(M')=\mathrm{reg}(M)$.
(2) *If* $\mathrm{reg}(M) < \mathrm{reg}(M'')$, *then* $\mathrm{reg}(M')=\mathrm{reg}(M'')+1$.
(3) *If* $\mathrm{reg}(M)=\mathrm{reg}(M'')$, *then* $\mathrm{reg}(M') \leqq \mathrm{reg}(M)+1$.

COROLLARY 5. *Let M be a graded module over a graded ring A and let $a \in P$ be a homogeneous M-regular element. Then we have*

$$\mathrm{reg}(M/aM) = \mathrm{reg}(M) + \deg(a) - 1.$$

PROOF. Apply Prop. 4 to the exact sequence

$$0 \longrightarrow M(-d) \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0, \quad d=\deg(a).$$

For example, if $A=k[X_1,\dots, X_n]/(f_1,\dots,f_r)$ is a complete intersection of type $(e_1,\dots, e_r)$ over a field k, then $\mathrm{reg}(A)=\sum_{i=1}^r e_i-r$, and if $A=k[X_1,\dots, X_n]$ is a polynomial ring with $\deg X_i=d_i>0$, then $\mathrm{reg}(A)=-\sum_{i=1}^n d_i+n$.

REMARK. Using the theorem on cohomology and base change (cf. Hartshorne [9], p. 290, Th. 12.11), we can prove the following semicontinuity theorem for regularity: Let A be a homogeneous algebra over a ring R and M a finitely generated graded A-module which is flat over R. Then:

(1) The function $y\mapsto \mathrm{reg}(M\otimes_R k(y))$ defined on $\mathrm{Spec}(R)$ is upper semicontinuous, namely, the set $\{y\in \mathrm{Spec}(R);\ M\otimes_R k(y)$ is m-regular$\}$ is open for every integer m.

(2) $\mathrm{reg}(M)=\sup\{\mathrm{reg}(M\otimes_R k(y));\ y\in \mathrm{Spec}(R)\}$.

(3) If I is an ideal of R which is contained in the Jacobson radical of R, then we have $\mathrm{reg}(M\otimes_R R/I)=\mathrm{reg}(M)$. Using these facts, problems on M sometimes can be reduced to those of graded modules over a homogeneous algebra over a field.

The following theorem gives a characterization of flat 0-regular homogeneous algebras.

THEOREM 6. *Let A be a homogeneous algebra over a ring R and suppose that A_1 is R-projective. Then* $\mathrm{reg}(A)=0$ *if and only if* $A\cong S_R(A_1)$, *the symmetric algebra of an R-module A_1.*

PROOF. We may assume that R is a local ring. If $A=R[X_1,\dots, X_n]$ with $\deg X_i=1$, then $\mathrm{reg}(A)=\mathrm{reg}(R)=0$ by Cor. 5. Conversely, suppose that $\mathrm{reg}(A)=0$ and put $A=S/I$, where $S=R[X_1,\dots, X_n]$ with $\deg X_i=1$, $n=\mathrm{rank}_R(A_1)$ and I is a homogeneous ideal of S. Then $\mathrm{reg}(I)\leqq 1$ by Prop. 4 and $I_0=I_1=0$. Hence, by Th. 2, we have $I_n=S_{n-1}I_1=0$ for every $n\geqq 2$. Therefore $I=0$ and $A=R[X_1,\dots, X_n]$. Q.E.D.

COROLLARY 7. *Let A be homogeneous algebra over a field k. Then the following conditions are equivalent:*

(1) $\mathrm{reg}(A)=0$.

(2) $A\cong k[X_1,\dots, X_n]$ *with* $\deg X_i=1$.

(3) *A is regular.*

THEOREM 8. *Let A be a homogeneous algebra over a ring R and $M=\bigoplus_{n\geqq 0} M_n$ a non-zero positively graded finitely generated graded A-module. Then:*

(1) $\mathrm{reg}(M) \geqq 0$.

(2) *Assume that M is R-flat, A is 1-regular, R-flat and every finitely generated projective R-module is free. If* $\mathrm{reg}(M)=0$ *then, M has a graded free resolution*

$$\cdots \longrightarrow F_n(-n) \longrightarrow \cdots \longrightarrow F_1(-1) \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \quad F_n = A^{b_n}.$$

If $\mathrm{reg}(A)=0$, *then the converse also holds.*

PROOF. (1): We may assume that R is a local ring with infinite residue field. Then there is a maximal M-regular sequence $a_1, \dots, a_r$ such that $a_i \in A_1$ for every i. Since $\mathrm{H}^0_P(M/(a_1, \dots, a_r)M) \neq 0$, there exists an integer $n \geqq 0$ such that $[\mathrm{H}^0_P(M/(a_1, \dots, a_r)M)]_n \neq 0$. Therefore we have $\mathrm{reg}(M) = \mathrm{reg}(M/(a_1, \dots, a_r)M) \geqq n \geqq 0$.

(2): Suppose that $\mathrm{reg}(M)=0$. Then since, by Th. 2, M is generated by M_0, we have an exact sequence

$$0 \longrightarrow N_0 \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \quad \text{where } F_0 = A^{b_0},\ b_0 = \mathrm{rank}_R(M_0).$$

Since F_0 is 1-regular and M is 0-regular, N_0 is 1-regular by Prop. 4, and we have $[N_0]_0 = 0$. Therefore $N_0(1)$ is 0-regular and positively graded. Thus, by the same argument as above, we have an exact sequence

$$0 \longrightarrow N_1 \longrightarrow F_1 \longrightarrow N_0(1) \longrightarrow 0, \quad F_1 = A^{b_1}.$$

Continuing this process, we get the desired free resolution of M. Conversely, if $\mathrm{reg}(A)=0$, R is a field and M has a free resolution

$$0 \longrightarrow F_n(-n) \longrightarrow \cdots \longrightarrow F_1(-1) \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \quad F_i = A^{b_i},$$

then using Prop. 4, we can easily show that $\mathrm{reg}(M)=0$. General case follows from Remark after Cor. 5. Q.E.D.

Let A be a homogeneous algebra over a field k and put $A=S/I$, where $S=k[X_1, \dots, X_v]$, $v=\mathrm{emb}(A)$ and I is a homogeneous ideal. We define the *initial degree* $\mathrm{i}(A)$ of A by $\mathrm{i}(A) = \min\{t;\ I_t \neq 0\}$ (cf. [19]).

COROLLARY 9. *Notation being as above, suppose that A is not regular. Then:*

(1) $\mathrm{reg}(A) \geqq \mathrm{i}(A) - 1$.

