

Submanifolds of Kaehlerian manifolds and metric compound structures

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(Received December 17, 1982)

Introduction

The research of submanifolds in Kaehlerian and almost complex manifolds is a wide and interesting branch of differential geometry and many differential geometers have concerned themselves with it.

In 1955, J. A. Schouten and K. Yano [15, 16, 17] introduced the notion of *invariant* (or *holomorphic*) submanifolds in an almost complex manifold and proved that an invariant submanifold in a Kaehlerian manifold is itself Kaehlerian and minimal. In 1963, Y. Tashiro [19] showed that a real hypersurface in an almost complex manifold has an almost contact structure, and later [20] introduced the notion of *semi-invariant* submanifolds.

D. E. Blair, G. D. Ludden and K. Yano [4] first studied, in 1970, the structure induced on certain submanifolds of codimension 2 in almost Hermitian manifolds or certain hypersurfaces in almost contact metric manifolds, which is nowadays called an (f, g, u, v, λ) -structure. The structure have been researched in the papers [3, 5, 6, 9, 10, 12, 25, 26, 30] of them and S. S. Eum, S. Goldberg, S. Ishihara, U. -H. Ki and M. Okumura. K. Yano and U. -H. Ki [29] have recently studied the $(f, g, u, v, w, \lambda, \mu, \nu)$ -structure induced on submanifolds of codimension 3 in almost Hermitian manifolds. Submanifolds of other kinds in almost Hermitian manifolds are *anti-invariant* (see K. Yano and M. Kon [27]), *generic* (see K. Yano and M. Kon [28]) and *CR-submanifolds* (see A. Bejancu [1], D. E. Blair and B. Y. Chen [2]).

In order to see the above-mentioned submanifolds from an integrated viewpoint, Y. Tashiro and the present author [23] introduced the notion of *metric compound structure* (f, g, v, f^\perp) on a Riemannian manifold, which is an abstraction of the structure induced on submanifolds in almost Hermitian manifolds. Each of the structures is characterized by the rank r of the matrix v , or the dimension of the normal distribution D^r defined by v . In the previous paper, we proved that, if $r=1$, the structure defines an almost contact metric one on the manifold, and studied in details properties of submanifolds with such a structure in an even-dimensional Euclidean space.

In the present paper, we shall see that some scalar fields are associated with a metric compound structure of rank r and these scalar fields are used to classify

invariant, anti-invariant, CR - and other submanifolds in almost Hermitian manifolds. In the case of $r=2$, we shall define the structure (f, g, D^2, λ) , which is equivalent to the (f, g, u, v, λ) -structure if $\lambda \neq \pm 1$ almost everywhere. The main purpose is to investigate geometric structures of Riemannian manifolds and submanifolds of Kaehlerian manifolds with (f, g, D^2, λ) -structure satisfying some additional properties. Conditions for such a manifold to be a warped product having a contact metric or Sasakian part and to be a space of constant curvature will be obtained in explicit ways using an adapted coordinate system.

We shall recall the notion of metric compound structures from [23] in Paragraph 1. After giving a brief survey of metric compound structures of rank r on Riemannian manifolds, we discuss conditions for the structures to induce (f, g, D^2, λ) -structures in Paragraph 2. In Paragraph 3, we shall treat λ -curves and λ -hypersurfaces of Riemannian manifolds admitting a general concircular scalar field λ . In Paragraph 4, a Riemannian manifold with (f, g, D^2, λ) -structure will be dealt with and it will be shown that, under certain conditions, the manifold is a space of constant curvature or a sphere. In Paragraph 5, we shall obtain fundamental formulas on submanifolds with (f, g, D^2, λ) -structure in Kaehlerian manifolds and investigate properties of pseudo-umbilical, minimal and CR -submanifolds. In Paragraph 6, we shall prove that any λ -hypersurface of a submanifold with (f, g, D^2, λ) -structure is AS -homothetic to Sasakian manifolds if there is an umbilical 2-section and that under additional assumptions such a submanifold is a warped product having a Sasakian part, a space of constant curvature or a sphere. Paragraph 7 will be devoted to researches of properties of submanifolds with normal (f, g, D^2, λ) -structure.

Throughout this paper we assume that manifolds and quantities are differentiable of class C^∞ . Unless otherwise stated, indices run over the following ranges

$$\kappa, \lambda, \mu, \nu, \dots = 1, 2, 3, \dots, m,$$

$$h, i, j, k, \dots = 1, 2, 3, \dots, n,$$

$$p, q, r, s, \dots = n+1, n+2, \dots, m,$$

$$a, b, c, d, \dots = 2, 3, \dots, n$$

respectively and summation convention is applied to repeated indices on their own ranges.

The author would like to express his deep appreciation to Professor Y. Tashiro who gave him extremely valuable suggestions and personal support during his studies of these problems, and to Professor K. Okamoto who gave him continuous encouragements and advices to improve this paper.

1. Metric compound structures

Let \tilde{M} be an almost Hermitian manifold of dimension m with structure (G, J) , where G is the almost Hermitian metric tensor and J the almost complex structure. The structure (G, J) satisfies the equation

$$J^2 = -I,$$

I being the identity tensor field of \tilde{M} , and

$$(1.1) \quad \begin{aligned} G(J\tilde{X}, J\tilde{Y}) &= G(\tilde{X}, \tilde{Y}), \\ G(J\tilde{X}, \tilde{Y}) + G(\tilde{X}, J\tilde{Y}) &= 0 \end{aligned}$$

for any vector fields \tilde{X} and \tilde{Y} on \tilde{M} .

