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On the limit orders of operator ideals*)

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Introduction

Four types of limit orders of operator ideals $\mathfrak{A}$ were introduced in 1971–2 by A. Pietsch with respective purposes, and these limit orders have been playing an important role in the theory of operator ideals ([20], [21], [22], [23], [10], [11], [12]). They are the S-, D-, I- and L-limit orders, $\lambda_S(\mathfrak{A})$ ([21]), $\lambda_D(\mathfrak{A})$ ([22]), $\lambda_I(\mathfrak{A})$ and $\lambda_L(\mathfrak{A})$ ([20]), which are defined by using Sobolev embeddings, (certain) diagonal operators between ℓ_u-spaces, identity and Littlewood operators between ℓ_u^n-spaces, respectively. (The last limit order is originally denoted by $\lambda_A(\mathfrak{A})$. We shall, however, adopt the above notation $\lambda_L(\mathfrak{A})$ and call it the L-limit order.) H. König [11] showed in 1974 the following remarkable relations among them: For a complete quasi-normed operator ideal $[\mathfrak{A}, \mathrm{A}]$,

$$(1) \qquad \lambda_I(\mathfrak{A}, u, v) = \lambda_D(\mathfrak{A}, u, v)$$

and

$$(2) \qquad \lambda_S(\mathfrak{A}, u, v; N) = N\left(\lambda_D(\mathfrak{A}, u, v) + \frac{1}{u} - \frac{1}{v}\right)$$

for $1 \leq u, v \leq \infty$. Thus, in Pietsch ([23], 14.4.1) the D-limit order is referred to simply as the limit order and denoted by $\lambda(\mathfrak{A})$. In this paper, we are concerned with the limit and L-limit orders. They are defined for $1 \leq u, v \leq \infty$ respectively by

$$(3) \qquad \lambda(\mathfrak{A}, u, v) := \inf \{\lambda > 0;\ D_\lambda \in \mathfrak{A}(\ell_u, \ell_v)\}$$

and

$$(4) \quad \lambda_L(\mathfrak{A}, u, v) := \inf \{\lambda > 0;\ ^\exists c = c(u,v,\lambda)\ s.t.\ \mathrm{A}(A_{2^n}\colon \ell_u^{2^n} \to \ell_v^{2^n}) \leq c(2^n)^\lambda\ (n = 0,1,2,\ldots,)\},$$

where $D_\lambda(\{\xi_n\}) = \{n^{-\lambda}\xi_n\}$ and A_{2^n} are the Littlewood matrices ([15]), that is,

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$$A_{2^0} = [1],\ A_{2^{n+1}} = \begin{bmatrix} A_{2^n} & A_{2^n} \\ A_{2^n} & -A_{2^n} \end{bmatrix} \qquad (n = 0, 1, 2, \ldots).$$

The limit order $\lambda(\mathfrak{A}, u, v)$ provides two kinds of criteria such that a diagonal operator from ℓ_u into ℓ_v belongs to $\mathfrak{A}$:

(a) *If* $\lambda > \lambda(\mathfrak{A}, u, v)$ (*resp.* $\lambda < \lambda(\mathfrak{A}, u, v)$), *then* $D_\lambda \in \mathfrak{A}(\ell_u, \ell_v)$ (*resp.* $D_\lambda \notin \mathfrak{A}(\ell_u, \ell_v)$).

(b) *Let* $1/r > \lambda(\mathfrak{A}, u, v)$. *Then, for every* $\sigma = \{\sigma_n\} \in \ell_r$ *the diagonal operator* $D_\sigma\colon \ell_u \to \ell_v$, $D_\sigma(\{\xi_n\}) = \{\sigma_n \xi_n\}$, *belongs to* $\mathfrak{A}$. *More precisely,*

$$\lambda(\mathfrak{A}, u, v) = \inf\{1/r \geq 0;\ \sigma \in \ell_r \Longrightarrow D_\sigma \in \mathfrak{A}(\ell_u, \ell_v)\} \tag{5}$$

([23], Proposition 14.4.2).

The first objective of this paper is to obtain, by generalizing (1), a nearly necessary and sufficient condition in order that a diagonal operator between ℓ_u-spaces belongs to a given quasi-normed operator ideal. The second objective is to investigate some properties of the $\boldsymbol{\alpha}$-limit order of $\mathfrak{A}$ which we shall deane by

$$\lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) := \inf\{\lambda > 0;\ \ D_{\{\alpha_n^{-\lambda}\}} \in \mathfrak{A}(\ell_u, \ell_v)\} \qquad (1 \leq u, v \leq \infty),$$

where $\boldsymbol{\alpha} = \{\alpha_n\}$ is an arbitrary fixed sequence of positive numbers which is strictly increasing and divergent to ∞, and $D_{\{\alpha_n^{-\lambda}\}}(\{\xi_n\}) = \{\alpha_n^{-\lambda}\xi_n\}$. The introduction of the $\boldsymbol{\alpha}$-limit order is motivated by the fact that there are some examples for which the above criteria given by $\lambda(\mathfrak{A})$ are of little avail. The last objective is to investigate the L-limit order, which has not yet been treated in detail.

Section 1 is devoted to some preliminary definitions and results, which are quoted for the most part from the monograph [23]. In Section 2 we study a couple of sequence spaces $\ell_{r,\infty}(\boldsymbol{\alpha})$ and $\ell^0_{r,\infty}(\boldsymbol{\alpha})$ to some extent for later use. The former is a generalization of the Lorentz sequence space $\ell_{r,\infty}$ and particularly useful in Sections 4 and 5. In Section 3 we generalize (1) to obtain the nearly necessary and sufficient condition stated above (Theorem 1 and its Corollary). In Section 4 we discuss the $\boldsymbol{\alpha}$-limit order, where the identities generalizing respectively (1) and (5) are shown (Theorems 3 and 2). In Section 5 the $\boldsymbol{\alpha}$-defects of normed operator ideals are considered, whose notion is based on König [12]. Under a certain assumption on $\boldsymbol{\alpha} = \{\alpha_n\}$, it is obtained that the condition $\lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) + \lambda_{\boldsymbol{\alpha}}(\mathfrak{A}^*, v, u) = 1$ implies

$$\lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) = \lim_{n\to\infty} \frac{\log \mathbf{A}\ (I_n : \ell_u^n \longrightarrow \ell_v^n)}{\log \alpha_n}$$

(Corollary to Theorem 5). In Section 6, we obtain several criteria given by the L-limit order $\lambda_L(\mathfrak{A})$ (and $\lambda(\mathfrak{A})$ as well) such that a certain type of block diagonal matrix operator between ℓ_u-spaces belongs to $\mathfrak{A}$; in particular, we obtain results analogous to (1), (a), and (5) (Theorem 6, its Corollary, and Theorem 7), which remain valid if the underlying ℓ_u-spaces are replaced by the Lorentz sequence spaces $\ell_{u,s}$ (Theorem 6′, its Corollary, and Theorem 7′). In Theorem 8 we introduce another type of limit order $\mu(\mathfrak{A})$ and compare it with $\lambda_L(\mathfrak{A})$ and $\lambda(\mathfrak{A})$. In the rest of this section, we give a representation of $\lambda_L(\mathfrak{L}, u, v)$ by means of $\ell_u^{2^n}(\mathscr{L}_p)$-spaces ($\mathfrak{L}$ is the ideal of all bounded linear operators between arbitrary Banach spaces), which is closely related with the Clarkson inequalities (Corollary to Theorem 9). In the final section we deal with a relation between $\lambda_L(\mathfrak{A})$ and $\lambda(\mathfrak{A})$ (cf. (1) and (2)): It is shown that

$$\begin{aligned}&\lambda(\mathfrak{A}, u, v) + \max\{\min(1/u, 1/u'), \min(1/v, 1/v')\}\\ &\quad\leq \lambda_L(\mathfrak{A}, u, v)\\ &\quad\leq \lambda(\mathfrak{A}, u, v) + \min\{\max(1/u, 1/u'), \max(1/v, 1/v')\}\end{aligned}$$

for $1 \leq u, v \leq \infty$, $1/u + 1/u' = 1/v + 1/v' = 1$, which is best possible for most values of u and v (Theorem 10 and Remark 4).

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§1. Preliminaries

The space of (bounded linear) operators from a Banach space E into another Banach space F is denoted by $\mathfrak{L}(E, F)$, while the class of all operators between arbitrary Banach spaces is denoted by $\mathfrak{L}$. A subclass $\mathfrak{A}$ of $\mathfrak{L}$ is called an *operator ideal* (cf. [23], 1.1.1; [22], 1.1.1) if the components

$$\mathfrak{A}(E, F) := \mathfrak{A} \cap \mathfrak{L}(E, F)$$

satisfy the following conditions:

(OI_1) If $a \in E'$, the dual space of E, and $y \in F$, then $a \otimes y \in \mathfrak{A}(E, F)$.
(OI_2) If $S_1, S_2 \in \mathfrak{A}(E, F)$, then $S_1 + S_2 \in \mathfrak{A}(E, F)$.
(OI_3) If $T \in \mathfrak{L}(E_0, E)$, $S \in \mathfrak{A}(E, F)$, and $R \in \mathfrak{L}(F, F_0)$, then $RST \in \mathfrak{A}(E_0, F_0)$.

Every component of an operator ideal is a linear space ([23], Proposition 1.1.2).

A mapping $\mathbf{A}$ from an operator ideal $\mathfrak{A}$ into the set of non-negative real numbers is called a (ideal) *quasi-norm* (cf. [23], 6.1.1; [22], 8.1.1) if it has the following properties:

(QN_1) $\mathbf{A}(a \otimes y) = \|a\| \|y\|$ for $a \in E'$ and $y \in F$.
(QN_2) There exists a constant $c_{\mathbf{A}} \geq 1$ such that

$$\mathbf{A}(S_1 + S_2) \le c_{\mathbf{A}}[\mathbf{A}(S_1) + \mathbf{A}(S_2)] \qquad \text{for} \quad S_1, S_2 \in \mathfrak{A}(E, F).$$

(QN_3) $\mathbf{A}(RST) \le \|R\|\mathbf{A}(S)\|T\|$ for $T \in \mathfrak{L}(E_0, E)$, $S \in \mathfrak{A}(E, F)$, and $R \in \mathfrak{L}(F, F_0)$.

In particular, $\mathbf{A}$ is called a *norm* if $c_{\mathbf{A}} = 1$ in (QN_2). A quasi-norm $\mathbf{A}$ is called a *p-norm* ($0 < p \le 1$) (cf. [23], 6.2.1) if the following p-triangle inequality holds:

$$\mathbf{A}(S_1 + S_2)^p \le \mathbf{A}(S_1)^p + \mathbf{A}(S_2)^p \qquad \text{for} \quad S_1, S_2 \in \mathfrak{A}(E, F).$$

A *quasi-normed operator ideal* $[\mathfrak{A}, \mathbf{A}]$ is an operator ideal $\mathfrak{A}$ with a quasi-norm $\mathbf{A}$. Each of its components is a usual quasi-normed space (cf. [23], 6.1.2). We always assume the completeness for quasi-normed operator ideals, that is, every component of theirs is complete (cf. [23], 6.1.3).

LEMMA A ([23]), Theorem 6.2.5). *Every quasi-normed operator ideal has an equivalent p-norm.*

For a normed operator ideal $[\mathfrak{A}, \mathbf{A}]$ its *adjoint operator ideal* $\mathfrak{A}^*$ is defined as follows (cf. [23], 9.1.1): An operator $S \in \mathfrak{L}(E, F)$ belongs to $\mathfrak{A}^*$ if and only if there exists a constant $\rho \ge 0$ such that

$$|\operatorname{trace}(SXL_0B)| \le \rho\|X\|\mathbf{A}(L_0)\|B\|$$

for all $B \in \mathfrak{L}(F, F_0)$, $L_0 \in \mathfrak{A}(F_0, E_0)$, and $X \in \mathfrak{L}(E_0, E)$, B and X being of finite rank, where E_0 and F_0 are arbitrary Banach spaces. The infimum of all such ρ is denoted by $\mathbf{A}^*(S)$. Then, $[\mathfrak{A}^*, \mathbf{A}^*]$ is a normed operator ideal ([23], 9.1.3).

Let now the sequence spaces ℓ_u, ℓ_u^n ($1 \le u \le \infty$), and c_0 be those as usual. For $\sigma = \{\sigma_n\} \in \ell_\infty$ let $D_\sigma = D_{\{\sigma_n\}}$ be the diagonal operator between ℓ_u-spaces defined by $D_\sigma(\{\xi_n\}) = \{\sigma_n \xi_n\}$. The *limit order* of an operator ideal $\mathfrak{A}$ and the *L-limit order* of a quasi-normed operator ideal $[\mathfrak{A}, \mathbf{A}]$ are defined by (3) and (4) respectively ([23], 14.4.1; [20]). The *I-limit order* of a quasi-normed operator ideal $[\mathfrak{A}, \mathbf{A}]$ is defined by

$$\lambda_I(\mathfrak{A}, u, v)$$
$$:= \inf\{\lambda > 0;\ ^\exists c = c(u,v,\lambda) \ s.t.\ \mathbf{A}(I_n\colon \ell_u^n \to \ell_v^n) \le cn^\lambda \quad (n = 1, 2, \dots)\},$$

where I_n are the identity operators ([20]). For an operator ideal $\mathfrak{A}$, let

$$\ell_{(\mathfrak{A},u,v)} := \{\sigma \in \ell_\infty;\ D_\sigma \in \mathfrak{A}(\ell_u, \ell_v)\} \qquad (1 \le u, v \le \infty)$$

(cf. [22], 4.10.1). If $\mathfrak{A}$ is a quasi-normed operator ideal with the quasi-norm $\mathbf{A}$, put $\|\sigma\|_{\mathbf{A}} = \mathbf{A}(D_\sigma)$ for $\sigma \in \ell_{(\mathfrak{A},u,v)}$. Then, $\ell_{(\mathfrak{A},u,v)}$ becomes a complete quasi-normed space with $\|\cdot\|_{\mathbf{A}}$ (cf. [12], p. 99). Let $\boldsymbol{N}$ (resp. $\boldsymbol{N}_0$) be the set of positive (resp. non-negative) integers.

LEMMA B (cf. [12]). (i) $\ell_{(\mathfrak{A},u,v)}$ *is symmetric: If* $\{\sigma_n\}\in\ell_{(\mathfrak{A},u,v)}$, *then* $\{\sigma_{\pi(n)}\}\in\ell_{(\mathfrak{A},u,v)}$ *for any permutation* π *on* $\boldsymbol{N}$.

(ii) $\{|\sigma_n|\}\in\ell_{(\mathfrak{A},u,v)}$ *if and only if* $\{\sigma_n\}\in\ell_{(\mathfrak{A},u,v)}$.