(2) $\mathrm{reg}(A) = \mathrm{i}(A) - 1$ *if and only if A has a graded minimal free resolution*

$$0 \longrightarrow S^{b_r} \xrightarrow{f_r} S^{b_{r-1}} \longrightarrow \cdots \longrightarrow S^{b_1} \xrightarrow{f_1} S \longrightarrow A \longrightarrow 0,$$

where $\deg f_1 = \mathrm{i}(A)$, $\deg f_i = 1$ $(i \geqq 2)$.

PROOF. Put $t=\mathrm{i}(A)$. Since we have $\mathrm{reg}(A) = \mathrm{reg}(I) - 1$ by Prop. 4, we have

$\mathrm{reg}(A)=\mathrm{reg}(I)-1\geqq \mathrm{i}(A)-1$, and $\mathrm{reg}(A)=\mathrm{i}(A)-1\Leftrightarrow \mathrm{reg}(I(t))=0\Leftrightarrow I(t)$ has a free resolution

$$0\longrightarrow S^{b_r}\xrightarrow{f_r} S^{b_{r-1}}\longrightarrow\cdots\longrightarrow S^{b_2}\xrightarrow{f_2} S^{b_1}\xrightarrow{g} I(t)\longrightarrow 0,$$

where $\deg g=0$, $\deg f_i=1$ ($i\geqq 2$) by Th. 8. This proves our assertions. Q.E.D.

COROLLARY 10. $\mathrm{reg}(A)=1$ *if and only if A has a minimal free resolution*

$$0\longrightarrow S^{b_r}\xrightarrow{f_r} S^{b_{r-1}}\longrightarrow\cdots\longrightarrow S^{b_1}\xrightarrow{f_1} S\longrightarrow A\longrightarrow 0,$$

where $\deg f_1=2$, $\deg f_i=1$ ($i\geqq 2$).

REMARK. (1) Th. 8, (2) (in case A is regular and R is a field) and Cor. 9, (2) were obtained by D. Eisenbud and S. Goto [2] independently, and on the other hand Cor. 9 is a generalization of a Schenzel's theorem for Cohen-Macaulay algebras (cf. [19]).

(2) Schenzel also showed that if A is a Gorenstein homogeneous algebra which is not a hypersurface, then $\mathrm{reg}(A)\geqq 2(\mathrm{i}(A)-1)$ and the equality holds if and only if A has a minimal free resolution

$$0\longrightarrow S^{b_r}\xrightarrow{f_r} S^{b_{r-1}}\longrightarrow\cdots\longrightarrow S^{b_1}\xrightarrow{f_1} S\longrightarrow A\longrightarrow 0,$$

where $\deg f_1=\deg f_r=\mathrm{i}(A)$, $\deg f_i=1$ ($1<i<r$). He called a Cohen-Macaulay (resp. Gorenstein) homogeneous algebra with $\mathrm{reg}(A)=\mathrm{i}(A)-1$ (resp. $\mathrm{reg}(A)=2(\mathrm{i}(A)-1)$) an extremal Cohen-Macaulay algebra (resp. an extremal Gorenstein algebra) and determined their Betti numbers completely.

REMARK. Let $(R, \mathfrak{m}, k)$ be a local ring such that its associated graded ring $A=\mathrm{G}_{\mathfrak{m}}(R)$ is 1-regular. Then, by Th. 8, (2), we have an exact sequence

$$\cdots\longrightarrow A^{b_n}(-n)\longrightarrow\cdots\longrightarrow A^{b_1}(-1)\longrightarrow A\longrightarrow k\longrightarrow 0.$$

From this, it is easy to show that we also have an exact sequence

$$\cdots\longrightarrow R^{b_n}\longrightarrow\cdots\longrightarrow R^{b_1}\longrightarrow R\longrightarrow k\longrightarrow 0.$$

Hence we have $1=\mathrm{F}(k, T)=(\sum_{n=0}^{\infty}(-1)^n b_n T^n)\mathrm{F}(A, T)$, where $\mathrm{F}(A, T)$ is the Hilbert series $\sum_{n=0}^{\infty}(\dim_k A_n)T^n$ of A, so that R satisfies the following formula (sometimes called the Fröberg formula, cf. [4]) for the Poincaré series $\mathrm{P}(R, T)=\sum_{n=0}^{\infty} b_n T^n=\sum_{n=0}^{\infty}\dim_k \mathrm{Tor}_n^R(k, k)T^n$ of R:

$$\mathrm{P}(R, T)\mathrm{F}(\mathrm{G}_{\mathfrak{m}}(R), -T)=1.$$

For Cohen-Macaulay modules, there are useful characterizations of regularity in terms of their Hilbert functions and Hilbert series (cf. Schenzel [19], Goto and

Watanabe [6]). We recall the results which we shall use later. For a graded module M over a graded ring A, we define $\mathrm{a}(M)$ by $\mathrm{a}(M)=\mathrm{reg}(M)-\dim(M)$. If A is a Cohen-Macaulay algebra over a field, then our $\mathrm{a}(A)$ coincides with the invariant $\mathrm{a}(A)$ defined in Goto and Watanabe [6].

Let A be a homogeneous algebra over an artinian local ring R and $M=\oplus_{n\in \boldsymbol{Z}} M_n$ a finitely generated graded A-module. We denote by $\mathrm{H}(M, n)$ and $\mathrm{h}(M, n)$ the Hilbert function and the Hilbert polynomial of M respectively. Therefore $\mathrm{H}(M, n)=\ell(M_n)$ and $\mathrm{h}(M, n)=\sum_{i=0}^{\infty}(-1)^i\ell(\mathrm{H}^i(X, \tilde{M}(n))$ for every $n\in \boldsymbol{Z}$, where $\ell(N)$ denotes the length of an R-module N and $X=\mathrm{Proj}(A)$. We also denote by $\mathrm{F}(M, T)$ the Hilbert series $\sum_{n\in\boldsymbol{Z}} \mathrm{H}(M, n)T^n$ of M.

PROPOSITION (cf. [19]). *Suppose that M is Cohen-Macaulay. Then*:

(1) *For an integer m, the following conditions are equivalent.*

(a) $\mathrm{a}(A)<m$.

(b) $\mathrm{H}(M, m)=\mathrm{h}(M, m)$.

(c) $\mathrm{H}(M, n)=\mathrm{h}(M, n)$ *for every* $n\geqq m$.

(2) *If* $\mathrm{F}(M, T)=f_M(T)/(1-T)^d$, *where* $d=\dim(M)$ *and* $f_M(T)\in \boldsymbol{Z}[T, T^{-1}]$, *then we have* $\mathrm{reg}(M)=\deg f_M(T)$.