Let M be a differentiable manifold of dimension n and ι an immersion of M into \tilde{M} . In terms of local coordinates (x^h) of M and (y^k) of \tilde{M} , the immersion ι is locally expressed by the parametric equations

$$y^k = y^k(x^h).$$

If we put

$$B_i^k = \partial_i y^k, \quad \partial_i = \partial/\partial x^i,$$

then $B_j = (B_j^k)$ are n local vector fields on M spanning the tangent space $T_x(M)$ at every point x of M . A Riemannian metric tensor $g = (g_{ji})$ of M is naturally induced from G of \tilde{M} :

$$g_{ji} = G_{\lambda\kappa} B_j^\lambda B_i^\kappa.$$

We can choose $m - n$ mutually orthogonal unit normal vector fields $C_p = (C_p^k)$ to M . Then the vectors B_i and C_p span the tangent space $T_x(\tilde{M})$ of \tilde{M} at every point x of M and the matrix

$$B = (B_i^k, C_p^k)$$

is regular. We have

$${}^t B G B = \begin{pmatrix} g_{ji} & 0 \\ 0 & \delta_{qp} \end{pmatrix},$$

and $\delta_{qp} = G_{\lambda\kappa} C_q^\lambda C_p^\kappa$ form the induced metric of the normal space $N_x(M)$ of M at each point x . We put

$$(1.2) \quad F = B^{-1} J B = \begin{pmatrix} f_i^h & -v_q^h \\ v_{pi} & f_{qp} \end{pmatrix}.$$

Then the map $f=(f_i^h)$ is an endomorphism of the tangent bundle of M and $f^\perp=(f_{qp})$ is that of the normal bundle of M . The matrix $v=(v_q^h)$ is a map of the normal bundle into the tangent bundle of M , that is, $v_q^h N_q$ for any vector $N=(N_q)$ normal to M are tangent components of JN . Since $J_{\lambda\kappa}$ is skew-symmetric, we have the relations $v_{pi}=v_p^h g_{ih}$ and the tensor fields

$$(1.3) \quad f_{ji} = G(JB_j, B_i) = J_{\lambda\kappa} B_j^\lambda B_i^\kappa, \quad f_{qp} = G(JC_q, C_p) = J_{\lambda\kappa} C_q^\lambda C_p^\kappa$$

are skew-symmetric in their indices.

The transforms of the tangent vectors B_i and the normal vectors C_p of M by J are expressed in the form

$$(1.4) \quad J_{\lambda}{}^\kappa B_i^\lambda = f_i^h B_h^\kappa + v_{pi} C_p^\kappa$$

and

$$(1.5) \quad J_{\lambda}{}^\kappa C_q^\lambda = -v_q^h B_h^\kappa + f_{qp} C_p^\kappa$$

respectively. The matrix (1.2) satisfies $F^2 = -I$, and consequently the quantities f_i^h , v_q^h and f_{qp} do the relations

$$(1.6) \quad f_j^i f_i^h = -\delta_j^h + v_{pj} v_p^h,$$

$$(1.7) \quad f_j^i v_{pi} = -v_{qj} f_{qp} = f_{pq} v_{qj},$$

$$(1.8) \quad v_q^i f_i^h = -f_{qp} v_p^h,$$

$$(1.9) \quad f_{rq} f_{qp} = -\delta_{rp} + v_r^i v_{pi},$$

where δ_j^h and δ_{qp} are components of I . The equation (1.1) is equivalent to

$$(1.10) \quad g_{kh} f_j^k f_i^h = g_{ji} - v_{pj} v_{pi}.$$

We shall denote by $N(M)$ the normal bundle of M and note here that the components v_q^h of v are regarded as components of $m-n$ tangent vector fields or those of n vector fields with respect to C_p in $N(M)$ whether q or h is fixed. The $(m-n)(m-n-1)/2$ scalar fields f_{qp} on M are regarded as components of a tensor field of type (0, 2) associated with the normal bundle $N(M)$.

Now, removing the ambient manifold \tilde{M} , we consider a Riemannian manifold M of dimension n admitting a metric tensor g , a tensor field $f=(f_i^h)$ of type (1, 1), a set $v=(v_q^h)$ of $m-n$ vector fields and a set $f^\perp=(f_{qp})$ of $(m-n)(m-n-1)/2$ scalar fields. If they satisfy the relations (1.6) to (1.10), then we say that M has a *metric compound structure* and the totality (f, g, v, f^\perp) a *metric compound structure* on M . If we put

$$(1.11) \quad F = \begin{pmatrix} f_i^h & -v_q^h \\ v_{pi} & f_{qp} \end{pmatrix}, \quad G = \begin{pmatrix} g_{ji} & 0 \\ 0 & \delta_{qp} \end{pmatrix}$$

then the set (G, F) defines an almost Hermitian structure on the product manifold $M \times R^{m-n}$ of the manifold M and an $(m-n)$ -dimensional Euclidean space R^{m-n} .

The Nijenhuis tensor $S=[F, F]$ of a metric compound structure (f, g, v, f^\perp) is written as

$$(1.12) \quad S_{ji}^h = f_j^k(\partial_k f_i^h - \partial_i f_k^h) - f_i^k(\partial_k f_j^h - \partial_j f_k^h) + v_{pj} \partial_i v_p^h - v_{pi} \partial_j v_p^h,$$

$$(1.13) \quad S_{jipq} = f_j^k(\partial_k v_{pi} - \partial_i v_{pk}) - f_i^k(\partial_k v_{pj} - \partial_j v_{pk}) - v_{qj} \partial_i f_{qp} + v_{qi} \partial_j f_{qp},$$

$$(1.14) \quad S_{jq}^h = -f_j^k \partial_k v_q^h + v_q^k(\partial_k f_j^h - \partial_j f_k^h) + f_{qp} \partial_j v_p^h,$$

$$(1.15) \quad S_{jpq} = f_j^k \partial_k f_{qp} + v_q^k(\partial_k v_{pj} - \partial_j v_{pk}) + f_{qs} \partial_j f_{sp},$$

$$(1.16) \quad S_{rq}^h = v_r^k \partial_k v_q^h - v_q^k \partial_k v_r^h,$$

$$(1.17) \quad S_{rqp} = -v_r^k \partial_k f_{qp} + v_q^k \partial_k f_{qp},$$

see [23].