(iii) *For a quasi-normed operator ideal* $[\mathfrak{A}, \mathbf{A}]$, *the inclusion map* $(\ell_{(\mathfrak{A},u,v)}, \|\cdot\|_{\mathbf{A}})\hookrightarrow\ell_\infty$ *is continuous.*

They are easily derived from the definition of (quasi-normed) operator ideals (cf. [23], Proposition 6.1.4 for (iii)).

Let $1\le u\le\infty$, $1\le s<\infty$ or $1\le u<\infty$, $s=\infty$. The *Lorentz sequence space* $\ell_{u,s}$ is the space of all $\{\sigma_n\}\in c_0$ such that

$$\|\{\sigma_n\}\|_{u,s}=\begin{cases}(\sum_{n=1}^{\infty} n^{s/u-1}|\sigma_n|^{*s})^{1/s} & (1\le u\le\infty,\ 1\le s<\infty),\\ \sup_n n^{1/u}|\sigma_n|^* & (1\le u<\infty,\ s=\infty)\end{cases}$$

is finite, where $\{|\sigma_n|^*\}$ is the non-increasing rearrangement of $\{|\sigma_n|\}$ (cf. [23], 13.9.1; [16]). $\|\cdot\|_{u,s}$ is a norm (resp. quasi-norm) if $1\le s\le u\le\infty$ (resp. $1\le u<s\le\infty$) ([7], Proposition 1; see also [23], 13.9.5). Clearly $\ell_{u,u}$ coincides with ℓ_u. For $u=s=\infty$, we put $\ell_{\infty,\infty}=\ell_\infty$.

LEMMA C ([23], Proposition 13.9.4; [16]). *Let* $1\le u_1<u_2\le\infty$ *and* $1\le s_1, s_2\le\infty$. *Then,*

$$\ell_{u_1,s_1}\subset\ell_{u_2,s_2}$$

and the inclusion map $\ell_{u_1,s_1}\hookrightarrow\ell_{u_2,s_2}$ *is continuous.*

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive numbers. We write $\alpha_n\prec\beta_n$ if $\alpha_n\le c\beta_n$ ($\forall n\in\boldsymbol{N}$) with some c.

§2. The spaces $\ell_{r,\infty}(\boldsymbol{\alpha})$ and $\ell^0_{r,\infty}(\boldsymbol{\alpha})$

DEFINITION 1. *Let* $\boldsymbol{\alpha}=\{\alpha_n\}$ *be an arbitrary fixed sequence of positive numbers which is strictly increasing and divergent to* ∞. *Let* $0<r<\infty$. *We define*

$$\ell_{r,\infty}(\boldsymbol{\alpha}):=\{\sigma=\{\sigma_n\}\in c_0;\ \|\sigma\|_{r,\infty;\boldsymbol{\alpha}}:=\sup\alpha_n^{1/r}|\sigma_n|^*<\infty\},$$

where $\{|\sigma_n|^*\}$ *is the non-increasing rearrangement of* $\{|\sigma_n|\}$; *and*

$$\ell^0_{r,\infty}(\boldsymbol{\alpha}):=\{\sigma=\{\sigma_n\}\in c_0;\ \|\sigma\|^0_{r,\infty;\boldsymbol{\alpha}}:=\sup\alpha_n^{1/r}|\sigma_n|<\infty\}.$$

For $r=\infty$, *let* $\ell_{\infty,\infty}(\boldsymbol{\alpha})=\ell^0_{\infty,\infty}(\boldsymbol{\alpha})=\ell_\infty$.

$\ell_{r,\infty}(\boldsymbol{\alpha})$ is a generalization of the Lorentz sequence space $\ell_{r,\infty}$. $\ell^0_{r,\infty}(\boldsymbol{\alpha})$ is a Banach space, as is easily seen.

LEMMA 1 ([7], Lemma 1). *Let* $\{\sigma_n\}, \{\mu_n\} \in c_0$. *Let* $\{|\sigma_{\phi(n)}|\}, \{|\mu_{\psi(n)}|\}$, *and* $\{|\sigma_{\omega(n)}+\mu_{\omega(n)}|\}$ *be the non-increasing rearrangements of* $\{|\sigma_n|\}, \{|\mu_n|\}$, *and* $\{|\sigma_n+\mu_n|\}$ *respectively. Then, for any* $n \in \boldsymbol{N}$

$$|\sigma_{\omega(2n)} + \mu_{\omega(2n)}| \le |\sigma_{\omega(2n-1)} + \mu_{\omega(2n-1)}| \le |\sigma_{\phi(n)}| + |\mu_{\psi(n)}| .$$

PROPOSITION 1. *Let* $0<r<\infty$. *Assume* $\alpha_{2n} \le c\alpha_n (\forall n \in \boldsymbol{N})$ *with some constant c. Then,* $\ell_{r,\infty}(\boldsymbol{\alpha})$ *is a quasi-normed space;*

$$\|\sigma+\mu\|_{r,\infty;\alpha} \le c^{1/r}(\|\sigma\|_{r,\infty;\alpha} + \|\mu\|_{r,\infty;\alpha}) \quad \text{for any} \quad \sigma, \mu \in \ell_{r,\infty}(\boldsymbol{\alpha}). \tag{6}$$

PROOF. Let us show (6). Let $\sigma=\{\sigma_n\}, \mu=\{\mu_n\} \in \ell_{r,\infty}(\boldsymbol{\alpha})$. Then, by Lemma 1

$$\begin{aligned}\|\sigma + \mu\|_{r,\infty;\alpha} &= \sup \alpha_n^{1/r}|\sigma_{\omega(n)} + \mu_{\omega(n)}| \\ &= \max\{\sup \alpha_{2n-1}^{1/r}|\sigma_{\omega(2n-1)} + \mu_{\omega(2n-1)}|, \sup \alpha_{2n}^{1/r}|\sigma_{\omega(2n)} + \mu_{\omega(2n)}|\} \\ &\le c^{1/r} \sup \alpha_n^{1/r}(|\sigma_{\phi(n)}| + |\mu_{\psi(n)}|) \\ &\le c^{1/r}(\|\sigma\|_{r,\infty;\alpha} + \|\mu\|_{r,\infty;\alpha}).\end{aligned}$$

REMARK 1. (i) *Without the condition* $\alpha_{2n} \prec \alpha_n$, $\ell_{r,\infty}(\boldsymbol{\alpha})$ *fails to become a linear space.*

(ii) $\|\cdot\|_{r,\infty;\alpha}$ *is not a norm.*

PROOF. (i) Let us assume that $\{\alpha_{2n}/\alpha_n\}$ is not bounded. Then, for each $k \in \boldsymbol{N}$ there exists $n_k \in \boldsymbol{N}$ such that $\alpha_{2n_k} > k\alpha_{n_k}$. Put $\sigma_{2n-1} = \alpha_n^{-1/r}$, $\sigma_{2n}=0$, and $\mu_{2n}=\alpha_n^{-1/r}$, $\mu_{2n-1}=0$ for $n \in \boldsymbol{N}$. Then, clearly $\sigma=\{\sigma_n\}, \mu=\{\mu_n\} \in \ell_{r,\infty}(\boldsymbol{\alpha})$, while $\sigma+\mu \notin \ell_{r,\infty}(\boldsymbol{\alpha})$ because

$$\alpha_{2n_k}^{1/r}(\sigma_{\omega(2n_k)} + \mu_{\omega(2n_k)}) = \alpha_{2n_k}^{1/r} \cdot \alpha_{n_k}^{-1/r} > k^{1/r} \longrightarrow \infty \qquad (k \longrightarrow \infty).$$

(ii) Take two positive numbers a and b such that $1<a/b<(\alpha_2/\alpha_1)^{1/r}$, and put $\sigma=(a, b, 0,\dots)$ and $\mu=(b, a, 0,\dots)$. Then, $\|\sigma\|_{r,\infty;\alpha} = \|\mu\|_{r,\infty;\alpha} = \max\{\alpha_1^{1/r}a, \alpha_2^{1/r}b\} = \alpha_2^{1/r}b$. Therefore

$$\|\sigma + \mu\|_{r,\infty;\alpha} = \alpha_2^{1/r}(a+b) > 2\alpha_2^{1/r}b = \|\sigma\|_{r,\infty;\alpha} + \|\mu\|_{r,\infty;\alpha}.$$

LEMMA 2 ([7], Lemma 4). *Let* $\{\sigma_n^{(k)}\}_{n,k}$ *be a double sequence such that* $\lim_{n\to\infty} \sigma_n^{(k)}=0$ *for each* $k \in \boldsymbol{N}$, *and* $\lim_{k\to\infty} \sigma_n^{(k)}=\sigma_n$ *(uniformly in n). Then,* $\lim_{n\to\infty} \sigma_n=0$, *and for each* $n \in \boldsymbol{N}$

$$|\sigma_{\phi(n)}| \le \liminf_{k\to\infty} |\sigma_{\phi_k(n)}^{(k)}|,$$

where $\{|\sigma_{\phi(n)}|\}$ *and* $\{|\sigma_{\phi_k(n)}^{(k)}|\}_n$ *are the non-increasing rearrangements of* $\{|\sigma_n|\}$ *and* $\{|\sigma_n^{(k)}|\}_n$ *respectively.*

PROPOSITION 2. *Let $0<r\leq\infty$ and let $\alpha_{2n}\prec\alpha_n$. Then, $\ell_{r,\infty}(\boldsymbol{\alpha})$ is complete.*

PROOF. Let $0<r<\infty$. Let $\{\sigma^{(k)}\}$, $\sigma^{(k)}=\{\sigma_n^{(k)}\}_n$, be an arbitrary Cauchy sequence in $\ell_{r,\infty}(\boldsymbol{\alpha})$. Then, for any $\varepsilon>0$ there exists $k_0\in \boldsymbol{N}$ such that for any $j,\ k\geq k_0$

$$(7)\qquad \|\sigma^{(j)}-\sigma^{(k)}\|_{r,\infty;\alpha}=\sup_n \alpha_n^{1/r}|\sigma^{(j)}_{\omega_{j,k}(n)}-\sigma^{(k)}_{\omega_{j,k}(n)}|<\varepsilon,$$

where $\{|\sigma^{(j)}_{\omega_{j,k}(n)}-\sigma^{(k)}_{\omega_{j,k}(n)}|\}_n$ is the non-increasing rearrangement of $\{|\sigma_n^{(j)}-\sigma_n^{(k)}|\}_n$. In particular, we have

$$\sup_n|\sigma_n^{(j)}-\sigma_n^{(k)}|<\alpha_1^{-1/r}\varepsilon \qquad \text{for any}\quad j,\ k\geq k_0,$$

whence there exists a sequence $\sigma=\{\sigma_n\}$ such that $\sigma_n=\lim_{k\to\infty}\sigma_n^{(k)}$ (uniformly in n). Let k be an arbitrary positive integer with $k\geq k_0$ and be fixed. Then, applying Lemma 2 to $\{\sigma_n^{(j)}-\sigma_n^{(k)}\}_n$, we have

$$(8)\qquad |\sigma_{\omega_k(n)}-\sigma^{(k)}_{\omega_k(n)}|\leq \liminf_{j\to\infty}|\sigma^{(j)}_{\omega_{j,k}(n)}-\sigma^{(k)}_{\omega_{j,k}(n)}| \quad \text{for each } n\in \boldsymbol{N},$$

where $\{|\sigma_{\omega_k(n)}-\sigma^{(k)}_{\omega_k(n)}|\}_n$ denotes the non-increasing rearrangement of $\{|\sigma_n-\sigma_n^{(k)}|\}_n$. Consequently, by (7) and (8) we have for any $k\geq k_0$

$$\begin{aligned}\|\sigma-\sigma^{(k)}\|_{r,\infty;\alpha}&=\sup_n\alpha_n^{1/r}|\sigma_{\omega_k(n)}-\sigma^{(k)}_{\omega_k(n)}|\\&\leq \sup_n\liminf_{j\to\infty}\alpha_n^{1/r}|\sigma^{(j)}_{\omega_{j,k}(n)}-\sigma^{(k)}_{\omega_{j,k}(n)}|\\&\leq \liminf_{j\to\infty}\sup_n\alpha_n^{1/r}|\sigma^{(j)}_{\omega_{j,k}(n)}-\sigma^{(k)}_{\omega_{j,k}(n)}|\\&=\liminf_{j\to\infty}\|\sigma^{(j)}-\sigma^{(k)}\|_{r,\infty;\alpha}\\&\leq\varepsilon,\end{aligned}$$

and hence $\{\sigma_n\}=\{\sigma_n-\sigma_n^{(k)}\}+\{\sigma_n^{(k)}\}\in\ell_{r,\infty}(\boldsymbol{\alpha})$, which completes the proof.

LEMMA 3. *Let $\{\alpha_n\}$ be a non-decreasing sequence of positive numbers which tends to ∞. Let $\{\sigma_n\}$ be a zero-sequence of positive numbers, and $\{\sigma_{\phi(n)}\}$ its non-increasing rearrangement. Then, if $\{\alpha_n\sigma_n\}$ is bounded, so is $\{\alpha_n\sigma_{\phi(n)}\}$. The converse is false.*

PROOF. Let m be an arbitrary positive integer and fixed. If $m\leq\phi(m)$, then

$$\alpha_m\sigma_{\phi(m)}\leq\alpha_{\phi(m)}\sigma_{\phi(m)}\leq\sup_n\alpha_n\sigma_n.$$

If $m>\phi(m)$, then there exists $k\in\boldsymbol{N}$ such that $1\leq k<m$ and $m\leq\phi(k)$, whence

$$\alpha_m\sigma_{\phi(m)}\leq\alpha_{\phi(k)}\sigma_{\phi(k)}\leq\sup_n\alpha_n\sigma_n.$$

Consequently, if $\{\alpha_n\sigma_n\}$ is bounded, so is $\{\alpha_n\sigma_{\phi(n)}\}$.