Suppose that $\mathrm{hd}(M)<\infty$ and let

$$0\longrightarrow F_r\longrightarrow F_{r-1}\longrightarrow\cdots\longrightarrow F_1\longrightarrow F_0\longrightarrow M\longrightarrow 0$$

be a graded minimal free resolution of M, where $F_i=\oplus_{j=1}^{b_i} A(-a_{ij})$ and $a_{i1}\leqq\cdots\leqq a_{ib_i}$. Then $\mathrm{F}(M, T)=g_M(T)\mathrm{F}(A, T)$, where $g_M(T)=\sum_{i=0}^{r}(-1)^i(\sum_{j=1}^{b_i} T^{a_{ij}})\in \boldsymbol{Z}[T, T^{-1}]$. It is easy to see that $a_{i1}\leqq a_{i+1,1}$($a_{i1}<a_{i+1,1}$ if R is a field) for every i. Moreover, if we assume that M is perfect, i.e., $\underline{\mathrm{Ext}}_A^i(M, A)=0$ for every $i\neq r$, then

$$0\longrightarrow F_0^*\longrightarrow F_1^*\longrightarrow\cdots\longrightarrow F_r^*\longrightarrow \underline{\mathrm{Ext}}_A^r(M, A)\longrightarrow 0$$

is a minimal free resolution of $\underline{\mathrm{Ext}}_A^r(M, A)$, and hence we have $a_{ib_i}\leqq a_{i+1,b_{i+1}}$ ($a_{ib_i}<a_{i+1,b_{i+1}}$ if R is a field) for every i ($M^*=\underline{\mathrm{Hom}}_A(M, A)$). Let A be a Cohen-Macaulay homogeneous algebra over a field k and let

$$0\longrightarrow F_r\longrightarrow F_{r-1}\longrightarrow\cdots\longrightarrow F_1\longrightarrow F_0\longrightarrow A\longrightarrow 0$$

be a graded minimal free resolution of A as an $S=k[X_1,\dots, X_v]$-module, $v=\mathrm{emb}(A)$, where $F_i=\oplus_{j=1}^{b_i} S(-a_{ij})$, $a_{i1}\leqq\cdots\leqq a_{ib_i}$ Then we have $\deg g_A(T)=a_{rb_r}$.

PROPOSITION (cf. [19]). (1) *Notation being as above, suppose that A and M are Cohen-Macaulay. Then*

$$\mathrm{a}(M)=\deg g_M(T)+\mathrm{a}(A).$$

(2) *Let A be a Cohen-Macaulay homogeneous algebra over a field and*

$F(A, T)=g_A(T)/(1-T)^v$, $v=\mathrm{emb}(A)$ *its Hilbert series. Then*

$$\mathrm{reg}(A) = \deg g_A(T) - \mathrm{emb}(A) + \dim(A).$$

If A is a Gorenstein algebra over a field, then $\mathrm{a}(A)$ is characterized by the property $K_A \cong A(\mathrm{a}(A))$, where K_A denotes the canonical module of A (cf. [6], (3.1.4)). If A is a Gorenstein homogeneous algebra over a field k and let

$$0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

be a graded minimal free resolution of A as an $S=k[X_1,\ldots, X_v]$-module, $v=\mathrm{emb}(A)$, then $F_i \cong F_{r-i}{}^*(-\mathrm{reg}(A)-r)$ (as graded S-modules).

Using these facts and a similar method to that in [19], we can obtain some informations about the minimal free resolutions of Cohen-Macaulay homogeneous algebras with $\mathrm{reg}(A)=\mathrm{i}(A)$. We omit the proof. (Similar results can be obtained for Gorenstein homogeneous algebras with $\mathrm{reg}(A)=2\mathrm{i}(A)-1$(e.g., $\mathrm{reg}(A)=3$ and A is not a hypersurface), but we don't state them.)

Let A be a homogeneous algebra over a field k which is not regular, and put $\mathrm{emb}(A)=v$, $\dim(A)=d$, $v-d=r$, $\mathrm{e}(A)=e$, $\mathrm{reg}(A)=m$, $\mathrm{i}(A)=t$ and $S=k[X_1,\ldots, X_v]$. Let

$$0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

be a graded minimal free resolution of A (as an S-module), where $F_i=\bigoplus_{j=1}^{b_i} S(-a_{ij})$, $F_0=S$ and $a_{i1} \leqq \cdots \leqq a_{ib_i}$.

THEOREM 11. *Suppose that A is a Cohen-Macaulay algebra with* $\mathrm{reg}(A)=\mathrm{i}(A)$. *Then we have*

$$\mathrm{F}(A, T) = \frac{\sum_{i=0}^{t-1}\binom{r+i-1}{i}T^i + \left(e - \binom{r+t-1}{r}\right)T^t}{(1-T)^d},$$

$$\mathrm{r}(A) = e - \binom{r+t-1}{r},$$

and one of the following two cases occurs:

(1) *For some j $(2 \leqq j \leqq r)$,*

$$F_i = \begin{cases} S^{b_i}(-t-i+1) & \text{if } 1 \leqq i < j \\ S^{b_i}(-t-i) & \text{if } j \leqq i \leqq r, \end{cases}$$

i.e., A has a minimal free resolution

$$0 \longrightarrow S^{b_r} \xrightarrow{f_r} \cdots \longrightarrow S^{b_j} \xrightarrow{f_j} S^{b_{j-1}} \longrightarrow \cdots \xrightarrow{f_1} S \longrightarrow A \longrightarrow 0,$$

where $\deg f_1 = t$, $\deg f_j = 2$, $\deg f_i = 1$ $(i \neq 1, j)$. *In this case we have*

$$b_i = \begin{cases} \binom{t-1+r}{t-1+i}\binom{t+i-2}{i-1} - \binom{r}{i-1}\left(e - \binom{r+t-1}{r}\right) & \text{if } 1 \leqq i < j \\ -\binom{t-1+r}{t+i}\binom{t+i-1}{i} + \binom{r}{i}\left(e - \binom{r+t-1}{r}\right) & \text{if } j \leqq i \leqq r. \end{cases}$$