2. Metric compound structures of rank r

Let M be a Riemannian manifold of dimension n having a metric compound structure (f, g, v, f^\perp) . We assume that the rank of $v=(v_q^h)$ is equal to r ($0 \leq r \leq \min \{n, m-n\}$) almost everywhere on M , and call it a *metric compound structure of rank r* . The phrase “almost everywhere on M ” means “on the whole manifold M except a border subset of M ”. There exist linearly independent non-vanishing vector fields $V_{(\alpha)}=(v_{(\alpha)}^h)$ on M and $N_{(\alpha)}=(N_{(\alpha)p})$ in R^{m-n} such that

$$(2.1) \quad v_q^h = N_{(\alpha)q} v_{(\alpha)}^h,$$

where $\alpha=1, 2, \dots, r$. Moreover we may normalize the vector fields $N_{(\alpha)}$ in R^{m-n} such as

$$(2.2) \quad N_{(\alpha)q} N_{(\beta)q} = \delta_{\alpha\beta}.$$

If we put

$$(2.3) \quad \lambda_{\alpha\beta} = f_{qp} N_{(\alpha)q} N_{(\beta)p},$$

then these are $r(r-1)/2$ scalar fields on M , and we have the relations

$$(2.4) \quad f_j^i f_i^h = -\delta_j^h + v_{(\alpha)j} v_{(\alpha)}^h,$$

$$(2.5) \quad f_j^i v_{(\alpha)i} = \lambda_{\alpha\beta} v_{(\beta)j},$$

$$(2.6) \quad v_{(\alpha)}^i f_i^h = -\lambda_{\alpha\beta} v_{(\beta)}^h,$$

$$(2.7) \quad f_{rq} f_{qp} = -\delta_{rp} + N_{(\alpha)r} N_{(\beta)p} v_{(\alpha)}^i v_{(\beta)i}$$

by virtue of (1.6) to (1.9), (2.1) and (2.2).

We define the distributions D^r and D^{n-r} of the tangent bundle $T(M)$ of M by

$$D^r = \text{span} \{V_{(1)}, V_{(2)}, \dots, V_{(r)}\},$$

$$D^{n-r} = \{X \in T(M) \mid g(X, V_{(\alpha)}) = 0\},$$

which are orthogonal complementary to one another. The complex structure F defined by (1.11) on the product manifold $M \times R^{m-n}$ is written as

$$F = \begin{pmatrix} f_i^h & -N_{(\alpha)q} v_{(\alpha)}^h \\ N_{(\alpha)p} v_{(\alpha)i} & f_{qp} \end{pmatrix},$$

and hence the transform of any vector field X of M by F is expressed in the form

$$FX = fX + v_{(\alpha)}(X)N_{(\alpha)},$$

where $v_{(\alpha)}$ is the associated 1-form of $V_{(\alpha)}$ for each α . For any vector field $X \in D^{n-r}$, we see that $FX \in D^{n-r}$. Therefore D^{n-r} is a holomorphic distribution and of even-dimension. Thus we have

THEOREM 2.1. *If a Riemannian manifold M has a metric compound structure of rank r , then M is of even- or odd-dimension according as r is even or odd.*

In the sequel we consider the case where $r=2$. For convenience sake, we shall denote the vector fields $V_{(1)}$, $V_{(2)}$, $N_{(1)}$ and $N_{(2)}$ by $U=(u^h)$, $V=(v^h)$, $\alpha=(\alpha_p)$ and $\beta=(\beta_p)$ respectively. Then the equation (2.4) is reduced to

$$(2.8) \quad f_j^i f_i^h = -\delta_j^h + u_j u^h + v_j v^h,$$

or equivalently

$$(2.9) \quad g_{kh} f_j^k f_i^h = g_{ji} - u_j u_i - v_j v_i.$$

If we denote

$$(2.10) \quad \lambda = \lambda_{12} = f_{qp} \alpha_q \beta_p,$$

then the equation (2.5) is reduced to

$$(2.11) \quad f_j^i u_i = \lambda v_j, \quad f_j^i v_i = -\lambda u_j.$$

If $\lambda \neq 0$, then it follows from (2.11) that

$$(2.12) \quad u_i v^i = 0,$$

$$(2.13) \quad u_i u^i = v_i v^i$$

because f_{ji} are skew-symmetric in i and j . Moreover, by means of (2.8) and (2.13), we have

$$(2.14) \quad u_i u^i = v_i v^i = 1 - \lambda^2.$$

If $\lambda = 0$, then it follows from (2.8) and (2.11) that

$$u_i u^i = v_i v^i = 1.$$

If a tensor field f of type $(1, 1)$, a metric g , two vector fields U and V and a function λ on M satisfy the relations (2.8), (2.9), (2.11), (2.12) and (2.14), then the totality of them is called an (f, g, u, v, λ) -structure by K. Yano and M. Okumura [30] (see, also [4]). The rank of f is equal to n or $n - 2$ whether the function $1 - \lambda^2$ vanishes or not. We can see that two vector fields \bar{U} and \bar{V} in D^2 given by

$$(2.15) \quad \bar{U} = U \cos \theta - V \sin \theta, \quad \bar{V} = U \sin \theta + V \cos \theta,$$

θ being a function, constitute the above-mentioned structure together with f, g and λ . Therefore the structure is essentially concerned with the distribution D^2 and the function λ , and we shall call it an (f, g, D^2, λ) -structure. Two vector fields having the same properties as U and V will be called a *canonical pair* of the structure.