For the latter assertion, put $\mu_n=1/\alpha_n$. We show that for a certain rearrangement $\{\mu_{\pi(n)}\}$ of $\{\mu_n\}$, $\{\alpha_n\mu_{\pi(n)}\}$ is not bounded. We may assume $\alpha_n \geq 1$ for all $n \in \boldsymbol{N}$. We choose a sequence $\{n_k\}$ of positive integers inductively as follows. Let n_1 be the smallest $n \in \boldsymbol{N}$ such that $\alpha_1^2 < \alpha_n$. If we have chosen $\{n_1, \ldots, n_{k-1}\}$, let n_k be the smallest $n \in \boldsymbol{N}$ such that $\alpha_{n_{k-1}}^2 < \alpha_n$ (hence $n_{k-1} < n_k$). Let π: $\boldsymbol{N} \to \boldsymbol{N}$ be a bijection such that $\pi(n_k) = n_{k-1}$ (put $n_0 = 1$). Then, $\{\alpha_n\mu_n\}$ is bounded, but $\{\alpha_n\mu_{\pi(n)}\}$ is not so because

$$\alpha_{n_k}\mu_{\pi(n_k)} = \frac{\alpha_{n_k}}{\alpha_{n_{k-1}}} > \alpha_{n_{k-1}} \longrightarrow \infty \qquad (k \longrightarrow \infty).$$

By Lemma 3 we have immediately

PROPOSITION 3. *Let* $0 < r < \infty$. *Then,*

$$\ell^0_{r,\infty}(\boldsymbol{\alpha}) \subsetneq \ell_{r,\infty}(\boldsymbol{\alpha}), \qquad \|\cdot\|_{r,\infty;\boldsymbol{\alpha}} \leq \|\cdot\|^0_{r,\infty;\boldsymbol{\alpha}}.$$

§ 3. A nearly necessary and sufficient condition such that a diagonal operator belongs to $[\mathfrak{A}, \mathrm{A}]$

The identity

$$(1) \qquad \lambda(\mathfrak{A}, u, v) = \lambda_I(\mathfrak{A}, u, v)$$

follows from the fact that

(i) if $D_{\{n^{-\lambda}\}} \in \mathfrak{A}(\ell_u, \ell_v)$, then there exists $c = c(u, v, \lambda)$ such that

$$(9) \qquad \mathrm{A}(I_n: \ell_u^n \longrightarrow \ell_v^n) \leq cn^\lambda \qquad (^\forall n \in \boldsymbol{N}),$$

and coversely,

(ii) if (9) holds with some c, then for any $\varepsilon > 0$ $D_{\{n^{-(\lambda+\varepsilon)}\}} \in \mathfrak{A}(\ell_u, \ell_v)$.

We generalize these assertions in the following theorem. The proof of its essential part is based on Pietsch's simplified proof of (1) ([23], Theorem 14.4.3).

THEOREM 1. *Let* $1 \leq u, v \leq \infty$. *Let* $\boldsymbol{\alpha} = \{\alpha_n\}$ *be a non-decreasing sequence of positive numbers which tends to* ∞.

(i) *If* $D_{\{\alpha_n^{-1}\}}$ *belongs to* $\mathfrak{A}(\ell_u, \ell_v)$, *then there exists* $c = c(u, v, \boldsymbol{\alpha})$ *such that*

$$(10) \qquad \mathrm{A}(I_n: \ell_u^n \longrightarrow \ell_v^n) \leq c\alpha_n \qquad (^\forall n \in \boldsymbol{N}).$$

(ii) *If* (10) *holds with some* c, *then for any* $\varepsilon > 0$ $D_{\{\alpha_n^{-(1+\varepsilon)}\}}$ *belongs to* $\mathfrak{A}(\ell_u, \ell_v)$.

PROOF. (i) Put $D = D_{\{\alpha_n^{-1}\}}$. Let $D_n(\{\xi_i\}_{1 \leq i \leq n}) = \{\alpha_i^{-1}\xi_i\}_{1 \leq i \leq n}$. Then, by (QN_3) we have

$$\mathrm{A}(D_n: \ell_u^n \longrightarrow \ell_v^n) \leq \mathrm{A}(D: \ell_u \longrightarrow \ell_v),$$

and hence

$$\mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \le \mathbf{A}(D_n\colon \ell_u^n \longrightarrow \ell_v^n)\|D_n^{-1}\colon \ell_u^n \longrightarrow \ell_u^n\| \\ \le \mathbf{A}(D)\,\alpha_n.$$

(ii) By Lemma A we may assume that $[\mathfrak{A}, \mathbf{A}]$ is a p-normed operator ideal (for some $0<p\le 1$). Let

$$N_k := \{n \in \boldsymbol{N};\ 2^{k-1} < \alpha_n \le 2^k\} \qquad (k = 1, 2,\dots)$$

and

$$N_0 := \{n \in \boldsymbol{N};\ 0 < \alpha_n \le 1\}.$$

Let $n_k = \operatorname{card} N_k$, the cardinal number of N_k $(k \in \boldsymbol{N}_0)$. We first assume that $n_k \ne 0$ for each $k \in \boldsymbol{N}_0$. Put

$$q_n^{(k)} = \begin{cases} 1 & (n \in N_k), \\ 0 & (n \notin N_k), \end{cases}$$

and let Q_k be the diagonal operator defined by $\{q_n^{(k)}\}_n$, i.e., $Q_k(\{\xi_n\}) = \{q_n^{(k)}\xi_n\}_n$. Then, we have

$$\mathbf{A}(Q_k\colon \ell_u \longrightarrow \ell_v) \le c\alpha_{n_k} \qquad (k = 0, 1, 2,\dots)$$

by the assumption (10) and the property (QN_3) of quasi-normed (in particular, p-normed) operator ideals. Therefore, for any $\varepsilon > 0$

$$\sum_{k=0}^{\infty} \mathbf{A}(2^{-\varepsilon k}\alpha_{n_k}^{-1}Q_k\colon \ell_u \longrightarrow \ell_v)^p = \sum_{k=0}^{\infty} 2^{-\varepsilon kp}\alpha_{n_k}^{-p}\mathbf{A}(Q_k\colon \ell_u \longrightarrow \ell_v)^p \\ \le c^p \sum_{k=0}^{\infty} (2^{-\varepsilon p})^k < \infty.$$

Consequently, the operator

$$S := \sum_{k=0}^{\infty} 2^{-\varepsilon k}\,\alpha_{n_k}^{-1}\,Q_k : \ell_u \longrightarrow \ell_v$$

is well-defined and belongs to $\mathfrak{A}$ because $[\mathfrak{A}, \mathbf{A}]$ is complete. Next, we put

$$\sigma_n = 2^{\varepsilon k}\alpha_{n_k}\alpha_n^{-(1+\varepsilon)} \qquad \text{for} \quad n \in N_k, \quad k = 0, 1, 2, \dots.$$

Then $\{\sigma_n\}$ is bounded. Indeed, let $n \in N_k$. Then $2^{k-1} < \alpha_n$. Since $n_k < n_0 + n_1 + \cdots + n_k$ and $\{\alpha_n\}$ is non-decreasing, we have $\alpha_{n_k} \le 2^k$, whence $\alpha_{n_k} \le 2\cdot 2^{k-1} < 2\alpha_n$. Therefore, we have

$$2^{\varepsilon k}\alpha_{n_k} = 2^{\varepsilon}2^{\varepsilon(k-1)}\alpha_{n_k} \le 2^{\varepsilon}\alpha_n^{\varepsilon}(2\alpha_n) = 2^{1+\varepsilon}\alpha_n^{1+\varepsilon},$$

or $\sigma_n \le 2^{1+\varepsilon}$. Consequently, the diagonal operator $D_{\{\sigma_n\}}\colon \ell_u \to \ell_u$ belongs to $\mathfrak{L}$.

Since the operator $D_{\{\alpha_n^{-(1+\varepsilon)}\}}: \ell_u \to \ell_v$ is the composition of $D_{\{\sigma_n\}}: \ell_u \to \ell_u \in \mathfrak{L}$ and $S: \ell_u \to \ell_v \in \mathfrak{A}$, we have $D_{\{\alpha_n^{-(1+\varepsilon)}\}} \in \mathfrak{A}(\ell_u, \ell_v)$ as desired.

In the case where there exist k with $n_k = \operatorname{card} N_k = 0$, we have only to take instead of $\{n_k\}$ the subsequence $\{n_{k_i}\}$ consisting of non-zero terms of $\{n_k\}$ in the above proof. This completes the proof.

By Theorem 1 and Lemma B we have immediately the following

COROLLARY. *Let $\{\alpha_n\}$ be a sequence (of real or complex numbers) with $\lim_{n\to\infty} |\alpha_n| = \infty$ and $\{^*|\alpha_n|\}$ the non-decreasing rearrangement of $\{|\alpha_n|\}$.*

(i) *If $D_{\{\alpha_n^{-1}\}} \in \mathfrak{A}(\ell_u, \ell_v)$, then there exists c such that*

$$\mathbf{A}(I_n: \ell_u^n \longrightarrow \ell_v^n) \le c(^*|\alpha_n|) \qquad (^\forall n \in \boldsymbol{N}).$$

(ii) *If*

$$\mathbf{A}(I_n: \ell_u^n \longrightarrow \ell_v^n) \le c(^*|\alpha_n|)^\mu \qquad (^\forall n \in \boldsymbol{N})$$

with some c and μ $(0<\mu<1)$, then $D_{\{\alpha^{-1}\}} \in \mathfrak{A}(\ell_u, \ell_v)$.

§4. The $\boldsymbol{\alpha}$-limit order of operator ideals

DEFINITION 2. *Let $\boldsymbol{\alpha} = \{\alpha_n\}$ be an arbitrary fixed sequence of positive numbers which is strictly increasing and divergent to ∞. We define the $\boldsymbol{\alpha}$-limit order of an operator ideal $\mathfrak{A}$ by*

$$\lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) := \inf \{\lambda > 0;\ D_{\{\alpha_n^{-\lambda}\}} \in \mathfrak{A}(\ell_u, \ell_v)\}$$

for $1 \le u, v \le \infty$.

If $\boldsymbol{\beta} = \{\beta_n\}$ is another sequence with the same property as $\boldsymbol{\alpha}$, and if $\alpha_n \prec \beta_n$, then $\lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) \ge \lambda_{\boldsymbol{\beta}}(\mathfrak{A}, u, v)$. In particular, if $\alpha_n \prec n$ and $n \prec \alpha_n$, $\lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v)$ coincides with $\lambda(\mathfrak{A}, u, v)$. We easily obtain

PROPOSITION 4. *If $\lambda > \lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v)$ (resp. $\lambda < \lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v)$), then $D_{\{\alpha_n^{-\lambda}\}} \in \mathfrak{A}(\ell_u, \ell_v)$ (resp. $D_{\{\alpha_n^{-\lambda}\}} \notin \mathfrak{A}(\ell_u, \ell_v)$).*

The following theorem generalizes (5) ([23], Proposition 14.4.2).

THEOREM 2. *Let $1 \le u, v \le \infty$. Then,*

$$\begin{aligned}(11)\qquad \lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) &= \inf \{1/r \ge 0;\ \sigma \in \ell_{r,\infty}(\boldsymbol{\alpha}) \Longrightarrow D_\sigma \in \mathfrak{A}(\ell_u, \ell_v)\} \\ &= \inf \{1/r \ge 0;\ \sigma \in \ell^0_{r,\infty}(\boldsymbol{\alpha}) \Longrightarrow D_\sigma \in \mathfrak{A}(\ell_u, \ell_v)\} \\ &= \inf \{1/r \ge 0;\ \sigma = \{\sigma_n\} \in \ell_{r,\infty}(\boldsymbol{\alpha}),\ \sigma_1 \ge \sigma_2 \ge \cdots > 0 \\ &\qquad\qquad \Longrightarrow D_\sigma \in \mathfrak{A}(\ell_u, \ell_v)\} \\ &= \inf \{1/r \ge 0;\ \sigma = \{\sigma_n\} \in \ell^0_{r,\infty}(\boldsymbol{\alpha}),\ \sigma_1 \ge \sigma_2 \ge \cdots > 0 \\ &\qquad\qquad \Longrightarrow D_\sigma \in \mathfrak{A}(\ell_u, \ell_v)\}.\end{aligned}$$

PROOF. The last equality is trivial. We write the first three terms of the right-hand side of (11) as m_1, m_2, and m_3 in that order. Let us show

$$(12)\qquad \lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) \geq m_1 \geq m_2 \geq m_3 \geq \lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v).$$

Let $\lambda > \lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v)$. Then, by Proposition 4

$$D_{\{\alpha_n^{-\lambda}\}} \in \mathfrak{A}(\ell_u, \ell_v).$$

Put $r = 1/\lambda$ and let $\{\sigma_n\} \in \ell_{r,\infty}(\boldsymbol{\alpha})$. Then, $\{\alpha_n^\lambda \sigma_{\phi(n)}\}$ is bounded and hence

$$D_{\{\alpha_n^\lambda \sigma_{\phi(n)}\}} \in \mathfrak{L}(\ell_u, \ell_u),$$

where ϕ is so defined that $\{|\sigma_{\phi(n)}|\}$ is the non-increasing rearrangement of $\{|\sigma_n|\}$. Therefore

$$D_{\{\sigma_{\phi(n)}\}} = D_{\{\alpha_n^{-\lambda}\}} \circ D_{\{\alpha_n^\lambda \sigma_{\phi(n)}\}} \in \mathfrak{A}(\ell_u, \ell_v).$$

Consequently, by Lemma B we have $D_{\{\sigma_n\}} \in \mathfrak{A}(\ell_u, \ell_v)$, which implies the first inequality in (12). The second inequality is an immediate consequence of Proposition 3. The third one is trivial. For the last, assume that $\sigma = \{\sigma_n\} \in \ell_{r,\infty}(\boldsymbol{\alpha})$, $\sigma_1 \geq \sigma_2 \geq \cdots > 0$ implies $D_\sigma \in \mathfrak{A}(\ell_u, \ell_v)$, and put $\sigma_n = \alpha_n^{-1/r}$. Then we have $m_3 \geq \lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v)$, which completes the proof.