(2) *For some* j $(1 \leqq j \leqq r)$,

$$F_i = \begin{cases} S^{b_i}(-t-i+1) & \text{if } \ 1 \leqq i < j \\ S^{b_i}(-t-i) & \text{if } \ j < i \leqq r, \end{cases}$$

and $F_j = S^u(-t-j+1) \oplus S^w(-t-j)$. *In this case we have*

$$b_i = \begin{cases} \binom{t-1+r}{t-1+i}\binom{t+i-2}{i-1} - \binom{r}{i-1}\left(e - \binom{r+t-1}{r}\right) & \text{if } \ 1 \leqq i < j \\ -\binom{t-1+r}{t+i}\binom{t+i-1}{i} + \binom{r}{i}\left(e - \binom{r+t-1}{r}\right) & \text{if } \ j < i \leqq r, \end{cases}$$

$$u = \binom{t-1+r}{t-1+j}\binom{t+j-2}{j-1} - \binom{r}{j-1}\left(e - \binom{r+t-1}{r}\right),$$

$$w = -\binom{t-1+r}{t+j}\binom{t+j-1}{j} + \binom{r}{j}\left(e - \binom{r+t-1}{r}\right). \quad \left(\binom{a}{b} = 0 \text{ if } a < b.\right)$$

COROLLARY 12. *Let A be a Cohen-Macaulay homogeneous algebra with* $\operatorname{reg}(A) = 2$. *Then we have*

$$\mathrm{F}(A, T) = \frac{1 + rT + (e-r-1)T^2}{(1-T)^d},$$

and one of the following three cases occurs:

(1) *A is an extremal Cohen-Macaulay algebra which has a minimal free resolution*

$$0 \longrightarrow S^{b_r} \xrightarrow{f_r} S^{b_{r-1}} \longrightarrow \cdots \longrightarrow S^{b_1} \xrightarrow{f_1} S \longrightarrow A \longrightarrow 0,$$

where $\deg f_1 = 3$, $\deg f_i = 1$ $(i \geqq 2)$ *and*

$$b_i = \frac{i(i+1)}{2}\binom{r+2}{i+2}, \qquad 1 \leqq i \leqq r,$$

$$\mathrm{e}(A) = (r+2)(r+1)/2,$$

$$\mathrm{r}(A) = (r+1)r/2.$$

(2) *For some j $(2 \leqq j \leqq r)$, A has a minimal free resolution*

$$0 \longrightarrow S^{b_r} \xrightarrow{f_r} \cdots \longrightarrow S^{b_j} \xrightarrow{f_j} S^{b_{j-1}} \longrightarrow \cdots \xrightarrow{f_1} S \longrightarrow A \longrightarrow 0,$$

where $\deg f_1 = \deg f_j = 2$, $\deg f_i = 1$ $(i \neq 1, j)$, *and we have*

$$b_i = \begin{cases} i\binom{r+1}{i+1} - (e-r-1)\binom{r}{i-1} & \text{if } 1 \leqq i < j \\ -(i+1)\binom{r+1}{i+1} + (e-r-1)\binom{r}{i} & \text{if } j \leqq i \leqq r, \end{cases}$$

$$\mathrm{r}(A) = e-r-1.$$

(3) *For some* j $(1 \leqq j \leqq r)$, *we have*

$$F_i = \begin{cases} S^{b_i}(-i-1) & \text{if } 1 \leqq i < j \\ S^{b_i}(-i-2) & \text{if } j < i \leqq r, \end{cases}$$

and $\quad F_j = S^u(-j-1) \oplus S^w(-j-2)$, *where*

$$b_i = \begin{cases} i\binom{r+1}{i+1} - (e-r-1)\binom{r}{i-1} & \text{if } 1 \leqq i < j \\ -(i+1)\binom{r+1}{i+2} + (e-r-1)\binom{r}{i} & \text{if } j < i \leqq r, \end{cases}$$

$$u = j\binom{r+1}{j+1} - (e-r-1)\binom{r}{j-1},$$

$$w = -(j+1)\binom{r+1}{j+2} - (e-r-1)\binom{r}{j},$$

$$\mathrm{r}(A) = e-r-1.$$

Now we consider about upper bounds for regularity. Let A be a homogeneous algebra over a field k, and put $\mathrm{emb}(A)=v$, $\dim(A)=d$, $\mathrm{e}(A)=e$ and $\mathrm{reg}(A)=m$.

PROPOSITION 13. (1) *If A is Cohen-Macaulay, then*

$$\mathrm{reg}(A) \leqq \mathrm{e}(A) + \dim(A) - \mathrm{emb}(A).$$

Moreover, the equality holds if and only if

$$\mathrm{F}(A, T) = \frac{1 + (v-d)T + T^2 + \cdots + T^m}{(1-T)^d}, \; m \geqq 1.$$

(2) *If A is Gorenstein,* $\mathrm{reg}(A) \geqq 3$ *and is not a hypersurface, then*

$$\mathrm{reg}(A) \leqq \mathrm{e}(A)/2 + \dim(A) - \mathrm{emb}(A) + 2.$$

(3) *If A is Cohen-Macaulay* (*resp. Gorenstein*), *then* $\mathrm{reg}(A)=1$ *if and only if* $\mathrm{emb}(A)=\mathrm{e}(A)+\dim(A)-1$ *and* $\mathrm{e}(A)\geqq 2$ (*resp. A is a quadric hypersurface*).

(4) *If A is Gorenstein, then* $\mathrm{reg}(A)=2$ (*resp.* $\mathrm{reg}(A)=3$) *if and only if* $\mathrm{emb}(A)=\mathrm{e}(A)+\dim(A)-2$ (*resp.* $\mathrm{emb}(A)=\mathrm{e}(A)/2+\dim(A)-1$).

PROOF. First, we may assume that k is an infinite field. Then, dividing A by a maximal regular sequence whose elements are of degree one, we may assume that A is artinian. Thus $A=A_0\oplus A_1\oplus\cdots\oplus A_m$ and $\mathrm{e}(A)=\ell(A_0)+\ell(A_1)+\cdots+\ell(A_m)=1+v+\ell_2+\cdots+\ell_m$, where $\ell_i=\ell(A_i)\geqq 1$. Moreover, if A is Gorenstein, then $\ell_i=\ell_{m-i}$ by duality, and hence we have $e=2(1+v)+\ell_2+\cdots+\ell_{m-2}$ if $m\geqq 3$. (1): $e=1+v+\ell_2+\cdots+\ell_m\geqq 1+v+(m-1)=v-m$ and the equality holds if and only if $\ell_2=\cdots=\ell_m=1$. Since other assertions are similarly proved, we omit their proofs. For the assertion (2), we use the following fact (cf. Stanley [20], p. 61, Remark (c)): If A is a homogeneous algebra over a field and $\ell(A_n)=1$ for some $n\geqq 1$, then we have $\ell(A_i)\leqq 1$ for every $i\geqq n$. Q. E. D.