The relation (1.7) is rewritten as

$$f_j^i u_i \alpha_p + f_j^i v_i \beta_p = u_j f_{qp} \alpha_q + v_j f_{qp} \beta_q.$$

By use of (2.11), (2.12) and (2.14), it follows from this equation that

$$(2.16) \quad f_{qp} \alpha_p = -\lambda \beta_q, \quad f_{qp} \beta_p = \lambda \alpha_q,$$

and from (2.7) that

$$(2.17) \quad f_{rq} f_{qp} = -\delta_{rp} + (1 - \lambda^2)(\alpha_r \alpha_p + \beta_r \beta_p).$$

We now put

$$(2.18) \quad u_p^\perp = (1 - \lambda^2)^{1/2} \beta_p, \quad v_p^\perp = (1 - \lambda^2)^{1/2} \alpha_p,$$

and define a 2-plane section D_N^\perp in the product $M \times R^{m-n}$ by

$$D_N^\perp = \text{span} \{ \alpha, \beta \}.$$

Then we see from the relations (2.16), (2.17) and (2.18) that the tensor field f^\perp , the metric $g^\perp = (\delta_{qp})$ and the vectors U^\perp and V^\perp define an $(f^\perp, g^\perp, D_N^\perp, \lambda)$ -structure in R^{m-n} at every point of M .

Conversely we can prove that, if a metric compound structure (f, g, v, f^\perp) defines an $(f^\perp, g^\perp, D_N^2, \lambda)$ -structure in R^{m-n} at every point of M , then the metric compound structure introduces an (f, g, D^2, λ) -structure on M . Thus we have the following

THEOREM 2.2. *Let M be a Riemannian manifold of dimension $n \geq 2$ having a metric compound structure (f, g, v, f^\perp) . Then the following statements are equivalent:*

- (1) M has a metric compound structure of rank 2,
- (2) M admits an (f, g, D^2, λ) -structure,
- (3) The Euclidean space R^{m-n} admits an $(f^\perp, g^\perp, D_N^2, \lambda)$ -structure.

In the case of a metric compound structure of rank 2, the equation (1.12) is rewritten as

$$(2.19) \quad \begin{aligned} S_{ji}{}^h &= f_j{}^k(\partial_k f_i{}^h - \partial_i f_k{}^h) - f_i{}^k(\partial_k f_j{}^h - \partial_j f_k{}^h) \\ &\quad + u_j \partial_i u^h - u_i \partial_j u^h + v_j \partial_i v^h - v_i \partial_j v^h \\ &\quad + (\alpha_p \partial_i \beta_p)(u_j v^h - v_j u^h) - (\alpha_p \partial_j \beta_p)(u_i v^h - v_i u^h) \end{aligned}$$

by a simple computation. Corresponding to the two vector fields \bar{U} and \bar{V} of D^2 given by (2.15), the vectors of the metric compound structure in R^{m-n} are expressed by

$$\bar{\alpha} = \alpha \cos \theta - \beta \sin \theta, \quad \bar{\beta} = \alpha \sin \theta + \beta \cos \theta.$$

We can prove that the tensor defined by (2.19) with respect to the vector fields $\bar{U}, \bar{V}, \bar{\alpha}$ and $\bar{\beta}$ is identical with $S_{ji}{}^h$. If $S_{ji}{}^h = 0$ identically, then the (f, g, D^2, λ) -structure on M is said to be *normal*. If we consider the metric compound structure of rank 2 on the product manifold $M \times R^2$ of the manifold M and a 2-dimensional Euclidean space R^2 , and choose the vectors α and β such as

$$\alpha = (1, 0) \quad \text{and} \quad \beta = (0, 1),$$

then our definition of the normality is the same due to K. Yano and M. Okumura [30].

REMARK 2.3. Let M be a Riemannian manifold with metric compound structure of rank r . If $r=0$, then the tensor field f itself is a complex structure on M . Y. Tashiro and the author [23] have studied the case of $r=1$. In this case, the function λ_{11} vanishes identically and M admits an almost contact metric structure. In the case where $r=3$, by putting $V_{(1)}=U, V_{(2)}=V, V_{(3)}=W, \lambda_{32}=\lambda, \lambda_{13}=\mu$ and $\lambda_{21}=v$, we see that M admits the so-called $(f, g, u, v, w, \lambda, \mu, v)$ -structure introduced by K. Yano and U.-H. Ki [29].

3. λ -curves and λ -hypersurfaces

For the sake of discussions in the later paragraphs, we consider here a scalar field λ on a Riemannian manifold (M, g) such that the gradient vector field of λ is a general concircular one and investigate properties of the scalar field λ and the manifold M . The Riemannian connection of M will be denoted by ∇ .

Let A be the gradient vector field of a scalar field λ . A point of M will be called a *stationary* or *ordinary* point of λ whether A vanishes at the point or not. If the vector field A satisfies the equation

$$\nabla_A A = \alpha A$$

with a scalar field α on M , then the trajectories of A are geodesic arcs in a neighborhood of an ordinary point of λ . The connected component of a regular hypersurface defined by $\lambda = \text{constant}$ will be called a λ -hypersurface and the geodesic containing a trajectory of the vector field A a λ -curve.