By Theorem 1, we immediately obtain the following generalization of the identity (1), i.e.,

$$\lambda(\mathfrak{A}, u, v) = \inf\{\lambda > 0;\ ^\exists c = c(u,v,\lambda)\ s.t.\ \mathbf{A}(I_n\colon \ell_u^n \to \ell_v^n) \leq c n^\lambda\ (^\forall n \in \boldsymbol{N})\}.$$

THEOREM 3. *Let* $[\mathfrak{A}, \mathbf{A}]$ *be a quasi-normed operator ideal, and let* $1 \leq u, v \leq \infty$. *Then,*

$$\lambda_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) = \inf\{\lambda > 0;\ ^\exists c = c(u,v,\lambda)\ s.t.\ \mathbf{A}(I_n\colon \ell_u^n \to \ell_v^n) \leq c\alpha_n^\lambda\ (^\forall n \in \boldsymbol{N})\}.$$

Now, W. Linde and Pietsch [14] introduced the ideal $[\mathfrak{P}_\gamma, \Pi_\gamma]$ of absolutely γ-summing operators as follows. Let γ_n denote the Gaussian measure on the n-dimensional Euclidean space $\boldsymbol{R}^n$ which is defined on every Borel set B by

$$\gamma_n(B) = (2\pi)^{-n/2} \int_B \exp\{-\textstyle\sum_{i=1}^n \tau_i^2/2\} d\tau_1 \cdots d\tau_n.$$

An operator $S \in \mathfrak{L}(E, F)$, E and F being real Banach spaces, is called *absolutely γ-summing* if there exists a constant $\rho \geq 0$ such that for every $x_1, x_2, \ldots, x_n \in E$,

$$\left\{\int_{\boldsymbol{R}^n} \|\textstyle\sum_{i=1}^n \tau_i S x_i\|^2 d\gamma_n(\tau)\right\}^{1/2} \leq \rho \sup\left[\{\textstyle\sum_{i=1}^n |\langle x_i, a\rangle|^2\}^{1/2};\ \|a\| \leq 1,\ a \in E'\right].$$

The infimum of all such ρ is denoted by $\Pi_\gamma(S)$. $[\mathfrak{P}_\gamma, \Pi_\gamma]$ is a normed operator ideal ([14], Theorems 1 and 2). They proved

PROPOSITION A ([14], Theorem 9). *Let* $2 \leq u \leq \infty$. *Let* $\sigma=\{\sigma_n\}$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. *Then,* D_σ *belongs to* $\mathfrak{P}_\gamma(\ell_u, \ell_\infty)$ *if and only if*

$$\sup \sigma_n \sqrt{\log(n+1)} < \infty.$$

REMARK 2. *Let* $2 \leq u \leq \infty$ *and let* $\boldsymbol{\alpha}=\{\alpha_n\}$, $\alpha_n=\log(n+1)$. *Then, Proposition A with Lemma B implies that*

$$D_\sigma \in \mathfrak{P}_\gamma(\ell_u, \ell_\infty) \quad \textit{if and only if} \quad \sigma \in \ell_{2,\infty}(\boldsymbol{\alpha}),$$

or

$$\ell_{(\mathfrak{P}_\gamma, u, \infty)} = \ell_{2,\infty}(\boldsymbol{\alpha}).$$

EXAMPLE 1. *Let u and* $\boldsymbol{\alpha}=\{\alpha_n\}$ *be as in Remark* 2. *Then,*

$$\lambda_\alpha(\mathfrak{P}_\gamma, u, \infty) = \frac{1}{2} \tag{13}$$

while

$$\lambda(\mathfrak{P}_\gamma, u, \infty) = 0. \tag{14}$$

In fact, from Proposition A it follows that

$$\begin{aligned} &D_{\{\alpha^{-\lambda}\}} \in \mathfrak{P}_\gamma(\ell_u, \ell_\infty) \quad (\textit{resp.}\ D_{\{\alpha_n^{-\lambda}\}} \notin \mathfrak{P}_\gamma(\ell_u, \ell_\infty)) \\ &\textit{provided}\ \lambda > 1/2 \quad (\textit{resp.}\ \lambda < 1/2), \end{aligned} \tag{15}$$

which implies (13). (14) is also derived immediately from Proposition A. Let us here recall the following criteria given by $\lambda(\mathfrak{A}, u, v)$:

(a) *If* $\lambda > \lambda(\mathfrak{A}, u, v)$ (*resp.* $\lambda < \lambda(\mathfrak{A}, u, v)$), *then* $D_\lambda \in \mathfrak{A}(\ell_u, \ell_v)$ (*resp.* $D_\lambda \notin \mathfrak{A}(\ell_u, \ell_v)$).

(b) *Let* $1/r > \lambda(\mathfrak{A}, u, v)$. *Then, for every* $\sigma \in \ell_r$, D_σ *belongs to* $\mathfrak{A}(\ell_u, \ell_v)$.

Since $\lambda(\mathfrak{P}_\gamma, u, \infty)=0$, the behavior (15) of $\{\alpha_n^{-\lambda}\}$ can not be described by these criteria (a) and (b). (Note that $\{\alpha_n^{-\lambda}\}=\{\log^{-\lambda}(n+1)\} \notin \ell_r$ for any $r>0$.) On the other hand, by Proposition 4, (15) is well expressed by $\lambda_\alpha(\mathfrak{P}_\gamma, u, \infty)=1/2$. (Compare also Proposition A or Remark 2 with (b); cf. Theorem 2.) Thus, in this case, the $\boldsymbol{\alpha}$-limit order $\lambda_\alpha(\mathfrak{A})$ is more appropriate than $\lambda(\mathfrak{A})$ for the ideal $\mathfrak{A}=\mathfrak{P}_\gamma$.

Let us next recall the definitions of the ideals $\mathfrak{N}_0$ and $\mathfrak{A}_p$ ($p>0$) of strictly nuclear and $\mathfrak{A}_p$-operators respectively. Let $S \in \mathfrak{L}(E, F)$ and let $a_n(S)$ be its n-th approximation number, i.e., $a_n(S):=\inf\{\|S-L\|;\ L \in \mathfrak{L}(E, F) \text{ and } \operatorname{rank}(L)<n\}$. S is called a *strictly nuclear operator* (resp. an $\mathfrak{A}_p$-*operator*) if $\{a_n(S)\} \in \ell_0 := \bigcap_{p>0} \ell_p$ (resp. $\{a_n(S)\} \in \ell_p$) (cf. [23], 18.7.1 (resp. 14.2.4)). By Proposition 14.4.9 in [23] and Proposition 6 in [2] the limit order of $\mathfrak{A}_p$ for $0<p<1$ is given by

$$(16)\quad \lambda(\mathfrak{A}_p, u, v) = \begin{cases} \dfrac{1}{p} - \dfrac{1}{u} + \dfrac{1}{v} & (1 \le v \le u \le \infty), \\ \dfrac{1}{p} & (1 \le u \le v \le 2 \text{ or } 2 \le u \le v \le \infty), \\ \max\left\{\dfrac{1}{p} + \dfrac{1}{2} - \dfrac{1}{u}, \dfrac{1}{p} + \dfrac{1}{v} - \dfrac{1}{2}\right\} & (1 \le u \le 2 \le v \le \infty). \end{cases}$$

EXAMPLE 2. (i) *For all* $1 \le u, v \le \infty$

$$\lambda(\mathfrak{N}_0, u, v) = \infty,$$

which only asserts

$$D_\lambda \notin \mathfrak{N}_0(\ell_u, \ell_v) \qquad \textit{for all} \quad \lambda > 0$$

and

$$\ell_r \not\subset \ell_{(\mathfrak{N}_0, u, v)} \qquad \textit{for all} \quad r > 0.$$

(ii) *Put* $\boldsymbol{\alpha} = \{\alpha_n\}$, $\alpha_n = \alpha^n$ $(\alpha > 1)$. *Then, for all* $1 \le u, v \le \infty$

$$\lambda_{\boldsymbol{\alpha}}(\mathfrak{N}_0, u, v) = 0,$$

which means that

$$D_{\{\alpha_n^{-\lambda}\}} \in \mathfrak{N}_0(\ell_u, \ell_v) \qquad \textit{for} \quad \textit{all} \quad \lambda > 0$$

or

$$\ell_{r,\infty}(\boldsymbol{\alpha}) \subset \ell_{(\mathfrak{N}_0, u, v)} \qquad \textit{for} \quad \textit{all} \quad r > 0.$$

(iii) *Let* $1 \le u \le v \le \infty$. *Then, there does not exist a sequence* $\boldsymbol{\alpha} = \{\alpha_n\}$, $0 < \alpha_n \nearrow \infty$, *such that* $0 < \lambda_{\boldsymbol{\alpha}}(\mathfrak{N}_0, u, v) < \infty$.

PROOF. (i) Since $\mathfrak{N}_0 = \bigcap_{p>0} \mathfrak{A}_p$ (cf. [23], 18.7.2), we have by (16)

$$\lambda(\mathfrak{N}_0, u, v) \ge \lambda(\mathfrak{A}_p, u, v) \longrightarrow \infty \qquad (p \longrightarrow 0).$$

(ii) Let $D_\sigma \in \mathfrak{L}(\ell_u, \ell_v)$, $\sigma = \{\sigma_n\}$, $\sigma_1 \ge \sigma_2 \ge \cdots \ge 0$. Then, by Theorem 1.27 in C. V. Hutton [6] (see also [23], Theorem 11.11.4),

$$(17)\qquad \frac{1}{2}\sigma_n \le a_n(D_\sigma) \le \sigma_n \qquad \text{for} \quad n \in \boldsymbol{N}$$

if $1 \le u \le v \le \infty$; and

$$(18)\qquad a_n(D_\sigma) = \left(\textstyle\sum_{k=n}^{\infty} \sigma_k^r\right)^{1/r} \qquad \text{for} \quad n \in \boldsymbol{N}$$

if $1 \le v < u \le \infty$, where $1/r = 1/v - 1/u$. Applying (17) and (18) to $D_{\{\alpha_n^{-\lambda}\}}\colon \ell_u \to \ell_v$, we have for $1 \le u, v \le \infty$

$$\{a_k(D_{\{\alpha_n^{-\lambda}\}})\}_k \in \ell_0 \qquad (^\forall\lambda > 0),$$

or $D_{\{\alpha_n^{-\lambda}\}} \in \mathfrak{N}_0(\ell_u, \ell_v)$ ($^\forall\lambda>0$). Hence $\lambda_{\boldsymbol{\alpha}}(\mathfrak{N}_0, u, v)=0$.

(iii) Suppose that $\lambda_{\boldsymbol{\alpha}}(\mathfrak{N}_0, u, v)<\infty$ for some $\boldsymbol{\alpha}=\{\alpha_n\}$, $0<\alpha_n\nearrow\infty$. Then, there exists a $\lambda>0$ such that $D_{\{\alpha_n^{-\lambda}\}} \in \mathfrak{N}_0(\ell_u, \ell_v)$, i.e., $\{a_k(D_{\{\alpha_n^{-\lambda}\}})\}_k \in \ell_0$, which is also valid for all $\lambda>0$ by (17). Hence $\lambda_{\boldsymbol{\alpha}}(\mathfrak{N}_0, u, v)=0$.

§5. The $\boldsymbol{\alpha}$-defect of $\mathfrak{A}$ and $\boldsymbol{\alpha}$-limit order of $\mathfrak{A}^*$

In this section, let $\boldsymbol{\alpha}=\{\alpha_n\}$ be a fixed strictly increasing sequence of positive numbers such that $\alpha_n\to\infty$ $(n\to\infty)$ and $\alpha_{2n}\prec\alpha_n$; and let $[\mathfrak{A}, \mathrm{A}]$ be a normed operator ideal. It should be noted that for normed operator ideals $[\mathfrak{A}, \mathrm{A}]$

$$0 \le \lambda(\mathfrak{A}, u, v) \le 1 \qquad \text{for} \quad 1 \le u, v \le \infty \tag{19}$$

([23], Theorem 6.7.2 and Propositions 14.4.4 and 22.4.6). In König [12] the defect $d(\mathfrak{A}, u, v)$ of $\mathfrak{A}$ is defined by

$$d(\mathfrak{A}, u, v) = \inf\left\{\frac{1}{r} - \frac{1}{s} ;\ \ell_r \subset \ell_{(\mathfrak{A},u,v)} \subset \ell_s\right\}.$$

As is easily shown (cf. Lemma C), it is represented as

$$\begin{aligned} d(\mathfrak{A}, u, v) &= \inf\left\{\frac{1}{r} - \frac{1}{s} ;\ \ell_{r,\infty} \subset \ell_{(\mathfrak{A},u,v)} \subset \ell_{s,\infty}\right\} \\ &= \inf\left\{\frac{1}{r} - \frac{1}{s} ;\ \ell^0_{r,\infty} \subset \ell_{(\mathfrak{A},u,v)} \subset \ell_{s,\infty}\right\}, \end{aligned}$$

where $\ell^0_{r,,\infty}=\ell^0_{r,\infty}(\{n\})$.

Definition 3. *We define the* $\boldsymbol{\alpha}$*-defect of* $\mathfrak{A}$ *by*

$$\begin{aligned} d_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) :&= \inf\left\{\frac{1}{r} - \frac{1}{s} ;\ \ell_{r,\infty}(\boldsymbol{\alpha}) \subset \ell_{(\mathfrak{A},u,v)} \subset \ell_{s,\infty}(\boldsymbol{\alpha})\right\} \\ &= \inf\left\{\frac{1}{r} - \frac{1}{s} ;\ \ell^0_{r,\infty}(\boldsymbol{\alpha}) \subset \ell_{(\mathfrak{A},u,v)} \subset \ell_{s,\infty}(\boldsymbol{\alpha})\right\} \end{aligned}$$

for $1\le u, v\le\infty$.

The following theorem generalizes Proposition 1 in König [12].

Theorem 4. *For* $1\le u, v\le\infty$ *we have*

$$d_{\boldsymbol{\alpha}}(\mathfrak{A}, u, v) = \inf\{\lambda - \mu;\ \lambda, \mu \ge 0 \ s.t.\ ^\exists c, d > 0 \text{ with } d\alpha_n^\mu \le \mathrm{A}(I_n\colon \ell_u^n \to \ell_v^n) \le c\alpha_n^\lambda \ (^\forall n \in \boldsymbol{N})\}.$$

Proof. Let us first show the inequality "$\ge$". Suppose $\ell_{r,\infty}(\boldsymbol{\alpha})\subset\ell_{(\mathfrak{A},u,v)}\subset$

$\ell_{s,\infty}(\boldsymbol{\alpha})$. Then, the inclusion maps $I: \ell_{r,\infty}(\boldsymbol{\alpha}) \hookrightarrow \ell_{(\mathfrak{A},u,v)}$ and $J: \ell_{(\mathfrak{A},u,v)} \hookrightarrow \ell_{s,\infty}(\boldsymbol{\alpha})$ are closed. Let us show that for I. Let $\sigma^{(k)}=\{\sigma_n^{(k)}\}\to\sigma=\{\sigma_n\}$ $(k\to\infty)$ in $\ell_{r,\infty}(\boldsymbol{\alpha})$ and $\sigma^{(k)}\to\mu=\{\mu_n\}$ $(k\to\infty)$ in $\ell_{(\mathfrak{A},u,v)}$. Then, by Lemma B (iii)

$$\sup_n |\sigma_n^{(k)} - \mu_n| \le \|\sigma^{(k)} - \mu\|_{\mathbf{A}} \longrightarrow 0 \qquad (k \longrightarrow \infty).$$

Therefore

$$\begin{aligned}\sup_n |\sigma_n - \mu_n| &\le \sup_n |\sigma_n - \sigma_n^{(k)}| + \sup_n |\sigma_n^{(k)} - \mu_n| \\ &\le \alpha_1^{-1/r} \sup_n \alpha_n^{1/r} |\sigma_{\omega_k(n)} - \sigma_{\omega_k(n)}^{(k)}| + \sup_n |\sigma_n^{(k)} - \mu_n| \\ &\to 0 \qquad (k\to\infty),\end{aligned}$$

where $\{|\sigma_{\omega_k(n)}-\sigma_{\omega_k(n)}^{(k)}|\}_n$ is the non-increasing rearrangement of $\{|\sigma_n-\sigma_n^{(k)}|\}_n$. Hence we have $\sigma=\mu$, i.e., I is closed. Consequently, I and J are continuous by the closed graph theorem. (Note that $\ell_{r,\infty}(\boldsymbol{\alpha})$ is complete metrizable by Propositions 1 and 2.) Therefore, there exist some constants c and d such that

$$\|\cdot\|_{\mathbf{A}} \le c\|\cdot\|_{r,\infty;\boldsymbol{\alpha}} \qquad \text{on} \quad \ell_{r,\infty}(\boldsymbol{\alpha})$$

and

$$\|\cdot\|_{s,\infty;\boldsymbol{\alpha}} \le d^{-1}\|\cdot\|_{\mathbf{A}} \qquad \text{on} \quad \ell_{(\mathfrak{A},u,v)}\,.$$

Consequently, we have for all $n\in \boldsymbol{N}$

$$\begin{aligned} d\alpha_n^{1/s} &= d\|(\overbrace{1,\dots,1}^{n}, 0, \dots)\|_{s,\infty;\boldsymbol{\alpha}} \\ &\le \|(\overbrace{1,\dots,1}^{n}, 0, \dots)\|_{\mathbf{A}} \\ &= \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \\ &\le c\|(\overbrace{1,\dots,1}^{n}, 0,\dots)\|_{r,\infty;\boldsymbol{\alpha}} = c\alpha_n^{1/r}. \end{aligned}$$

Hence we have the inequality "$\ge$".