DEFINITION 14. If A is Cohen-Macaulay and $\mathrm{reg}(A)=\mathrm{e}(A)+\dim(A)-\mathrm{emb}(A)$, then we say that A is a *stretched Cohen-Macaulay algebra*.

REMARK. If $(R, \mathfrak{m})$ is a local ring such that its associated graded ring $\mathrm{G}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, then $\mathrm{G}_{\mathfrak{m}}(R)$ is a stretched Cohen-Macaulay algebra exactly when R is a stretched local ring in the sense of Sally [18].

EXAMPLE. (1) If A is Cohen-Macaulay and $\mathrm{emb}(A)=\mathrm{e}(A)+\dim(A)-1$, $\mathrm{e}(A)\geqq 2$ or $\mathrm{emb}(A)=\mathrm{e}(A)+\dim(A)-2$, then A is a stretched Cohen-Macaulay algebra. (Polynomial rings are not stretched Cohen-Macaulay algebras.)

(2) Every hypersurface is a stretched Cohen-Macaulay algebra. If A is a complete intersection, then A is a stretched Cohen-Macaulay algebra if and only if A is a hypersurface or a complete intersection of type (2, 2).

(3) $A=k[X_1,\dots, X_v]/((X_1,\dots, X_v)^{m+1}, X_iX_j\ (i\neq j), X_i^2\ (i\geqq 2))$ is an artinian stretched Cohen-Macaulay algebra with $\mathrm{reg}(A)=m$ and $\mathrm{emb}(A)=\mathrm{r}(A)=v$. In fact, $A=k\oplus(kx_1+\cdots+kx_v)\oplus kx_1^2\oplus kx_1^3\oplus\cdots\oplus kx_1^m$, where x_i is the image of X_i in A, and $\mathrm{Hom}_A(A/P, A)=kx_2+\cdots+kx_v+kx_1^m$. In particular, $A=k[X, Y, Z]/(X^3, Y^2, Z^2, XY, YZ, ZX)$ is an artinian stretched Cohen-Macaulay algebra with $\mathrm{reg}(A)=2$, $\mathrm{emb}(A)=\mathrm{r}(A)=3$ and $\mathrm{e}(A)=5$. Conversely, as was pointed out to us by S. Itoh, it is easy to show that every artinian stretched Cohen-Macaulay algebra is of the form $A=k[X, Y_1,\dots, Y_n]/((X, Y_1,\dots, Y_n)^{m+1}, XY_i, Y_iY_j-a_{ij}X^2 (1\leqq i, j\leqq n))$, $a_{ij}\in k$ or $A=k[X_1,\dots, X_n]/((X_1,\dots, X_n)^3, X_i^2, X_iX_j-a_{ij}X_1X_2\ (i<j,\ (i, j)\neq(1, 2)))$, $a_{ij}\in k$.

REMARK. It seems difficult to determine the precise upper bound for Gorenstein algebras (cf. Stanely [20], p. 70).

THEOREM 15. *Let A be a stretched Cohen-Macaulay algebra. If A is Gorenstein, then A is a hypersurface or* $\mathrm{reg}(A)=2$. *Moreover, if A is an integral domain, then the converse also holds.*

PROOF. If A is Gorenstein and $\mathrm{reg}(A)\geqq 3$, then since $\mathrm{F}(A, T)(1-T)^d=1+(v-d)T+\cdots+T^m$ with $v=\mathrm{emb}(A)$, $d=\dim(A)$ and $m=\mathrm{reg}(A)$, we have $v-d=1$, i.e., A is a hypersurface. Conversely, if A is a stretched Cohen-Macaulay domain with $\mathrm{reg}(A)=2$, then we have $\mathrm{F}(A, T)=(1+(v-d)T+T^2)/(1-T)^d$, so that $\mathrm{F}(A, T^{-1})=(-1)^d T^{d-2}\mathrm{F}(A, T)$. Hence by a theorem of Stanley (cf. [20], Th. 4.4) A is Gorenstein. Q.E.D.

REMARK. As the ring $A=k[X, Y, Z]/(X^3, Y^2, Z^2, XY, YZ, ZX)$ shows, if A is not an integral domain, then the converse of Th. 15 does not necessarily hold.

COROLLARY 16. (1) (Treger, Goto) *If A is a Cohen-Macaulay homogeneous integral domain with* $\mathrm{emb}(A)=\mathrm{e}(A)+\dim(A)-2$, *then A is Gorenstein.*
(2) *A is a Gorenstein homogeneous algebra with* $\mathrm{emb}(A)=\mathrm{e}(A)+\dim(A)-3$ *if and only if A is a hypersurface of degree four.*

PROOF. (1): In fact, A is a stretched Cohen-Macaulay algebra with $\mathrm{reg}(A)=2$. (2): In fact, A is a stretched Cohen-Macaulay algebra with $\mathrm{reg}(A)=3$. Q.E.D.

PROPOSITION 17. *Let A be a normal Cohen-Macaulay homogeneous algebra over an algebraically closed field. Then we have*

$$\mathrm{reg}(A) \leqq \min\{k\in \boldsymbol{Z};\ k \geqq (\mathrm{e}(A)-1)/(\mathrm{emb}(A)-\dim(A))\}.$$

PROOF. By Bertini's theorem (cf. Flenner [3]), there is an A-regular sequence $a_1,\ldots, a_{d-2}\in A_1$, $d=\dim(A)$ such that if we put $B=A/(a_1,\ldots, a_{d-2})A$, then $C=\mathrm{Proj}(B)$ is a smooth curve in $\boldsymbol{P}^r$, $r=\mathrm{emb}(A)-\dim(A)+1$, which is not contained in a hyperplane. Then by a theorem of Castelnuovo (cf. Szpiro [23], p. 52, Th. 1), we have $[\mathrm{H}^2_P(B)]_n=\mathrm{H}^1(C, \mathcal{O}_C(n))=0$ for every $n\geqq(e-1)/(r-1)-1$, $e=\mathrm{e}(B)$, and this implies that

$$\begin{aligned}\mathrm{reg}(A) = \mathrm{reg}(B) &\leqq \min\{k;\ k \geqq (\mathrm{e}(B)-1)/(\mathrm{emb}(B)-2)\}\\ &= \min\{k;\ k \geqq (\mathrm{e}(A)-1)/(\mathrm{emb}(A)-\dim(A))\}.\end{aligned}$$

Q.E.D.

The following two theorems are generalizations of some theorems which are proved for Cohen-Macaulay modules and algebras to Buchsbaum modules and algebras (for Buchsbaum modules, see Stückrad and Vogel [21], [22]).