Let W be a neighborhood of an ordinary point x of the scalar field λ in M . Then we can choose a local coordinate system (x^h) in W such that the hypersurfaces defined by $x^1 = \text{constant}$ are λ -hypersurfaces and the curves defined by the equations $x^a = \text{constant}$ are λ -curves. With respect to such a coordinate system (x^h) , we first have

$$(3.1) \quad g_{b1} = g_{1a} = 0,$$

because λ -curves are orthogonal to λ -hypersurfaces. Since the λ -curves are geodesics, we have the equations

$$\frac{d}{dx^1} \frac{dx^h}{dx^1} + \{j^h\} \frac{dx^j}{dx^1} \frac{dx^i}{dx^1} = \alpha \frac{dx^h}{dx^1}$$

along the first coordinate curves, where $\{j^h\}$ indicates the Christoffel symbol formed with the metric tensor g_{ji} . It follows from this equation and $dx^h/dx^1 = \delta_1^h$ that

$$\{1^h\} = \alpha \delta_1^h,$$

which is reduced to

$$g^{h1} \partial_1 g_{11} - (1/2) g^{hi} \partial_i g_{11} = \alpha \delta_1^h.$$

Putting $h = a$ in this equation, we see that g_{11} depends on x^1 only. Therefore, by a suitable choice of the first coordinate x^1 , we may suppose that

$$(3.2) \quad g_{11} = 1$$

in W . Hence we have $\alpha=0$ and x^1 may be regarded as the arc length of λ -curves in W . We shall call such a local coordinate system (x^1, x^a) an *adapted* one for the scalar field λ , and denote by prime the ordinary differentiation with respect to the arc length x^1 .

A scalar field λ is said to be *general concircular* if the gradient vector field A of λ satisfies the equation

$$(3.3) \quad \nabla_X A = \phi X + \psi g(X, A)A$$

for any vector field X and some functions ϕ and ψ on M ([14]). If the function ψ in (3.3) vanishes identically, then λ is said to be *concircular*. Moreover, if the equation (3.3) is expressed in the form

$$\nabla_X A = -k\lambda X,$$

k being a constant, then λ is said to be *special concircular* with characteristic constant k ([22], see also [11]).

Since the gradient vector field A of a general concircular scalar field λ on M satisfies the equation

$$\nabla_A A = (\phi + \psi g(A, A))A,$$

the trajectories of A are geodesic arcs. We shall prove

PROPOSITION 3.1. *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. If M admits a general concircular scalar field λ , then the underlying manifold of M is locally the product $R \times \bar{M}$ of a 1-dimensional Euclidean space R and an $(n-1)$ -dimensional Riemannian manifold \bar{M} , and the metric form of M is written as*

$$(3.4) \quad g_{ji} dx^j dx^i = (dx^1)^2 + \sigma^2 \bar{g}_{cb} dx^c dx^b$$

with respect to an adapted coordinate system (x^1, x^a) in a neighborhood of an ordinary point of λ , where $\bar{g}_{cb} dx^c dx^b$ is the metric form of \bar{M} and the function σ is given by

$$(3.5) \quad \sigma = \lambda' \exp\left(-\int \psi \lambda' dx^1\right).$$

PROOF. We put $\lambda_j = \nabla_j \lambda$ and $\lambda^h = g^{ih} \lambda_i$, the components of the gradient vector field A of λ . The equation (3.3) is then expressed as

$$(3.6) \quad \nabla_j \lambda_i = \phi g_{ji} + \psi \lambda_j \lambda_i$$

with respect to a local coordinate system (x^h) . Let $E = (e^h)$ be the unit vector field in the direction of A and denote by μ the length of A . Then we have

$$(3.7) \quad \lambda^h = \mu e^h.$$

Differentiating this equation covariantly, putting $\mu_j = \nabla_j \mu$ and making use of (3.6), we obtain

$$\mu_j e_i + \mu \nabla_j e_i = \phi g_{ji} + \psi \mu^2 e_j e_i$$

and, contracting this equation with e^i

$$(3.8) \quad \mu_j = (\phi + \psi \mu^2) e_j.$$

From these equations, we have the equation

$$(3.9) \quad \nabla_j e_i = -h(g_{ji} - e_j e_i),$$

where $h = -\phi/\mu$. A unit vector field E satisfying the equation (3.9) is called a *locally symmetric vector field of the first order* by A. G. Walker ([24]) if $\dim M \geq 4$.

Let $\bar{M}(x)$ be the λ -hypersurface through an ordinary point x of λ in a neighborhood W of M . Then the λ -hypersurface $\bar{M}(x)$ is regular in W and E is the unit normal vector field of $\bar{M}(x)$ through any point of $\bar{M}(x)$. We take a local coordinate system (x^a) in $\bar{M}(x)$ such that $\bar{M}(x)$ is expressed by the parametric equations

$$x^h = x^h(x^a)$$

in W . Then the induced metric tensor g_{cb}^* on $\bar{M}(x)$ is given by

$$g_{cb}^* = g_{ji} B_c^j B_b^i,$$

where $B_b^i = \partial_b x^i$. The second fundamental tensor h_{cb} of $\bar{M}(x)$ is defined by

$$(3.10) \quad h_{cb} = (\nabla_c B_b^i) e_i = -B_c^j B_b^i \nabla_j e_i,$$

and the equation of Gauss is written as

$$(3.11) \quad \nabla_c B_b^h = \partial_c B_b^h + \{^h_{ji}\} B_c^j B_b^i - \{^a_{cb}\}^* B_a^h = h_{cb} e^h,$$

where ∇ indicates the covariant differentiation of van der Waerden-Bortolotti in $\bar{M}(x)$ and $\{^a_{cb}\}^*$ is the Christoffel symbol formed with the induced metric tensor g_{cb}^* . It follows from (3.9) and (3.10) that the second fundamental tensor h_{cb} is equal to

$$(3.12) \quad h_{cb} = h g_{cb}^*.$$

Therefore each λ -hypersurface $\bar{M}(x)$ is totally umbilical and the mean curvature is equal to $h/(n-1)$.