To prove the converse inequality, assume that

$$d\alpha_n^{1/s} \le \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \le c\alpha_n^{1/r} \qquad (\forall n\in \boldsymbol{N}). \tag{20}$$

It is sufficient to show that for any $\varepsilon>0$

$$\ell_{r-\varepsilon,\infty}(\boldsymbol{\alpha}) \subset \ell_{(\mathfrak{A},u,v)} \subset \ell_{s,\infty}(\boldsymbol{\alpha}).$$

Let $\sigma=\{\sigma_n\}\in \ell_{r-\varepsilon,\infty}(\boldsymbol{\alpha})$ and let $\{|\sigma_{\phi(n)}|\}$ be the non-increasing rearragement of $\{|\sigma_n|\}$. Then, $\{\alpha_n^{1/(r-\varepsilon)}|\sigma_{\phi(n)}|\}$ is bounded, and hence

$$D_{\{\alpha_n^{1/(r-\varepsilon)}\sigma_{\phi(n)}\}} \in \mathfrak{L}(\ell_u, \ell_u).$$

Since $1/(r-\varepsilon)>\lambda_\alpha(\mathfrak{A}, u, v)$ by (20) and Theorem 3, we have

$$D_{\{\alpha_n^{-1/(r-\varepsilon)}\}}\in\mathfrak{A}(\ell_u, \ell_v)$$

by Proposition 4. Therefore

$$D_{\{\sigma_{\phi(n)}\}}=D_{\{\alpha_n^{-1/(r-\varepsilon)}\}}\circ D_{\{\alpha_n^{1/(r-\varepsilon)}\sigma_{\phi(n)}\}}\in\mathfrak{A}(\ell_u, \ell_v),$$

or $\{\sigma_{\phi(n)}\}\in\ell_{(\mathfrak{A},u,v)}$. Consequently, we have $\sigma\in\ell_{(\mathfrak{A},u,v)}$ by Lemma B. Let next $\sigma=\{\sigma_n\}\in\ell_{(\mathfrak{A},u,v)}$. If $\sigma\in c_0$, assume that $|\sigma_n|\geq|\sigma_{n+1}|>0$ ($^\forall n\in\boldsymbol{N}$), and put

$$D_\sigma^{(n)}(\{\xi_i\}_{1\leq i\leq n}) = \{\sigma_i\xi_i\}_{1\leq i\leq n}.$$

Then, by (20) and (QN_3) we have

$$\begin{aligned} d\alpha_n^{1/s} &\leq \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \\ &\leq \mathbf{A}(D_\sigma^{(n)}\colon \ell_u^n \longrightarrow \ell_v^n)\,\|(D_\sigma^{(n)})^{-1}\colon \ell_v^n \longrightarrow \ell_v^n\| \\ &\leq |\sigma_n|^{-1}\mathbf{A}(D_\sigma\colon \ell_u \longrightarrow \ell_v), \end{aligned}$$

or

$$\alpha_n^{1/s}|\sigma_n| \leq d^{-1}\mathbf{A}(D_\sigma\colon \ell_u \longrightarrow \ell_v)$$

for all $n\in\boldsymbol{N}$, i.e., $\sigma\in\ell_{s,\infty}(\boldsymbol{\alpha})$. If $\sigma\notin c_0$, there exists $\varepsilon_0>0$ such that $|\sigma_n|\geq\varepsilon_0$ for infinitely many $n\in\boldsymbol{N}$; let $\{n_k;\ k\in\boldsymbol{N}\}$ be the set of all such n ($n_k<n_{k+1}$ for all $k\in\boldsymbol{N}$). Put $\tilde{\sigma}_k=\sigma_{n_k}$. Then, by ($\mathrm{OI}_3$),

$$D_{\{\tilde{\sigma}_k\}}\in\mathfrak{A}(\ell_u, \ell_v).$$

Let now $\mu=\{\mu_k\}\in\ell_\infty$. Then, $\{\mu_k\tilde{\sigma}_k^{-1}\}$ is bounded, and hence

$$D_{\{\mu_k\tilde{\sigma}_k^{-1}\}}\in\mathfrak{L}(\ell_v, \ell_v).$$

Therefore we have

$$D_\mu=D_{\{\mu_k\tilde{\sigma}_k^{-1}\}}\circ D_{\{\tilde{\sigma}_k\}}\in\mathfrak{A}(\ell_u, \ell_v),$$

which implies $\ell_{(\mathfrak{A},u,v)}=\ell_\infty$. Since the inclusion map $\ell_{(\mathfrak{A},u,v)}\hookrightarrow\ell_\infty$ is continuous, by the open mapping theorem we have with some K

$$\begin{aligned} d\alpha_n^{1/s} &\leq \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \\ &= \|(\overbrace{1,\dots,1}^{n}, 0,\dots)\|_{\mathbf{A}} \\ &\leq K\|(\overbrace{1,\dots,1}^{n}, 0,\dots)\|_\infty=K \end{aligned}$$

for all $n\in\boldsymbol{N}$, from which it follows that $s=\infty$ and hence $\ell_{s,\infty}(\boldsymbol{\alpha})=\ell_\infty$. This completes the proof.

The next theorem is a generalization of Proposition 2 in König [12].

THEOREM 5. *Let* $1 \leq u, v \leq \infty$. *If* $\alpha_n \prec n$, *then*

$$\lambda_{\alpha}(\mathfrak{A}, u, v) + \lambda_{\alpha}(\mathfrak{A}^*, v, u) \geq 1 + d_{\alpha}(\mathfrak{A}, u, v).$$

If $n \prec \alpha_n$, *then the converse inequality holds.*

PROOF. Suppose that $\alpha_n \prec n$. By Corollary 5.3 in [4],

$$\mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \cdot \mathbf{A}^*(I_n\colon \ell_v^n \longrightarrow \ell_u^n) = n.$$

Hence

$$\begin{aligned}
\lambda_{\alpha}(\mathfrak{A}^*, v, u) &\geq \lambda(\mathfrak{A}^*, v, u) \\
&= \inf\{\nu \geq 0;\ \ \mathbf{A}^*(I_n\colon \ell_v^n \longrightarrow \ell_u^n) \leq \tilde{d} n^{\nu} \quad (^{\forall} n \in \boldsymbol{N})\} \\
&= \inf\{\nu \geq 0;\ \ \tilde{d}^{-1} n^{1-\nu} \leq \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \quad (^{\forall} n \in \boldsymbol{N})\} \\
&\geq \inf\{\nu \geq 0;\ \ d\alpha_n^{1-\nu} \leq \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \quad (^{\forall} n \in \boldsymbol{N})\} \\
&= \inf\{1-\mu \geq 0;\ \ \mu \geq 0,\ d\alpha_n^{\mu} \leq \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \quad (^{\forall} n \in \boldsymbol{N})\},
\end{aligned}$$

where one should observe that $\nu \leq 1$ may be assumed (cf. (19); more precisely, see [23], Theorem 6.7.2 and Lemma in 22.4.6). Therefore we have

$$\begin{aligned}
&\lambda_{\alpha}(\mathfrak{A}, u, v) + \lambda_{\alpha}(\mathfrak{A}^*, v, u) \\
&\quad \geq \inf\{\lambda \geq 0;\ \ \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \leq c\alpha_n^{\lambda} \quad (^{\forall} n \in \boldsymbol{N})\} \\
&\qquad + \inf\{1-\mu \geq 0;\ \ \mu \geq 0,\ d\alpha_n^{\mu} \leq \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \quad (^{\forall} n \in \boldsymbol{N})\} \\
&\quad = 1 + \inf\{\lambda - \mu;\ \ \lambda, \mu \geq 0,\ d\alpha_n^{\mu} \leq \mathbf{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \leq c\alpha_n^{\lambda} \quad (^{\forall} n \in \boldsymbol{N})\} \\
&\quad = 1 + d_{\alpha}(\mathfrak{A}, u, v).
\end{aligned}$$

If $n \prec \alpha_n$, then the converse inequalities "$\leq$" hold in place of "$\geq$" in the above proof.

Theorem 5, combined with Theorems 2, 3 and 4, yields

COROLLARY. *Let* $1 \leq u, v \leq \infty$. *If* $\alpha_n \prec n$, *then the condition*

$$\lambda_{\alpha}(\mathfrak{A}, u, v) + \lambda_{\alpha}(\mathfrak{A}^*, v, u) = 1$$

implies the following (i)–(iv), *which are mutually equivalent*:

(i) $d_{\alpha}(\mathfrak{A}, u, v) = 0$;

(ii) *There exists* $r > 0$ *such that for any* $\varepsilon > 0$

$$\ell_{r-\varepsilon,\infty}(\boldsymbol{\alpha}) \subset \ell_{(\mathfrak{A},u,v)} \subset \ell_{r+\varepsilon,\infty}(\boldsymbol{\alpha});$$

(iii) *There exists* $\lambda \geq 0$ *such that for any* $\varepsilon > 0$

$$d\alpha_n^{\lambda-\varepsilon} \leq \mathrm{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n) \leq c\alpha_n^{\lambda+\varepsilon} \qquad ({}^{\forall}n \in \boldsymbol{N})$$

with some constants c and d;

(iv)
$$\lambda_\alpha(\mathfrak{A}, u, v) = \lim_{n\to\infty} \frac{\log \mathrm{A}(I_n\colon \ell_u^n \longrightarrow \ell_v^n)}{\log \alpha_n}.$$

In (ii) *and* (iii), *we have* $\lambda = 1/r = \lambda_\alpha(\mathfrak{A}, u, v)$.

If $n \prec \alpha_n$, *then each of* (i)–(iv) *implies that*

$$\lambda_\alpha(\mathfrak{A}, u, v) + \lambda_\alpha(\mathfrak{A}^*, v, u) \leq 1.$$

This result generalizes Corollary 1 to Proposition 2 in König [12]. The proof is easy and is omitted.

§ 6. The L-limit order and block diagonal matrix operators

We recall that the L-limit order of a quasi-normed operator ideal $[\mathfrak{A}, \mathrm{A}]$ is defined by

(4) $\lambda_L(\mathfrak{A}, u, v)$

$$:= \inf\{\lambda > 0;\ {}^{\exists}c = c(u,v,\lambda)\ s.t.\ \mathrm{A}(A_{2^n}\colon \ell_u^{2^n} \to \ell_v^{2^n}) \leq c(2^n)^\lambda\ ({}^{\forall}n \in \boldsymbol{N}_0)\}.$$

Let us first show an identity analogous to (1) for the L-limit order. For $\lambda > 0$ we put

$$A_\lambda := \sum_{n=0}^{\infty} \oplus \frac{1}{(2^n)^\lambda} A_{2^n} = \begin{bmatrix} A_{2^0} & & & & 0 \\ & \frac{1}{2^\lambda} A_{2^1} & & & \\ & & \ddots & & \\ & & & \frac{1}{(2^n)^\lambda} A_{2^n} & \\ 0 & & & & \ddots \end{bmatrix}.$$

(Although the notations A_λ and A_{2^n} are not consistent, there will be no confusion.)

THEOREM 6. *For* $1 \leq u, v \leq \infty$

(21)
$$\lambda_L(\mathfrak{A}, u, v) = \inf\{\lambda > 0;\ A_\lambda \in \mathfrak{A}(\ell_u, \ell_v)\}.$$

PROOF. We write the right-hand side of (21) as λ_0. Suppose $A_\lambda \in \mathfrak{A}(\ell_u, \ell_v)$. Let

$$J_n\colon \ell_u^{2^n} \longrightarrow \ell_u = \sum_{k=0}^{\infty} \oplus \ell_u^{2^k}$$

and

$$P_n\colon \ell_v = \sum_{k=0}^{\infty} \oplus \ell_v^{2^k} \longrightarrow \ell_v^{2^n}$$

be the embedding and projection defined respectively by

$$J_n(\xi_1,\dots,\xi_{2^n}):=(0;\dots;0,\dots,0;\overbrace{\xi_1,\dots,\xi_{2^n}}^{2^n};0,\dots,0;\dots)$$

and

$$P_n(\xi_1;\dots;\xi_{2^n},\dots,\xi_{2^{n+1}-1};\dots):=(\xi_{2^n},\dots,\xi_{2^{n+1}-1}).$$

Then we have

$$\begin{aligned}\mathbf{A}(A_{2^n}\colon \ell_u^{2^n}\longrightarrow \ell_v^{2^n}) &\le (2^n)^\lambda \|J_n\colon \ell_u^{2^n}\longrightarrow \ell_u\|\mathbf{A}(A_\lambda\colon \ell_u\longrightarrow \ell_v)\|P_n\colon \ell_v\longrightarrow \ell_v^{2^n}\|\\ &= \mathbf{A}(A_\lambda\colon \ell_u\longrightarrow \ell_v)(2^n)^\lambda\end{aligned}$$

for all $n\in \boldsymbol{N}_0$. Hence, $\lambda_L(\mathfrak{A}, u, v)\le\lambda_0$.