Let A be a homogeneous algebra over a field k and $M=\bigoplus_{n\in \boldsymbol{Z}}M_n$ a finitely generated graded A-module. Then we say that M is a Buchsbaum module if M_P is a Buchsbaum A_P-module.

PROPOSITION 18. *Suppose that M is Buchsbaum and let $a_1,\dots, a_d \in A_1$ be a system of parameters for M (if k is an infinite field, such a system of parameters always exists). Put $\mathfrak{q}=(a_1,\dots, a_d)$. Then*

(1) $\quad \operatorname{reg}(M)=\deg \mathrm{F}(M/\mathfrak{q}M, T)$.

(2) $$\mathrm{F}(M, T)=\frac{\mathrm{F}(M/\mathfrak{q}M, T)-\sum_{i=0}^{d-1}\sum_{j=0}^{i}\binom{i}{j}(1-T)^{d-i-1}T^{j+1}\mathrm{F}(\mathrm{H}_P^j(M), T)}{(1-T)^d}.$$

PROOF. Put $a=a_1$. Then M/aM is a Buchsbaum module with $\dim(M/aM) =d-1$ and from the exact sequence

$$0 \longrightarrow \mathrm{H}_P^0(M)(-1) \longrightarrow M(-1) \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0,$$

we get

(a) $\mathrm{F}(M/aM, T) = (1-T)\mathrm{F}(M, T) + T\mathrm{F}(\mathrm{H}_P^0(M), T)$,

(b) $0 \longrightarrow \mathrm{H}_P^i(M) \longrightarrow \mathrm{H}_P^i(M/aM) \longrightarrow \mathrm{H}_P^{i+1}(M)(-1) \longrightarrow 0 \ (0 \leqq i \leqq d-2)$

and

$$0 \longrightarrow \mathrm{H}_P^{d-1}(M) \longrightarrow \mathrm{H}_P^{d-1}(M/aM) \longrightarrow \mathrm{H}_P^d(M)(-1) \xrightarrow{a} \mathrm{H}_P^d(M) \longrightarrow 0$$

are exact (cf. Goto and Shimoda [7], Lemma 2.6).

Using (b), it is easy to see that $\operatorname{reg}(M)=\operatorname{reg}(M/aM)$. This implies (1). On the other hand, if we put $M_i=M/(a_1,\dots, a_i)M$, then by (a), we have

$$(1-T)\mathrm{F}(M_i, T) = \mathrm{F}(M_{i+1}, T) - T\mathrm{F}(\mathrm{H}_P^0(M_i), T).$$

Therefore we get

$$(1-T)^d\mathrm{F}(M, T) = \mathrm{F}(M_d, T) - \textstyle\sum_{i=0}^{d-1}(1-T)^{d-i-1}T\mathrm{F}(\mathrm{H}_P^0(M_i), T).$$

Next, by induction on i, we have

$$\mathrm{F}(\mathrm{H}_P^k(M_i), T) = \sum_{j=0}^{i}\binom{i}{j}T^j\mathrm{F}(\mathrm{H}_P^{k+j}(M), T)$$

if $k \leqq d-i-1$ and $i \leqq d-1$; in particular,

$$\mathrm{F}(\mathrm{H}_P^0(M_i), T) = \sum_{j=0}^{i}\binom{i}{j}T^j\mathrm{F}(\mathrm{H}_P^j(M), T).$$

This completes the proof. Q.E.D.

THEOREM 19. *Let A be a Buchsbaum homogeneous algebra over a field k, Then:*

(1) $\mathrm{reg}(A) \leqq \mathrm{e}(A) + \dim(A) - \mathrm{emb}(A) + \mathrm{I}(A)$, *where* $\mathrm{I}(A)$ *is the invariant of a Buchsbaum algebra* A *(cf.* [22]), *and the equality holds if and only if*

$$\mathrm{F}(A, T) = \frac{1 + (v-d)T + T^2 + \cdots + T^m + G(T)}{(1-T)^d},$$

where $v = \mathrm{emb}(A)$, $d = \dim(A)$, $m = \mathrm{reg}(A)$ *and* $G(T) = \sum_{i=0}^{d-1} \sum_{j=0}^{i} \binom{i}{j} (1-T)^{d-i-1} T^{j+1} \mathrm{F}(\mathrm{H}_P^j(A), T)$. *(We call such a Buchsbaum algebra a stretched Buchsbaum algebra.)*

(2) $\mathrm{reg}(A) = 1$ *if and only if* $\mathrm{emb}(A) = \mathrm{e}(A) + \dim(A) + \mathrm{I}(A) - 1$ *and* A *is not regular.*

(3) *If* $\mathrm{emb}(A) = \mathrm{e}(A) + \dim(A) + \mathrm{I}(A) - 2$, *then* A *is a stretched Buchsbaum algebra with* $\mathrm{reg}(A) = 2$.

PROOF. We may assume that k is an infinite field. Then there is a minimal reduction $a_1, \dots, a_d$ of P such that $a_i \in A_1$ for every i (cf. Northcott and Rees [13]). Put $\mathfrak{q} = (a_1, \dots, a_d)$ and $A/\mathfrak{q} = B = B_0 \oplus \cdots \oplus B_m$, where $m = \mathrm{reg}(A) = \mathrm{reg}(B)$, Then we get

$$\begin{aligned} \mathrm{e}(A) + \mathrm{I}(A) &= \mathrm{e}(\mathfrak{q}) + \mathrm{I}(A) = \ell(A/\mathfrak{q}) \\ &= 1 + (v-d) + \ell(B_2) + \cdots + \ell(B_m) \\ &\geqq 1 + (v-d) + (m-1) = m + v - d, \end{aligned}$$

and the equality holds if and only if $\ell(B_i) = 1$ for every $i \geqq 2$. This proves (1). The proofs of (2) and (3) are similar, so that we omit them. Q.E.D.

Next, we state a result about the degree of defining equations of homogeneous algebras.

Let A be a homogeneous algebra over a field k which is not regular and let m be a positive integer. We say that A is defined by forms of degree at most m if $A = S/I$, $S = k[X_1, \dots, X_v]$, $v = \mathrm{emb}(A)$ and I is a homogeneous ideal which is generated by homogeneous elements of degree at most m.

PROPOSITION 20. (1) *Every homogeneous algebra* A *is defined by forms of degree at most* $\mathrm{reg}(A) + 1$.

(2) *If* A *is Gorenstein and is not a hypersurface, then* A *is defined by forms of degree at most* $\mathrm{reg}(A)$.