Now, for the general concircular scalar field λ , we choose an adapted coordinate system (x^1, x^a) in a neighborhood W of an ordinary point. The scalar fields λ and μ are differentiable functions of the first coordinate x^1 only. Since the last $n-1$ coordinates (x^a) of (x^h) is a local coordinate system of each λ -hypersurface in W , it is clear that

$$B_b^h = \delta_b^h, \quad e^h = \delta_1^h \quad \text{and} \quad g_{cb}^* = g_{cb}$$

on the λ -hypersurface $\bar{M}(x)$, and hence the equation (3.11) is reduced to

$$(3.13) \quad \{c_b^h\} - \{c_b^a\}^* \delta_a^h = h g_{cb} \delta_1^h.$$

The equation (3.13) for $h=1$ leads to

$$\{c_b^1\} = h g_{cb},$$

which is also reduced to

$$(3.14) \quad \partial_1 g_{cb} = -2h g_{cb}.$$

Therefore, putting

$$(3.15) \quad \sigma = \exp\left(-\int h dx^1\right),$$

we see that the components g_{cb} are written as

$$(3.16) \quad g_{cb} = \sigma^2 \bar{g}_{cb},$$

where \bar{g}_{cb} are independent of x^1 and form a metric tensor of an $(n-1)$ -dimensional manifold.

Since the length μ of the gradient vector field A of λ is constant on each λ -hypersurface through an ordinary point, all points of a λ -hypersurface are ordinary, that is, a λ -hypersurface is a closed submanifold. Let \bar{M} be an $(n-1)$ -dimensional Riemannian manifold diffeomorphic to the λ -hypersurface $\bar{M}(x)$ and having \bar{g}_{cb} as metric tensor. The manifold M is therefore locally diffeomorphic to the product $R \times \bar{M}$ of an open interval R with \bar{M} , and the metric form of M is given by (3.4).

It follows from the equations (3.6) and (3.7) that

$$\lambda' = \mu, \quad \lambda'' = \phi + \psi \lambda'^2,$$

and hence we obtain the relation

$$h = -(\lambda''/\lambda') + \psi \lambda'.$$

Substituting this relation into (3.15), we can derive the expression (3.5). This completes the proof of Proposition 3.1.

We notice that the function σ is dependent on all coordinates x^h in general. If the function ψ is dependent on the first coordinate x^1 only, so is σ . Then the manifold M and λ -hypersurfaces are homothetic to each other. Moreover, by a suitable choice of the arc length x^1 such that $\lambda' > 0$, we may consider that σ is a positive valued function on R . Therefore M is locally a warped product $R \times_{\sigma} \bar{M}$ of a 1-dimensional Euclidean space R and the $(n-1)$ -dimensional Riemannian manifold \bar{M} , to which each λ -hypersurface $\bar{M}(x)$ is homothetic.

Now we shall denote by $\{\bar{a}_{cb}\}$ the Christoffel symbol formed with \bar{g}_{cb} . Since the metric form of M is given by (3.4), where the function σ depends on all coordinates x^h in general, we have the relations

$$(3.17) \quad \begin{cases} \{11\} = \{a\} = \{c1\} = 0, \\ \{c1\} = (\sigma_1/\sigma)\delta_c^a, \quad \{cb\} = -\sigma\sigma_1\bar{g}_{cb}, \\ \{cb\} = \{\bar{a}_{cb}\} + (1/\sigma)(\sigma_c\delta_b^a + \sigma_b\delta_c^a - \sigma^a\bar{g}_{cb}), \end{cases}$$

where $\sigma_1 = \partial_1\sigma$, $\sigma_b = \partial_b\sigma$ and $\sigma^a = \bar{g}^{ae}\sigma_e$.

Components of the curvature tensors of M and \bar{M} will be denoted by $R_{kji}{}^h$ and $\bar{R}_{dcb}{}^a$ respectively, and the Riemannian connection of \bar{M} with respect to \bar{g} by $\bar{\nabla}$. Then the curvature tensor of M has non-trivial components

$$(3.18) \quad \begin{cases} R_{1c1}{}^a = (\sigma_{11}/\sigma)\delta_c^a, \quad R_{1cb}{}^1 = -\sigma\sigma_{11}\bar{g}_{cb}, \\ R_{acd}{}^1 = (\sigma_1\sigma_a - \sigma\partial_a\sigma_1)\bar{g}_{cb} - (\sigma_1\sigma_c - \sigma\partial_c\sigma_1)\bar{g}_{db}, \\ R_{acb}{}^a = \bar{R}_{dcb}{}^a - (1/\sigma)(\delta_d^a\sigma_{cb} - \delta_c^a\sigma_{db} + \sigma_d^a\bar{g}_{cb} - \sigma_c^a\bar{g}_{db}) \\ \quad - (\sigma_1^2 + \sigma_e\sigma^e/\sigma^2)(\delta_d^a\bar{g}_{cb} - \delta_c^a\bar{g}_{db}), \end{cases}$$

where we have put

$$\sigma_{cb} = \bar{\nabla}_c\sigma_b - (2/\sigma)\sigma_c\sigma_b \quad \text{and} \quad \sigma_c^a = \bar{g}^{ea}\sigma_{ce}.$$

Moreover, denoting by R_{ji} and \bar{R}_{cb} the components of the Ricci tensors of M and \bar{M} respectively, the Ricci tensor of M has non-trivial components

$$(3.19) \quad \begin{cases} R_{11} = -(n-1)\sigma_{11}/\sigma, \\ R_{c1} = (n-2)(\sigma_1\sigma_c - \sigma\partial_c\sigma_1)/\sigma, \\ R_{cb} = \bar{R}_{cb} - [\sigma\sigma_{11} + \sigma_e^e/\sigma + (n-2)(\sigma_1^2 + \sigma_e\sigma^e/\sigma^2)]\bar{g}_{cb} \\ \quad - (n-3)\sigma_{cb}/\sigma. \end{cases}$$