Conversely, let $\mathbf{A}(A_{2^n}\colon \ell_u^{2^n}\longrightarrow \ell_v^{2^n})\le c(2^n)^\lambda$ ($^\forall n\in \boldsymbol{N}_0$). By Lemma A we may assume that $[\mathfrak{A}, \mathbf{A}]$ is a p-normed operator ideal. Then, for any $\varepsilon>0$

$$\begin{aligned}\textstyle\sum_{n=0}^\infty \mathbf{A}(2^{-(\lambda+\varepsilon)n}A_{2^n}\colon \ell_u^{2^n}\longrightarrow \ell_v^{2^n})^p &\le c^p \textstyle\sum_{n=0}^\infty 2^{-(\lambda+\varepsilon)pn}.2^{\lambda pn}\\ &= c^p\textstyle\sum_{n=0}^\infty 2^{-\varepsilon pn}<\infty .\end{aligned}$$

Hence

$$A_{\lambda+\varepsilon}=\textstyle\sum_{n=0}^\infty \oplus\, 2^{-(\lambda+\varepsilon)n}A_{2^n}\in\mathfrak{A}(\ell_u, \ell_v),$$

which means $\lambda+\varepsilon\ge\lambda_0$. Consequently we have $\lambda_L(\mathfrak{A}, u, v)\ge\lambda_0$.

COROLLARY. *If* $\lambda>\lambda_L(\mathfrak{A}, u, v)$ (*resp.* $\lambda<\lambda_L(\mathfrak{A}, u, v)$), *then* $A_\lambda\in\mathfrak{A}(\ell_u, \ell_v)$ (*resp.* $A_\lambda\notin\mathfrak{A}(\ell_u, \ell_v)$).

PROOF. Let $\lambda>\lambda_L(\mathfrak{A}, u, v)$. Then, by Theorem 6 there exists a μ with $\lambda>\mu\ge\lambda_L(\mathfrak{A}, u, v)$ such that $A_\mu\in\mathfrak{A}(\ell_u, \ell_v)$. Put

$$E_{\lambda-\mu}:=\textstyle\sum_{n=0}^\infty \oplus\, 2^{-(\lambda-\mu)n}E_{2^n},$$

where E_{2^n} are the $2^n\times 2^n$ unit matrices. Since $E_{\lambda-\mu}\in\mathfrak{L}(\ell_u, \ell_u)$, we have $A_\lambda=A_\mu E_{\lambda-\mu}\in\mathfrak{A}(\ell_u, \ell_v)$.

Next, for $\sigma=\{\sigma_n\}_{n\in\boldsymbol{N}_0}\in\ell_\infty$ we put

$$A_\sigma:=\textstyle\sum_{n=0}^\infty\oplus\,\sigma_n A_{2^n}=\begin{bmatrix}\sigma_0 A_{2^0} & & & & 0\\ & \sigma_1 A_{2^1} & & & \\ & & \ddots & & \\ & & & \sigma_n A_{2^n} & \\ 0 & & & & \ddots\end{bmatrix}.$$

Such a type of operator is used, e.g., in [13]. The following result is analogous to (5).

THEOREM 7. *For* $1 \leq u, v \leq \infty$

$$\lambda_L(\mathfrak{A}, u, v) = \inf \{1/r \geq 0; \sigma \in \ell^0_{r,\infty}(\{2^n\}) \Longrightarrow A_\sigma \in \mathfrak{A}(\ell_u, \ell_v)\}.$$

PROOF. Let us assume that $\sigma \in \ell^0_{r,\infty}(\{2^n\})$ implies $A_\sigma \in \mathfrak{A}(\ell_u, \ell_v)$. Then, $A_\lambda \in \mathfrak{A}(\ell_u, \ell_v)$ for any $\lambda > 1/r$ because $\{2^{-\lambda n}\} \in \ell^0_{r,\infty}(\{2^n\})$. Hence we have the inequality "$\leq$" by Theorem 6.

Conversely, let $1/r > \lambda_L(\mathfrak{A}, u, v)$. Then $A_{1/r} \in \mathfrak{A}(\ell_u, \ell_v)$. Let $\sigma = \{\sigma_n\} \in \ell^0_{r,\infty}(\{2^n\})$. Then

$$D := \sum_{n=0}^{\infty} \oplus\, 2^{n/r}\sigma_n E_{2^n} \in \mathfrak{L}(\ell_u, \ell_u).$$

Therefore we have $A_\sigma = A_{1/r} D \in \mathfrak{A}(\ell_u, \ell_v)$.

The following lemma refines Pietsch's results implicitly shown in [20].

LEMMA 4 (cf. [20], Lemma 12, (5), and (5*)). *Let* $1 \leq u, v \leq \infty$. *Then,*

$$\|A_{2^n} \colon \ell_u^{2^n} \longrightarrow \ell_v^{2^n}\| = 2^{n\lambda(u,v)} \tag{22}$$

($\ell_u^{2^n}$*-spaces are assumed to be complex*), *where*

$$\lambda(u, v) = \lambda_L(\mathfrak{L}, u, v) = \begin{cases} 1/u' + 1/v - 1/2 & \text{if } 2 \leq u \leq \infty,\ 1 \leq v \leq 2, \\ 1/v & \text{if } 1 \leq u \leq 2,\ 1 \leq v \leq u', \\ 1/u' & \text{if } v' \leq u \leq \infty,\ 2 \leq v \leq \infty, \end{cases}$$

$1/u + 1/u' = 1/v + 1/v' = 1$. *In particular,*

$$\|A_{2^n} \colon \ell_u^{2^n} \to \ell_u^{2^n}\| = 2^{n \cdot \max(1/u, 1/u')}.$$

PROOF. The inequality "$\leq$" of (22) is obtained in the computation of (5) in [20]. Let $2 \leq u \leq \infty$, $1 \leq v \leq 2$. Put $A_{2^n} = [\varepsilon_{jk}^{(n)}]$. We define $\sigma^{(n)} = \{\sigma_k^{(n)}\} \in \ell_u^{2^n}$ inductively as follows. Let $\sigma_1^{(1)} = 2^{-1/2} e^{-i\pi/4}$, $\sigma_2^{(1)} = 2^{-1/2} e^{i\pi/4}$, and put $\sigma_{2k-1}^{(m+1)} = \sigma_1^{(1)} \sigma_k^{(m)}$, $\sigma_{2k}^{(m+1)} = \sigma_2^{(1)} \sigma_k^{(m)}$ $(k = 1, \ldots, 2^m;\ m = 1, \ldots, n-1)$. Then, $\|A_{2^n}\sigma^{(n)}\|_v = 2^{n/v}$. Indeed, we have $|\sum_{k=1}^{2^n} \varepsilon_{jk}^{(n)} \sigma_k^{(n)}| = 1$ for $j = 1, \ldots, 2^n$; we prove it by induction. The case $n = 1$ is trivial. Assume that $|\sum_{k=1}^{2^m} \varepsilon_{jk}^{(m)} \sigma_k^{(m)}| = 1$ for $j = 1, \ldots, 2^m$. Then, since

$$\sum_{k=1}^{2^m} \varepsilon_{jk}^{(m)} \sigma_{2^m+k}^{(m+1)} = e^{i\pi/2} \sum_{k=1}^{2^m} \varepsilon_{jk}^{(m)} \sigma_k^{(m+1)} \qquad (j = 1, \ldots, 2^m)$$

(note that $\sigma_{2^m+k}^{(m+1)} = e^{i\pi/2} \sigma_k^{(m+1)}$) and

$$2^{1/2} e^{i\pi/4} \sigma_k^{(m+1)} = \sigma_k^{(m)} \qquad (k = 1, \ldots, 2^m),$$

we have for $j=1,\ldots, 2^m$

$$\begin{aligned}|\textstyle\sum_{k=1}^{2^{m+1}} \varepsilon_{jk}^{(m+1)}\sigma_k^{(m+1)}| &= |\textstyle\sum_{k=1}^{2^m} \varepsilon_{jk}^{(m)}\sigma_k^{(m+1)} + \sum_{k=1}^{2^m} \varepsilon_{jk}^{(m)}\sigma_{2^m+k}^{(m+1)}| \\ &= 2^{1/2}|\textstyle\sum_{k=1}^{2^m} \varepsilon_{jk}^{(m)}\sigma_k^{(m+1)}| \\ &= |\textstyle\sum_{k=1}^{2^m} \varepsilon_{jk}^{(m)}\sigma_k^{(m)}| = 1:\end{aligned}$$

The proof for $j=2^m+1,\ldots, 2^{m+1}$ is immediate from this. On the other hand, $\|\sigma^{(n)}\|_u=(2^n)^{1/u-1/2}$. Consequently, we have $\|A_{2^n}\sigma^{(n)}\|_v=(2^n)^{1/u'+1/v-1/2}\|\sigma^{(n)}\|_u$. In the second and last cases, the vectors $(1, 0,\ldots, 0)$ and $(1,\ldots, 1)\in \ell_u^{2^n}$ satisfy the equation $\|A_{2^n}\xi\|_v=2^{n\lambda(u,v)}\|\xi\|_u$, respectively. Thus, we obtain (22). [Note: Combined with the computations of (5) and (5*) in [20], the inequality

$$\|A_{2^n}: \ell_u^{2^n} \longrightarrow \ell_v^{2^n}\| \mathbf{N}_1(A_{2^n}: \ell_v^{2^n} \longrightarrow \ell_u^{2^n}) \geq \operatorname{trace}(2^n E_{2^n}) = 2^{2n}$$

(cf. the proof of Lemma 12 in [20]) also yields (22) except the case $2\leq u\leq\infty$, $1\leq v\leq 2$, in which it yields only

$$c_G^{-1}\, 2^{n\lambda(u,v)} \leq \|A_{2^n}: \ell_u^{2^n} \longrightarrow \ell_v^{2^n}\| \leq 2^{n\lambda(u,v)},$$

c_G (>1) being the Grothendieck constant. Here $\mathbf{N}_1$ is the nuclear norm (see (33) in §7).]

Now, let B_{2^n} be arbitrary $2^n\times 2^n$ matrices ($n\in \boldsymbol{N}_0$) and put

$$B:=\sum_{n=0}^{\infty}\oplus B_{2^n} = \begin{bmatrix} B_{2^0} & & & & 0 \\ & B_{2^1} & & & \\ & & \ddots & & \\ & & & B_{2^n} & \\ 0 & & & & \ddots \end{bmatrix}.$$

We write $\|B_{2^n}\|_{s,t}$ for $\|B_{2^n}: \ell_s^{2^n} \longrightarrow \ell_t^{2^n}\|$, $1\leq s, t\leq\infty$. In the next theorem we introduce another limit order $\mu(\mathfrak{A})$.

THEOREM 8. *Let* $1\leq u, v\leq\infty$. *Let* $\kappa(t)=\min(1/t, 1/t')$ *for* $1\leq t\leq\infty$, *where* $1/t+1/t'=1$.

(i) *If* $\sup_{n\in\boldsymbol{N}_0}(2^n)^{\lambda-\kappa(t)}\|B_{2^n}\|_{t,t}<\infty$ *for* $t=u$ *or* v *with some* $\lambda>\lambda_L(\mathfrak{A}, u, v)$, *then*

$$B = \sum_{n=0}^{\infty}\oplus B_{2^n}\in \mathfrak{A}(\ell_u, \ell_v).$$

(ii) *Let*

$$\mu(\mathfrak{A}, u, v):= \inf\{\mu>0;\ \sup_{n\in\boldsymbol{N}_0}(2^n)^{\mu}\|B_{2^n}\|_{t,t}<\infty\ (t=u \text{ or } v) \Longrightarrow B\in\mathfrak{A}(\ell_u, \ell_v)\}.$$

Then,

$$\lambda(\mathfrak{A}, u, v) \leq \mu(\mathfrak{A}, u, v) \leq \lambda_L(\mathfrak{A}, u, v) - \max\{\kappa(u), \kappa(v)\}. \tag{23}$$

PROOF. (i) Let us assume that $\sup_{n\in N_0}(2^n)^{\lambda-\kappa(u)}\|B_{2^n}\|_{u,u}<\infty$. (The proof for $t=v$ is similar.) Put

$$C := \textstyle\sum_{n=0}^{\infty} \oplus\, 2^{(\lambda-1)n} A_{2^n} B_{2^n}.$$

Then, $C\in\mathfrak{L}(\ell_u, \ell_u)$ because by Lemma 4

$$\begin{aligned}\sup_n \|2^{(\lambda-1)n}A_{2^n}B_{2^n}\|_{u,u} &\leq \sup_n 2^{(\lambda-1)n}\|A_{2^n}\|_{u,u}\|B_{2^n}\|_{u,u}\\ &= \sup_n (2^n)^{\lambda-1}(2^n)^{\max(1/u,1/u')}\|B_{2^n}\|_{u,u}\\ &= \sup_n (2^n)^{\lambda-\kappa(u)}\|B_{2^n}\|_{u,u} < \infty.\end{aligned}$$

(Note that C is block diagonal.) On the other hand,

$$A_\lambda = \textstyle\sum_{n=0}^{\infty} \oplus 2^{-\lambda n} A_{2^n} \in \mathfrak{A}(\ell_u, \ell_v)$$

by the assumption $\lambda > \lambda_L(\mathfrak{A}, u, v)$. Since $A_{2^n}^2 = 2^n E_{2^n}$, we have

$$A_\lambda C = \textstyle\sum_{n=0}^{\infty} \oplus\, 2^{-n} A_{2^n}^2 B_{2^n} = \sum_{n=0}^{\infty} \oplus\, B_{2^n} = B,$$

and consequently, $B\in\mathfrak{A}(\ell_u, \ell_v)$.

(ii) From (i) it immediately follows that

$$\mu(\mathfrak{A}, u, v) \leq \lambda_L(\mathfrak{A}, u, v) - \kappa(t) \qquad \text{for} \quad t = u \text{ and } v,$$

or the second inequality of (23). Let us suppose that $\sup_{n\in N_0}(2^n)^{\mu}\|B_{2^n}\|_{t,t}<\infty$ for $t=u$ or v implies $B\in\mathfrak{A}(\ell_u, \ell_v)$. Put

$$B_{2^n} := \begin{bmatrix} (2^n)^{-\mu} & & & 0 \\ & (2^n+1)^{-\mu} & & \\ & & \ddots & \\ 0 & & & (2^{n+1}-1)^{-\mu} \end{bmatrix}.$$

Then, $D_\mu = \sum_{n=0}^{\infty} \oplus\, B_{2^n} \in \mathfrak{A}(\ell_u, \ell_v)$ since $\sup_{n\in N_0}(2^n)^{\mu}\|B_{2^n}\|_{t,t}<\infty$. Consequently, we have

$$\lambda(\mathfrak{A}, u, v) \leq \mu,$$

which implies the first inequality of (23).