PROOF. (1): Since $\mathrm{reg}(I) = \mathrm{reg}(A) + 1$ by Prop. 4, we have $S_i I_j = I_{i+j}$ for every $i \geqq 0$ and $j \geqq \mathrm{reg}(A) + 1$ ($\geqq \mathrm{i}(A)$) by Th. 2. (2): Let $0 \to F_r \to \cdots \to F_1 \to F_0 \to A \to 0$ be a graded minimal free resolution of A, where $F_i = \oplus_{j=1}^{b_i} S(-a_{ij})$, $a_{i1} \leqq \cdots \leqq a_{ib_i}$. Then A is defined by forms of degree at most $a_{1b_1} = -a_{r-1,1} +$

$\operatorname{reg}(A)+r \leqq -r+\operatorname{reg}(A)+r=\operatorname{reg}(A)$. Q.E.D.

Finally we give some examples.

(1) (Polarized varieties). Let X be a projective variety over an algebraically closed field k, and let $\mathscr{L}=\mathcal{O}_X(D)$ be an ample invertible sheaf on X. Put $A=A(X, \mathscr{L})=A(X, D)=\oplus_{n\geqq 0} \mathrm{H}^0(X, \mathcal{O}_X(nD))$. Then A is a noetherian graded k-algebra and $X \cong \operatorname{Proj}(A)$, $\mathscr{L}=\widetilde{A(1)}$. Moreover, we have $\mathrm{H}^i_P(A)=0$ for $i=0, 1$ and $\mathrm{H}^{i+1}_P(A) \cong \oplus_{n\in \boldsymbol{Z}} \mathrm{H}^i(X, \mathcal{O}_X(nD))$ for every $i \geqq 1$. Hence if $\dim(X) \geqq 1$, then A is Cohen-Macaulay (resp. Gorenstein) if and only if $\mathrm{H}^i(X, \mathcal{O}_X(nD))=0$ $(0<i<\dim(X), n\in \boldsymbol{Z})$ (resp. A is Cohen-Macaulay and $\omega_X \cong \mathcal{O}_X(n)$ for some $n \in \boldsymbol{Z}$, where ω_X is the dualizing sheaf of X). For these facts, see Goto and Watanabe [6].

Example 1 (Algebraic curves). If X is a projective curve with arithmetic genus $\mathrm{p_a}(X)$ and D is an ample divisor (i.e., $\deg D>0$), then $A(X, D)$ is a two-dimensional Cohen-Macaulay graded domain. Since $\mathrm{H}^1(\mathcal{O}_X(nD))=0$ if $\deg nD > 2\mathrm{p_a}(X)-2$, we have

$$\operatorname{reg} A(X, D) \leqq (2\mathrm{p_a}(X)-2)/\deg D + 2.$$

In particular, if $\deg D > 2\mathrm{p_a}(X)-2$(resp. $\deg D=2\mathrm{p_a}(X)-2$), then $\operatorname{reg} A(X, D) \leqq 2$ (resp. $\operatorname{reg} A(X, D) \leqq 3$, cf. Example 2). If $\deg D \geqq 2\mathrm{p_a}(X)+1$, then $A(X, D)$ is homogeneous (cf. Fujita [5], Cor. 1.11) and X is defined by forms of degree at most 3 if we embed X by D by Prop. 20 (cf. Homma [10]).

If $X=\boldsymbol{P}^1$ and $\deg D=n>0$, then $A(X, D) \cong k[X, Y]^{(n)}$ and $\operatorname{reg} A(X, D)=1$ for every $n \geqq 2$. Conversely if $\operatorname{reg} A(X, D) \leqq 1$ for some ample divisor D, then $\mathrm{H}^1(X, \mathcal{O}_X)=0$, i.e., $\mathrm{p_a}(X)=0$, and hence $X \cong \boldsymbol{P}^1$.

If X is an elliptic curve, then $A(X, D)$ is a normal Gorenstein algebra with $\operatorname{reg} A(X, D)=2$ and the converse is also true.

Example 2 (Canonical curves). Let X be a smooth projective curve of genus g which is not hyperelliptic, i.e., its canonical divisor K is very ample. Then the canonical ring $A=A(X, K)$ of X is a two-dimensional normal Gorenstein homogeneous algebra with $\operatorname{emb}(A)=g$, $\mathrm{e}(A)=2g-2$, $\operatorname{reg}(A)=3$, and $C=\operatorname{Proj}(A)$ is the canonical curve of X. In fact, A is homogeneous (cf. Saint-Donat [15]) and since $\omega_C=\mathcal{O}_X(1)$, A is a Gorenstein algebra with $\mathrm{a}(A)=1$. Hence canonical curves with $g \geqq 4$ are defined by forms of degree at most 3 by Prop. 20 (cf. [15]). Conversely, if A is a two-dimensional normal Gorenstein homogeneous algebra with $\operatorname{reg}(A)=3$, then $C=\operatorname{Proj}(A)$ is a canonical curve, since $\mathrm{a}(A)=3-2=1$ and $\omega_C \cong \mathcal{O}_C(1)$.

Example 3 (K–3 surfaces). Assume that $\operatorname{char} k=0$. Let X be a K–3 surface, i.e., a smooth projective surface such that $\mathrm{H}^1(X, \mathcal{O}_X)=0$ and $\omega_X \cong \mathcal{O}_X$. If D is a normally generated ample divisor (i.e., $A(X, D)$ is homogeneous) on X, then $A=A(X, D)$ is a three-dimensional Gorenstein homogeneous algebra with $\mathrm{reg}(A)=3$. In fact, since $\mathrm{H}^1(X, \mathcal{O}_X(nD))=\mathrm{H}^1(X, \mathcal{O}_X(-nD))=0$ for every $n>0$ by Kodaira's vanishing theorem and Serre duality, we have $\mathrm{H}^1(X, \mathcal{O}_X(nD))=0$ for every $n \in \boldsymbol{Z}$ and this implies that A is Cohen-Macaulay. Since $\omega_X \cong \mathcal{O}_X$, we have $\mathrm{a}(A)=0$ and $\mathrm{reg}(A)=3$. Conversely, if A is a three-dimensional Gorenstein homogeneous algebra with $\mathrm{reg}(A)=3$ which has an isolated singularity at its vertex, then $X=\mathrm{Proj}(A)$ is a K–3 surface.