If M is an Einstein manifold, that is,

$$(3.20) \quad R_{ji} = (n-1)kg_{ji},$$

then it follows from the second equation of (3.19) and (3.20) that σ_1/σ is a function of the first coordinate x^1 only and, consequently, from (3.5) that σ depends on x^1 only. Therefore the first equation of (3.19) gives

$$(3.21) \quad \sigma_{11} = -k\sigma,$$

and the third equation of (3.19) leads to

$$\bar{R}_{cb} = (n-2)(\sigma_1^2 + k\sigma^2)\bar{g}_{cb},$$

that is, \bar{M} is also an Einstein manifold. The converse is true. A similar argument can be developed in the case where M is a space of constant curvature by use of (3.18). The equation (3.21) shows that σ is a special concircular scalar field with characteristic constant k . Thus we have the following

LEMMA 3.2. *Let M be an $n(>2)$ -dimensional Riemannian manifold admitting a general concircular scalar field λ . Then M is an Einstein manifold (resp. a space of constant curvature) if and only if the manifold \bar{M} diffeomorphic to each λ -hypersurface is an Einstein manifold (resp. a space of constant curvature), and the function σ given by (3.5) is a special concircular scalar field and does not depend on \bar{M} .*

REMARK 3.3. In this lemma, if the scalar curvature of M is equal to a constant k , then that of \bar{M} is equal to the constant $\sigma_1^2 + k\sigma^2$ and vice versa. The stationary points of a concircular scalar field σ are isolated and the number of them is at most two. It is known (Y. Tashiro [21, 22]) that, in a neighborhood of a stationary point of σ , \bar{M} is isometric to an $(n-1)$ -dimensional sphere and M is a space of constant curvature.

4. Riemannian manifolds with (f, g, D^2, λ) -structures

In this paragraph we shall clarify the geometrical structure of a Riemannian manifold with (f, g, D^2, λ) -structure having certain properties.

We first prove following

LEMMA 4.1. *Let M be a Riemannian manifold of dimension $n>2$ with (f, g, D^2, λ) -structure. If a canonical pair U and V of D^2 satisfy the relations*

$$(4.1) \quad \nabla_j u_i - \nabla_i u_j = 2\tau f_{ji}$$

$$(4.2) \quad \nabla_j v_i - \nabla_i v_j = 2\rho f_{ji}$$

with scalar fields τ and ρ , and ρ does not vanish on M , then $\tau = -k\rho$, where k is a constant.

PROOF. Differentiating (4.2) covariantly, we have

$$2\rho \nabla_k f_{ji} + 2\rho_k f_{ji} = \nabla_k \nabla_j v_i - \nabla_k \nabla_i v_j.$$

If we take the cyclic sum of this equation with respect to the indices i, j and k and make use of Bianchi's identity, then we obtain

$$\rho(\nabla_k f_{ji} + \nabla_j f_{ki} + \nabla_i f_{kj}) = -(\rho_k f_{ji} + \rho_j f_{ik} + \rho_i f_{kj}).$$

Similarly, from (4.1), we also obtain

$$\tau(\nabla_k f_{ji} + \nabla_j f_{ik} + \nabla_i f_{kj}) = -(\tau_k f_{ji} + \tau_j f_{ik} + \tau_i f_{kj}).$$

From these equations, we have

$$\rho(\tau_k f_{ji} + \tau_j f_{ik} + \tau_i f_{kj}) = \tau(\rho_k f_{ji} + \rho_j f_{ik} + \rho_i f_{kj})$$

and, contracting this equation with f^{ji} and making use of (2.8), (2.12) and (2.14),

$$\begin{aligned} (4.3) \quad & \rho[(n-4+2\lambda^2)\tau_k + 2(u^i \tau_i)u_k + 2(v^i \tau_i)v_k] \\ & = \tau[(n-4+2\lambda^2)\rho_k + 2(u^i \rho_i)u_k + 2(v^i \rho_i)v_k]. \end{aligned}$$

Moreover, contracting this equation with u^k and v^k , we can obtain $\rho u^i \tau_i = \tau u^i \rho_i$ and $\rho v^i \tau_i = \tau v^i \rho_i$ respectively. Substituting these relations into (4.3), we find

$$\rho \tau_k = \tau \rho_k,$$

which implies that $\tau = k\rho$, k being a constant. This completes the proof.

If one of the scalar field τ and ρ is a constant, then so is the other and the tensor field (f_{ji}) is a closed 2-form of M . We can see from (4.1) and (4.2) that the set of all zero-points, if any, of the vector fields U and V is a border subset of M . Therefore the function $1 - \lambda^2$ does not vanish almost everywhere on M . We also prove the following

LEMMA 4.2. *Suppose that the function λ does not vanish almost everywhere on M in addition to the assumption of Lemma 4.1. Then the vector fields U and V of a canonical pair are infinitesimal conformal transformations if and only if they satisfy the equations*

$$(4.4) \quad \nabla_j u_i = \rho(kf_{ji} - \lambda g_{ji}),$$

$$(4.5) \quad \nabla_j v_i = \rho(f_{ji} + k\lambda g_{ji}).$$

In this case we have the equation

$$(4.6) \quad \lambda_i = \rho(u_i - kv_i).$$

PROOF. If the vector fields U and V are infinitesimal conformal transformations, then we have

$$(4.7) \quad \nabla_j u_i + \nabla_i u_j = 2\gamma g_{ji},$$

$$(4.8) \quad \nabla_j v_i + \nabla_i v_j = 2\zeta g_{ji},$$

where γ and ζ are scalar fields on M . Comparing (4.1) with (4.7) and (4.2) with (4.8) and taking account of Lemma 4.1, we obtain the equations

$$(4.9) \quad \nabla_j u_i = k\rho f_{ji} + \gamma g_{ji},$$

$$(4.10) \quad \nabla_j v_i = \rho f_{ji} + \zeta g_{ji},$$

respectively. On the other hand, differentiating (2.14) covariantly, we get

$$(4.11) \quad u^j \nabla_j u_i = v^j \nabla_j v_i = -\lambda \lambda_i$$

and, substituting (4.9) and (4.10) into this equation and making use of (2.11),

$$\gamma u_j + k\rho \lambda v_j = -\rho \lambda u_j + \zeta v_j.$$

Since U and V are linearly independent, it follows from this equation that $\gamma = -\rho \lambda$ and $\zeta = k\rho \lambda$. Thus the substitution of these relations into (4.9) and (4.10) yields the equations (4.4) and (4.5). The converse is trivial. The equation (4.6) follows also from (4.4) and (4.11). This completes the proof.