In Theorem 10 in the next section we shall show

$$\lambda(\mathfrak{A}, u, v) + \max\{\kappa(u), \kappa(v)\} \le \lambda_L(\mathfrak{A}, u, v) \le \lambda(\mathfrak{A}, u, v) + 1 - \max\{\kappa(u), \kappa(v)\}. \tag{24}$$

Combined with this, Theorem 8 yields

COROLLARY 1. *Let* $1 \le u, v \le \infty$. *Then, we have*

$$\lambda_L(\mathfrak{A}, u, v) - 1 + \max\{\kappa(u), \kappa(v)\} \le \mu(\mathfrak{A}, u, v) \le \lambda_L(\mathfrak{A}, u, v) - \max\{\kappa(u), \kappa(v)\} \tag{25}$$

and

$$\lambda(\mathfrak{A}, u, v) \le \mu(\mathfrak{A}, u, v) \le \lambda(\mathfrak{A}, u, v) + 1 - 2\max\{\kappa(u), \kappa(v)\}. \tag{26}$$

In particular,

$$\mu(\mathfrak{A}, u, v) = \lambda(\mathfrak{A}, u, v) = \lambda_L(\mathfrak{A}, u, v) - \frac{1}{2} \qquad \text{if } u=2 \text{ or } v=2. \tag{27}$$

Combined with (24), (26) and (27), Theorems 6, 7 and 8 yield criteria by $\lambda(\mathfrak{A}, u, v)$ such that a block diagonal matrix operator belongs to $\mathfrak{A}(\ell_u, \ell_v)$. Taking account of the fact that the limit order $\lambda(\mathfrak{A}, u, v)$ is extensively calculated for various special ideals $\mathfrak{A}$ (cf. [23], 14.4 and 22.4–6; [2]), these inequalities and identities would be useful. In particular, by Theorem 8 with (26) and (25) we obtain

COROLLARY 2. *Let* $1 \le u, v \le \infty$. *Let*

$$\sup_{n \in N_0} (2^n)^{\mu} \|B_{2^n}\|_{t,t} < \infty \qquad (t = u \quad \text{or} \quad v)$$

for some μ *with*

$$\mu > \lambda(\mathfrak{A}, u, v) + 1 - 2\max\{\kappa(u), \kappa(v)\}$$

or

$$\mu > \lambda_L(\mathfrak{A}, u, v) - \max\{\kappa(u), \kappa(v)\}.$$

Then,

$$B = \sum_{n=0}^{\infty} \oplus B_{2^n} \in \mathfrak{A}(\ell_u, \ell_v).$$

This result may be compared with the following one given by Pietsch [24] recently.

PROPOSITION B ([24], Theorem 1). *Let* $0 < p < \infty$ *and* $S \in \mathfrak{L}(E, F)$. *Then,*

$S \in \mathfrak{A}_p$ if and only if there exists a sequence $\{S_n\}$ in $\mathfrak{L}(E, F)$ with rank $(S_n) \leq 2^n$ *and $\sum_{n=0}^{\infty} 2^n \|S_n\|^p < \infty$ such that $S = \sum_{n=0}^{\infty} S_n$.*

Combining Corollary 2 and Proposition B, we have

COROLLARY 3. *Let $1 \leq u, v \leq \infty$. Let $0 < p < \infty$ and*

$$\frac{1}{p} > \lambda(\mathfrak{A}, u, v) + 1 - 2\max\{\kappa(u), \kappa(v)\}$$

or

$$\frac{1}{p} > \lambda_L(\mathfrak{A}, u, v) - \max\{\kappa(u), \kappa(v)\}.$$

Assume that

$$\sum_{n=0}^{\infty} 2^n \|B_{2^n}\|_{s,t}^p < \infty,$$

where $(s, t) = (u, u)$ or (v, v) if $u \leq v$ and $(s, t) = (u, v)$ if $u \geq v$. Then,

$$B = \sum_{n=0}^{\infty} \oplus B_{2^n} \in (\mathfrak{A} \cap \mathfrak{A}_p)(\ell_u, \ell_v).$$

The proof is immediate by observing that

$$\|B_{2^n}\|_{u,v} \leq \|B_{2^n}\|_{u,u}, \|B_{2^n}\|_{v,v} \qquad \text{if } u \leq v$$

and

$$\|B_{2^n}\|_{u,u}, \|B_{2^n}\|_{v,v} \leq \|B_{2^n}\|_{u,v} \qquad \text{if } u \geq v.$$

Now, we show that $\lambda_L(\mathfrak{A}, u, v)$ gives the same criteria as in Theorem 6 or its Corollary, and Theorem 7 for (block diagonal matrix) operators between Lorentz sequence spaces $\ell_{u,s}$ and $\ell_{v,t}$. Some results of this type for $\lambda(\mathfrak{A}, u, v)$ are obtained in [17] and [8].

The following lemma is easily derived from the property (QN_3) of quasi-normed operator ideals (cf. (1.4) in [10]).

LEMMA 5. *For $1 \leq u_1, u_2, v_1, v_2 \leq \infty$,*

$$|\lambda_L(\mathfrak{A}, u_1, v_1) - \lambda_L(\mathfrak{A}, u_2, v_2)| \leq \left|\frac{1}{u_1} - \frac{1}{u_2}\right| + \left|\frac{1}{v_1} - \frac{1}{v_2}\right|.$$

THEOREM 6′. *Let $1 \leq u, v, s, t \leq \infty$. Then,*

$$\lambda_L(\mathfrak{A}, u, v) = \inf\{\lambda > 0;\ A_\lambda \in \mathfrak{A}(\ell_{u,s}, \ell_{v,t})\}. \tag{28}$$

PROOF. Let us show the inequality "$\leq$". If $1 < u \leq \infty$ and $1 \leq v < \infty$, take arbitrary u_1 and v_1 with $1 < u_1 < u$ and $v < v_1 < \infty$. Then, the inclusion maps $\ell_{u_1} \hookrightarrow \ell_{u,s}$ and $\ell_{v,t} \hookrightarrow \ell_{v_1}$ are continuous by Lemma C. Hence, $A_\lambda \in \mathfrak{A}(\ell_{u,s}, \ell_{v,t})$

implies $A_\lambda \in \mathfrak{A}(\ell_{u_1}, \ell_{v_1})$. By Theorem 6 this implies

$$(29) \qquad \lambda_L(\mathfrak{A}, u_1, v_1) \le \inf\{\lambda > 0;\ A_\lambda \in \mathfrak{A}(\ell_{u,s}, \ell_{v,t})\}.$$

Letting $u_1 \to u$ and $v_1 \to v$, we have the desired inequality by Lemma 5. If $u=1$ or $v=\infty$, we have only to put $u_1=u=1$ or $v_1=v=\infty$.

In a similar way, we obtain the converse inequality of (29) for any u_1 and v_1 with $1 \le u \le u_1 \le \infty$ and $1 \le v_1 \le v \le \infty$, and hence the inequality "$\ge$" of (28).

COROLLARY. *Let $1 \le u, v, s, t \le \infty$. If $\lambda > \lambda_L(\mathfrak{A}, u, v)$ (resp. $\lambda < \lambda_L(\mathfrak{A}, u, v)$), then $A_\lambda \in \mathfrak{A}(\ell_{u,s}, \ell_{v,t})$ (resp. $A_\lambda \notin \mathfrak{A}(\ell_{u,s}, \ell_{v,t})$).*

By Theorem 6′ we have easily

THEOREM 7′. *Let $1 \le u, v, s, t \le \infty$. Then,*

$$\lambda_L(\mathfrak{A}, u, v) = \inf\{1/r \ge 0;\ \sigma \in \ell^0_{r,\infty}(\{2^n\}) \Longrightarrow A_\sigma \in \mathfrak{A}(\ell_{u,s}, \ell_{v,t})\}.$$

In the rest of this section, we give a representation of $\lambda_L(\mathfrak{L}, u, v)$ which is closely related with Clarkson's inequalities. Let $\mathscr{L}_p = \mathscr{L}_p(X, \mathscr{M}, \mu)$ be the usual (complex) $\mathscr{L}_p$-space, $1 \le p < \infty$, on an arbitrary but fixed measure space $(X, \mathscr{M}, \mu)$. Let $\ell^n_u(\mathscr{L}_p)$, $1 \le u \le \infty$, denote the direct sum of n copies of $\mathscr{L}_p$ with the norm

$$|||\boldsymbol{f}|||_{u(p)} = \begin{cases} (\sum_{j=1}^n \|f_j\|_p^u)^{1/u} & (1 \le u < \infty), \\ \max_{1 \le j \le n} \|f_j\|_p & (u = \infty) \end{cases}$$

for $\boldsymbol{f} = \{f_j\} \in \ell^n_u(\mathscr{L}_p)$. In [9] the author showed the following

THEOREM 9 ([9], Theorems 1 and 3). (i) *Let $1 < p < \infty$ and $1 \le u, v \le \infty$. Assume that $\mathscr{M}$ contains infinitely many (countable) mutually disjoint sets of finite positive measure. Then, for every $n \in \boldsymbol{N}_0$*

$$(30) \qquad \|A_{2^n}\colon \ell^{2^n}_u(\mathscr{L}_p) \longrightarrow \ell^{2^n}_v(\mathscr{L}_p)\| = 2^{nc(u,v;p)},$$

where

$$c(u, v; p) = \begin{cases} \dfrac{1}{u'} + \dfrac{1}{v} - \min\left(\dfrac{1}{p}, \dfrac{1}{p'}\right) & \text{if } \min(p, p') \le u \le \infty, \\ & \qquad 1 \le v \le \max(p, p'), \\ \dfrac{1}{v} & \text{if } 1 \le u \le \min(p, p'),\ 1 \le v \le u', \\ \dfrac{1}{u'} & \text{if } v' \le u \le \infty,\ \max(p, p') \le v \le \infty, \end{cases}$$

$1/p + 1/p' = 1/u + 1/u' = 1/v + 1/v' = 1$. *In particular,*

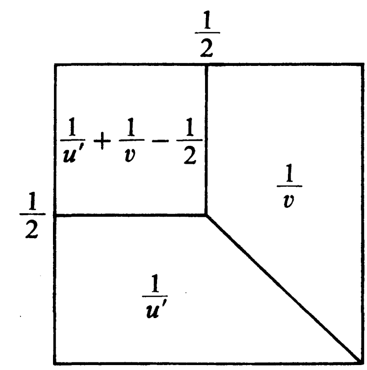
(31) $$\|A_{2^n}: \ell_u^{2^n}(\mathscr{L}_2) \longrightarrow \ell_v^{2^n}(\mathscr{L}_2)\| = 2^{n\lambda(u,v)},$$

$\lambda(u, v)$ being as in Lemma 4.

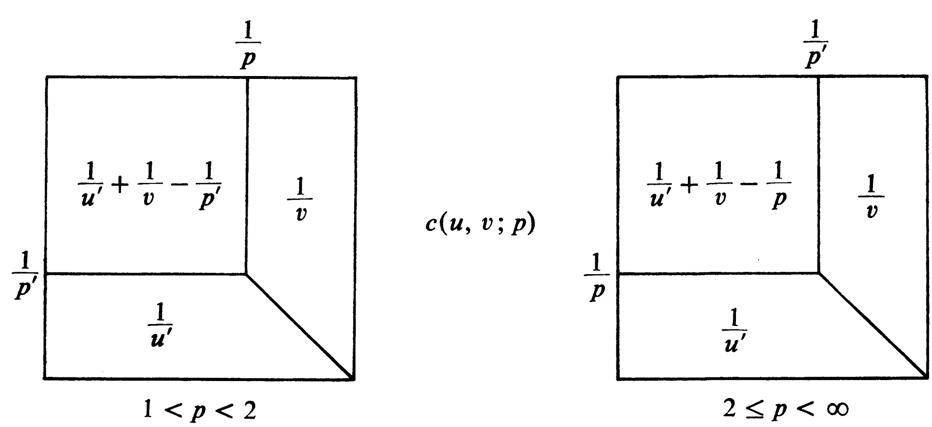
(ii) *Let $1 \leq q \leq p < \infty$. Assume that $\mu(X) < \infty$. Then,*

(32) $$\|A_{2^n}: \ell_p^{2^n}(\mathscr{L}_p) \longrightarrow \ell_q^{2^n}(\mathscr{L}_q)\| = \mu(X)^{1/q-1/p}2^{n\lambda(p,q)}.$$

To compare the norms (22) and (30) of A_{2^n} in $\ell_u^{2^n}$- and $\ell_u^{2^n}(\mathscr{L}_p)$-spaces, it is convenient to express $\lambda(u, v)$ and $c(u, v; p)$ graphically in the unit squares with the coordinates $1/u$ and $1/v$ as follows.