EXAMPLE 4 (Fano varieties). Assume that $\mathrm{char}\, k=0$. Let X be a Fano variety, i.e., a smooth projective variety whose anticanonical bundle $\omega_X^{-1}=\mathcal{O}_X(-K)$ is ample. Then the anticanonical ring $A=A(X, -K)$ of X is a Gorenstein algebra with $\mathrm{reg}(A)=\dim(A)-1$. In fact, by Kodaira's vanishing theorem and Serre duality, we get $\mathrm{H}^i(X, \mathcal{O}_X(n))=\mathrm{H}^i(X, \mathcal{O}_X(-nK))=0$ if either $i>0$, $n \geqq 0$ or $i<\dim(X)$, $n<0$. Hence A is Cohen-Macaulay and $\mathcal{O}_X(1)=\mathcal{O}_X(-K)$, and this implies that A is a Gorenstein algebra with $\mathrm{a}(A)=-1$. In particular, if X is a Del Pezzo surface (resp. a Fano 3-fold), then $A=A(X, -K)$ is a Gorenstein algebra with $\mathrm{reg}(A)=2$ (resp. $\mathrm{reg}(A)=3$).

(2) (Algebras with $\mathrm{reg}(A)=1$).

EXAMPLE 1 (Seminormal rings). If A is a reduced one-dimensional homogeneous algebra over a field k, then $\mathrm{reg}(A) \leqq 1$ if and only if A is seminormal, or equivalently A is isomorphic to $k[X_1,\dots, X_n]/(X_iX_j;\ i \neq j)$ (cf. Davis [1]). We generalize this example (cf. Leahy and Vitulli [11]). Let $\{I_s;\ 1 \leqq s \leqq r\}$ be a partition of $\{1,\dots, n\}$ and put

$$A = k[X_1,\dots, X_n, Y_1,\dots, Y_m]/(X_iX_j;\ i \in I_s, j \in I_t\ (s \neq t)).$$

Then we have the exact sequence

$$0 \longrightarrow A \longrightarrow \textstyle\prod_{s=1}^r A/\mathfrak{p}_s \xrightarrow{f} \prod^{r-1} A/\mathfrak{p} \longrightarrow 0,$$

where $A/\mathfrak{p}_s=k[X_i;\ i \in I_s, Y_1,\dots, Y_m]$, $A/\mathfrak{p}=k[Y_1,\dots, Y_m]$ and $f(\bar{a}_1,\dots, \bar{a}_r)=(\bar{a}_2-\bar{a}_1,\dots, \bar{a}_r-\bar{a}_1)$. Therefore we have $\mathrm{reg}(A) \leqq 1$ and it is easy to show that $\mathrm{F}(A, T)=\sum_{s=1}^r (1-T)^{-m_s}-(r-1)(1-T)^{-m}$, $\mathrm{emb}(A)=\sum_{s=1}^r m_s-(r-1)m=:v$, $\dim(A)=\max\{m_s;\ 1 \leqq s \leqq r\}=:d$, $\mathrm{depth}_P(A)=m+1$, $\mathrm{e}(A)=\mathrm{Card}\{s;\ m_s=d\}$, where $m_s=\mathrm{Card}(I_s)+m$. A has a minimal free resolution

$$0 \longrightarrow F_k \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0,$$

where $k=v-m-1$, $F_i=S^{b_i}(-i-1)$, $b_i=(r-1)\binom{v-m}{i+1}-\sum_{s=1}^r \binom{v-m_s}{i+1}$ for every $i \geqq 1$. A is Cohen-Macaulay (resp. Gorenstein) if and only if $A=k[X_1,\dots,$

X_n, $Y_1,\ldots, Y_m]/(X_iX_j;\ i\neq j)$ (in this case $\mathrm{r}(A)=n-1$) (resp. $A=k[X_1, X_2, Y_1,\ldots, Y_m]/(X_1X_2)$). A is a Buchsbaum algebra which is not Cohen-Macaulay if and only if $m=0$ and $\mathrm{Card}(I_s)=d\geqq 2$ for every s, and in this case we have $\mathrm{emb}(A)=dr$, $\mathrm{e}(A)=r$, $\mathrm{I}(A)=(d-1)(r-1)$, $\mathrm{F}(A, T)=r(1-T)^{-d}-(r-1)$, $b_i=(r-1)\binom{v}{i+1}-r\binom{v-d}{i+1}$ for every $i\geqq 1$.

EXAMPLE 2 (Varieties of minimal degree). Let X be a non-degenerate closed subvariety of $\boldsymbol{P}^n$ (i.e., X is not contained in a hyperplane) and let A be its homogeneous coordinate ring. In this case, A is a Cohen-Macaulay algebra with $\mathrm{reg}(A)\leqq 1$ if and only if $\deg X=\mathrm{codim} X+1$, and the complete classification of such varieties (so-called varieties of minimal degree) is classically known (cf. Saint-Donat [16]). Thus Cohen-Macaulay homogeneous domains over an algebraically closed field with $\mathrm{reg}(A)=1$ are completely calssified.

For other examples of Cohen-Macaulay algebras with $\mathrm{reg}(A)=1$ (and Gorenstein algebras with $\mathrm{reg}(A)=2$), see Sally [17].

(3) (Veronesean subrings). Let A be a Cohen-Macaulay homogeneous algebra over a field k with $\dim(A)=d\geqq 1$. Then for an integer $s\geqq 1$, the s-uple Veronesean subring $A^{(s)}=\oplus_{n\geqq 0} A_{ns}$ of A is also Cohen-Macaulay and we have $\dim(A^{(s)})=d$, $\mathrm{e}(A^{(s)})=\mathrm{e}(A)s^{d-1}$, $\mathrm{emb}(A^{(s)})=\mathrm{H}(A, s)$ and

$$\mathrm{a}(A^{(s)}) = [\mathrm{a}(A)/s], \qquad \text{where } [\quad] \text{ is the Gauss symbol.}$$

EXAMPLE 1. Put $B=k[X_1,\ldots, X_d]^{(s)}$. Then we have $\mathrm{e}(B)=s^{d-1}$, $\mathrm{emb}(B)=\binom{d+s-1}{s}$ and $\mathrm{reg}(B)=[d(s-1)/s]$. In particular, $\mathrm{reg}(B)=1$ if and only if either $d=2$, $s\geqq 2$ or $(d, s)=(3, 2)$.

EXAMPLE 2. Suppose that $\dim(A)=\mathrm{a}(A)=d\geqq 2$, and put $B=A^{(d)}$. Then $\mathrm{reg}(B)=d+1$ and B is Gorenstein if and only if A is so (cf. Goto and Watanabe [6], Cor. (3.1.5), Th. (3.2.1)). For example, if $A=k[X_0,\ldots, X_d]/(f)$, $\deg f=2d+1$, then $B=A^{(d)}$ is a Gorenstein homogeneous algebra with $\dim(B)=d$, $\mathrm{emb}(B)=\binom{2d}{d}$, $\mathrm{e}(B)=(2d+1)d^{d-1}$ and $\mathrm{reg}(B)=d+1$.

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