$\lambda(u, v)$



$c(u, v; p)$

By (30)–(32) and Lemma 4, we have

COROLLARY. (i) *Let $1 < p < \infty$ and $1 \leq u, v \leq \infty$. Assume that $\mathscr{M}$ contains* *infinitely many (countable) mutually disjoint sets of finite positive measure. Then,*

$$\lambda_L(\mathfrak{L}, u, v) + \delta(u, v; p)$$
$$= \inf\{\lambda > 0;\ {}^{\exists}c = c(u,v,\lambda) \text{ s.t. } \|A_{2^n}: \ell_u^{2^n}(\mathscr{L}_p) \to \ell_v^{2^n}(\mathscr{L}_p)\| \le c(2^n)^\lambda \quad ({}^{\forall}n \in \mathbf{N}_0)\},$$

where

$$\delta(u, v; p) = \begin{cases} \dfrac{1}{2} - \kappa(p) & \text{if } 1 \le v \le 2 \le u \le \infty, \\ \dfrac{1}{u'} - \kappa(p) & \text{if } \min(p, p') \le u \le 2,\ 1 \le v \le u', \\ \dfrac{1}{v} - \kappa(p) & \text{if } v' \le u \le \infty,\ 2 \le v \le \max(p, p'), \\ 0 & \text{if } 1 \le u \le \min(p, p') \text{ or } \max(p, p') \le v \le \infty, \end{cases}$$

$\kappa(p) = \min(1/p, 1/p')$. *In particular,*

$$\lambda_L(\mathfrak{L}, u, v)$$
$$= \inf\{\lambda > 0;\ {}^{\exists}c = c(u,v,\lambda) \text{ s.t. } \|A_{2^n}: \ell_u^{2^n}(\mathscr{L}_2) \to \ell_v^{2^n}(\mathscr{L}_2)\| \le c(2^n)^\lambda \quad ({}^{\forall}n \in \mathbf{N}_0)\}.$$

(ii) *Let* $1 \le v \le u < \infty$. *Assume that* $\mu(X) < \infty$. *Then,*

$$\lambda_L(\mathfrak{L}, u, v)$$
$$= \inf\{\lambda > 0;\ {}^{\exists}c = c(u,v,\lambda) \text{ s.t. } \|A_{2^n}: \ell_u^{2^n}(\mathscr{L}_u) \to \ell_v^{2^n}(\mathscr{L}_v)\| \le c(2^n)^\lambda \quad ({}^{\forall}n \in \mathbf{N}_0)\}.$$

For (i), observe that

$$\|A_{2^n}: \ell_u^{2^n}(\mathscr{L}_p) \longrightarrow \ell_v^{2^n}(\mathscr{L}_p)\| = 2^{n\delta(u,v;p)}\|A_{2^n}: \ell_u^{2^n} \longrightarrow \ell_v^{2^n}\|.$$

REMARK 3. *We write* $A_{2^n} = [\varepsilon_{ij}]$. *Then,* (30) *in Theorem* 9 *yields the inequality*

$$\Big(\sum_{i=1}^{2^n} \Big\|\sum_{j=1}^{2^n} \varepsilon_{ij} f_j\Big\|_p^v\Big)^{1/v} \le 2^{nc(u,v;p)}\Big(\sum_{j=1}^{2^n} \|f_j\|_p^u\Big)^{1/u} \qquad ({}^{\forall}f_1, \dots, f_{2^n} \in \mathscr{L}_p)$$

(*the usual modification is required if* $u = \infty$ *or* $v = \infty$). This includes as special cases all the following well-known inequalities given by J. A. Clarkson [3] and R. P. Boas [1]: For all f and g in $\mathscr{L}_p$,

$$(\|f+g\|_p^p + \|f-g\|_p^p)^{1/p} \le \begin{cases} 2^{1/p}(\|f\|_p^p + \|g\|_p^p)^{1/p} & \text{if } 1 < p < 2; \\ 2^{1/p'}(\|f\|_p^p + \|g\|_p^p)^{1/p} & \text{if } 2 \le p < \infty; \end{cases}$$

$$(\|f+g\|_p^{p'} + \|f-g\|_p^{p'})^{1/p'} \le 2^{1/p'}(\|f\|_p^p + \|g\|_p^p)^{1/p} \qquad \text{if } 1 < p < 2;$$

$$(\|f+g\|_p^p + \|f-g\|_p^p)^{1/p} \le 2^{1/p}(\|f\|_p^{p'} + \|g\|_p^{p'})^{1/p'} \qquad \text{if } 2 \le p < \infty,$$

where $1/p+1/p'=1$ (Clarkson [3], Theorem 2; see also E. Hewitt and K. Stromberg [5], §15); and including them except the first inequality for $1<p<2$,

$$(\|f+g\|_p^v+\|f-g\|_p^v)^{1/v}\le 2^{1/u'}(\|f\|_p^u+\|g\|_p^u)^{1/u}$$

holds for $1<u\le p\le v<\infty$ and $u'\le v$, $1/u+1/u'=1$ (Boas [1], Theorem 1).

§7. A relation between $\lambda_L(\mathfrak{A})$ and $\lambda(\mathfrak{A})$

THEOREM 10. *Let* $1\le u,\ v\le\infty$. *Let* $\kappa(t)=\min(1/t,\ 1/t')$, $1\le t\le\infty$, *where* $1/t+1/t'=1$. *Then,*

$$\lambda(\mathfrak{A},u,v)+\max\{\kappa(u),\kappa(v)\}\le\lambda_L(\mathfrak{A},u,v)\le\lambda(\mathfrak{A},u,v)+1-\max\{\kappa(u),\kappa(v)\}. \tag{24}$$

In particular,

$$\lambda_L(\mathfrak{A},u,v)=\lambda(\mathfrak{A},u,v)+\frac{1}{2}\qquad \text{if}\quad u=2\quad\text{or}\quad v=2.$$

PROOF. Let us first show the second inequality. Suppose that $\mathbf{A}(I_n\colon \ell_u^n\to\ell_v^n)\le cn^\lambda$ ($^\forall n\in\boldsymbol{N}$) with some c. Then, by Lemma 4

$$\mathbf{A}(A_{2^n}\colon \ell_u^{2^n}\longrightarrow\ell_v^{2^n})\le\begin{cases}\|A_{2^n}\colon\ell_u^{2^n}\longrightarrow\ell_u^{2^n}\|\mathbf{A}(I_{2^n}\colon\ell_u^{2^n}\longrightarrow\ell_v^{2^n}),\\ \mathbf{A}(I_{2^n}\colon\ell_u^{2^n}\longrightarrow\ell_v^{2^n})\|A_{2^n}\colon\ell_v^{2^n}\longrightarrow\ell_v^{2^n}\|\end{cases}$$

$$\le\begin{cases}c(2^n)^{\lambda+\max(1/u,1/u')},\\ c(2^n)^{\lambda+\max(1/v,1/v')}.\end{cases}$$

for all $n\in\boldsymbol{N}_0$. Since $\max(1/t,\ 1/t')=1-\kappa(t)$, we obtain the desired inequality.

The first inequality in (24) has already been obtained in Theorem 8; it can be also shown directly as follows. Let $\mathbf{A}(A_{2^n}\colon\ell_u^{2^n}\to\ell_v^{2^n})\le c(2^n)^\lambda$ ($^\forall n\in\boldsymbol{N}_0$). Then, using the identity $A_{2^n}^2=2^nE_{2^n}$ and Lemma 4, we have

$$\mathbf{A}(I_{2^n}\colon\ell_u^{2^n}\longrightarrow\ell_v^{2^n})\le c(2^n)^{\lambda-\kappa(t)}\qquad\text{for } t=u \text{ and } v.$$

Consequently, by (QN_3),

$$\lambda(\mathfrak{A},u,v)=\inf\{\lambda>0;\ ^\exists c=c(u,v,\lambda)\ s.t.\ \mathbf{A}(I_{2^n}\colon\ell_u^{2^n}\to\ell_v^{2^n})\le c(2^n)^\lambda(^\forall n\in\boldsymbol{N}_0)\}$$
$$\le\lambda-\max\{\kappa(u),\kappa(v)\}.$$

By Theorem 10 and (19) we have

COROLLARY. *If* $[\mathfrak{A}, \mathbf{A}]$ *is a normed operator ideal, then*

$$\max\{\kappa(u), \kappa(v)\} \leq \lambda_L(\mathfrak{A}, u, v) \leq 2 - \max\{\kappa(u), \kappa(v)\}$$

for $1 \leq u, v \leq \infty$.

We finally observe that (24) in Theorem 10 is best possible for most values of u and v in the sense that equality occurs in each inequality of (24) with suitable ideals. Let us first recall the definitions of the ideals $\mathfrak{N}_p$ and $\mathfrak{P}_p$ $(1 \leq p < \infty)$ of p-nuclear and absolutely p-summing operators respectively. An operator $S \in \mathfrak{L}(E, F)$ is called *p-nuclear* ([18]; [23], 18.2.1) if it is represented as

$$Sx = \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n \qquad \text{for all} \quad x \in E$$

with $\{a_n\} \subset E'$ and $\{y_n\} \subset F$ such that

$$(\sum_{n=1}^{\infty} \|a_n\|^p)^{1/p} < \infty$$

and

$$\sup\{(\sum_{n=1}^{\infty} |\langle y_n, b \rangle|^{p'})^{1/p'};\ \|b\| \leq 1,\ b \in F'\} < \infty.$$

Put

$$\text{(33)} \qquad \mathbf{N}_p(S) := \inf\,[(\sum_{n=1}^{\infty} \|a_n\|^p)^{1/p} \sup_{\|b\| \leq 1} (\sum_{n=1}^{\infty} |\langle y_n, b \rangle|^{p'})^{1/p'}],$$

where the infimum is taken over all such representations of S as above. An operator $S \in \mathfrak{L}(E, F)$ is called *absolutely p-summing* ([19]; [23], 17.3.1) if there exists a constant $\rho \geq 0$ such that for every finite system of elements $x_1, x_2, \ldots, x_n \in E$,

$$(\sum_{i=1}^{n} \|Sx_i\|^p)^{1/p} \leq \rho \sup\{(\sum_{i=1}^{n} |\langle x_i, a \rangle|^p)^{1/p};\ \|a\| \leq 1,\ a \in E'\}.$$

The infimum of all such ρ is denoted by $\Pi_p(S)$. $[\mathfrak{N}_p, \mathbf{N}_p]$ and $[\mathfrak{P}_p, \Pi_p]$ are normed operator ideals.

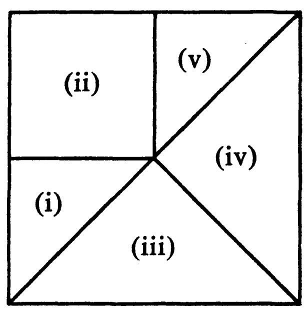
REMARK 4. *In the inequalities of* (24) *in Theorem* 10, *equality is attained as in the following table*:

	left	right
$1 \leq u, v \leq 2$	$\mathfrak{N}_1$	$\mathfrak{L}$
$1 \leq u \leq 2 \leq v \leq \infty$	$\mathfrak{L}$	
$1 \leq v \leq 2 \leq u \leq \infty$		$\mathfrak{N}_1, \mathfrak{P}_1$
$2 \leq u, v \leq \infty$	$\mathfrak{N}_1, \mathfrak{P}_1$	$\mathfrak{L}$

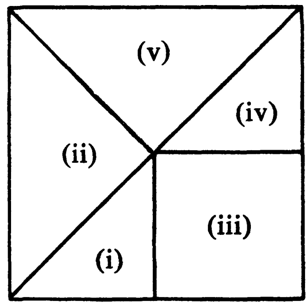
In fact, $\lambda(\mathfrak{A}, u, v)$ and $\lambda_L(\mathfrak{A}, u, v)$ are calculated for $\mathfrak{A}=\mathfrak{L}, \mathfrak{N}_p$, and $\mathfrak{P}_p$ in Pietsch [20] (see also [23], 22.4), from which we obtain the following.

$$(34)\qquad \lambda_L(\mathfrak{L}, u, v)=\lambda(\mathfrak{L}, u, v)+\begin{cases} \frac{1}{v'} & \text{if } \text{(i) } 0\le\frac{1}{u}\le\frac{1}{v}\le\frac{1}{2}, \\ \frac{1}{2} & \text{if } \text{(ii) } 0\le\frac{1}{u}\le\frac{1}{2}\le\frac{1}{v}\le 1, \\ \frac{1}{u'} & \text{if } \text{(iii) } 0\le\frac{1}{v}\le\min\left(\frac{1}{u}, \frac{1}{u'}\right), \\ \frac{1}{v} & \text{if } \text{(iv) } \frac{1}{2}\le\frac{1}{u}\le 1,\ \frac{1}{u'}\le\frac{1}{v}\le\frac{1}{u}, \\ \frac{1}{u} & \text{if } \text{(v) } \frac{1}{2}\le\frac{1}{u}\le 1,\ \frac{1}{u}\le\frac{1}{v}\le 1. \end{cases}$$

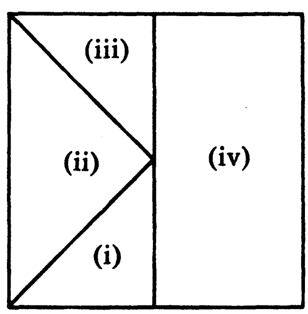
The classification in (34) is graphically expressed as



$$(35)\qquad \lambda_L(\mathfrak{N}_1, u, v)=\lambda(\mathfrak{N}_1, u, v)+\begin{cases} \frac{1}{u} & \text{if } \text{(i) } 0\le\frac{1}{v}\le\frac{1}{u}\le\frac{1}{2}, \\ \frac{1}{v} & \text{if } \text{(ii) } 0\le\frac{1}{u}\le\frac{1}{2},\ \frac{1}{u}\le\frac{1}{v}\le\frac{1}{u'}, \\ \frac{1}{2} & \text{if } \text{(iii) } 0\le\frac{1}{v}\le\frac{1}{2}\le\frac{1}{u}\le 1, \\ \frac{1}{v'} & \text{if } \text{(iv) } \frac{1}{2}\le\frac{1}{v}\le\frac{1}{u}\le 1, \\ \frac{1}{u'} & \text{if } \text{(v) } \max\left(\frac{1}{u}, \frac{1}{u'}\right)\le\frac{1}{v}\le 1: \end{cases}$$



$$(36)\quad \lambda_L(\mathfrak{P}_1, u, v) = \lambda(\mathfrak{P}_1, u, v) + \begin{cases} \frac{1}{u} & \text{if } (\mathrm{i})\ 0 \le \frac{1}{v} \le \frac{1}{u} \le \frac{1}{2}, \\ \frac{1}{v} & \text{if } (\mathrm{ii})\ 0 \le \frac{1}{u} \le \frac{1}{2},\ \frac{1}{u} \le \frac{1}{v} \le \frac{1}{u'}, \\ \frac{1}{u'} & \text{if } (\mathrm{iii})\ 0 \le \frac{1}{u} \le \frac{1}{2},\ \frac{1}{u'} \le \frac{1}{v} \le 1, \\ \frac{1}{2} & \text{if } (\mathrm{iv})\ \frac{1}{2} \le \frac{1}{u} \le 1: \end{cases}$$



Now, let $0 \le 1/u,\ 1/v \le 1/2$. Then, the inequalities (24) are precisely

$$(37)\quad \lambda(\mathfrak{A}, u, v) + \frac{1}{u} \le \lambda_L(\mathfrak{A}, u, v) \le \lambda(\mathfrak{A}, u, v) + \frac{1}{u'} \quad \text{if} \quad 0 \le \frac{1}{v} \le \frac{1}{u} \le \frac{1}{2}$$

and

$$(38)\quad \lambda(\mathfrak{A}, u, v) + \frac{1}{v} \le \lambda_L(\mathfrak{A}, u, v) \le \lambda(\mathfrak{A}, u, v) + \frac{1}{v'} \quad \text{if} \quad 0 \le \frac{1}{u} \le \frac{1}{v} \le \frac{1}{2}.$$

From (34)–(36) we conclude that both in (37) and (38), equality is attained on the left with $\mathfrak{A} = \mathfrak{N}_1$ and $\mathfrak{P}_1$, and on the right with $\mathfrak{A} = \mathfrak{L}$. This proves the assertion of Remark 4 for $2 \le u,\ v \le \infty$. The desired conclusion for the other cases is also derived from (34)–(36) in a similar way.

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