We define a vector field $\xi = (\xi^h)$ on M by

$$(4.12) \quad \xi^h = \rho(ku^h + v^h)$$

and denote the associated 1-form by η . Concerning the vector field ξ and the gradient vector field A of λ , we have

THEOREM 4.3. *Let M be a Riemannian manifold of dimension $n > 2$ with (f, g, D^2, λ) -structure, where λ does not vanish almost everywhere on M . Assume that U and V of a canonical pair are infinitesimal conformal transformations and satisfy the relations*

$$\nabla_j u_i - \nabla_i u_j = 2\tau f_{ji}, \quad \nabla_j v_i - \nabla_i v_j = 2\rho f_{ji}$$

with scalar fields τ and ρ , and ρ does not vanish. Then the vector fields $(1+k^2)^{-1/2} \rho^{-1} A$ and $(1+k^2)^{-1/2} \rho^{-1} \xi$ constitute a canonical pair of the structure, and A and ξ satisfy the equations

$$(4.13) \quad \nabla_j \lambda_i = -(1+k^2)\rho^2 \lambda g_{ji} + (\kappa/\rho^2) \lambda_j \lambda_i,$$

$$(4.14) \quad \nabla_j \eta_i = (1+k^2)\rho^2 \lambda f_{ji} + (\kappa/\rho^2) \lambda_j \eta_i,$$

where the scalar field κ is given by

$$(4.15) \quad \kappa = (1 - \lambda^2)^{-1} u^i \rho_i = - [k(1 - \lambda^2)]^{-1} v^i \rho_i.$$

PROOF. It follows from (2.12), (2.14), (4.6) and (4.12) that

$$(4.16) \quad \lambda_i \xi^i = \lambda^i \eta_i = 0,$$

$$(4.17) \quad \lambda_i \lambda^i = \eta_i \xi^i = (1 + k^2) \rho^2 (1 - \lambda^2),$$

$$(4.18) \quad \lambda_j \lambda_i + \eta_j \eta_i = (1 + k^2) \rho^2 (u_j \mu_i + v_j \nu_i).$$

By use of equation (2.11), we can verify the equations

$$(4.19) \quad f_j^i \lambda_i = \lambda \eta_j, \quad f_j^i \eta_i = - \lambda \lambda_j.$$

Comparing (2.8) and (2.9) with (4.18), we can obtain the equations

$$(4.20) \quad f_j^i f_i^h = - \delta_j^h + [(1 + k^2) \rho^2]^{-1} (\lambda_j \lambda^h + \eta_j \xi^h),$$

$$(4.21) \quad g_{kh} f_j^k f_i^h = g_{ji} - [(1 + k^2) \rho^2]^{-1} (\lambda_j \lambda_i + \eta_j \eta_i).$$

Thus we see from (4.16), (4.17), (4.19), (4.20) and (4.21) that $(1 + k^2)^{-1/2} \rho^{-1} A$ and $(1 + k^2)^{-1/2} \rho^{-1} \xi$ together with f and g constitute the (f, g, D^2, λ) -structure and hence they form a canonical pair.

Differentiating (4.6) covariantly and making use of (4.4) and (4.5), we have

$$(4.22) \quad \nabla_j \lambda_i = \rho_j (u_i - kv_i) - (1 + k^2) \rho^2 \lambda g_{ji}.$$

Since $\nabla_j \lambda_i$ and g_{ji} are symmetric in i and j , the equation (4.22) implies that

$$\rho_j (u_i - kv_i) = \rho_i (u_j - kv_j).$$

This equation means that ρ_j is proportional to $u_j - kv_j$, and we may put

$$(4.23) \quad \rho_j = \kappa (u_j - kv_j),$$

and the proportional factor κ is given by (4.15). Substituting (4.23) into (4.22), we have the equation (4.13). Similarly, differentiating (4.12) covariantly and making use of (4.4) and (4.5), we have

$$\nabla_j \eta_i = \rho_j (ku_i + v_i) + (1 + k^2) \rho^2 f_{ji}$$

and this equation substituted with (4.23) is equivalent to (4.14). This completes the proof of Theorem 4.3.

The equation (4.13) shows that the function λ is a general concircular scalar field on M . Thus, by Proposition 3.1, the underlying manifold of M is locally the product $R \times \bar{M}$ of a 1-dimensional Euclidean space R and an $(n - 1)$ -dimensional Riemannian manifold \bar{M} and the metric tensor g of M is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma^2 \bar{g}_{cb} \end{pmatrix}$$

with respect to an adapted coordinate system (x^1, x^a) for λ , where the function σ is expressed by

$$(4.24) \quad \sigma = \lambda' \exp\left(-\int (\kappa/\rho^2) \lambda' dx^1\right)$$

by virtue of (4.13). Now we prove the following

LEMMA 4.4. *Under the assumptions of Theorem 4.3, the manifold M is locally a warped product $R \times_a \bar{M}$ of a 1-dimensional Euclidean space R and the manifold \bar{M} homothetic to λ -hypersurfaces, where σ is a function on R given by $\sigma = (1 - \lambda^2)^{1/2}$.*

PROOF. We shall prove that the function κ/ρ^2 in the integrand of (4.24) is dependent only on the first coordinate x^1 . Since λ depends only on x^1 , it follows from (4.6) that

$$(4.25) \quad \lambda' = \rho(u_1 - kv_1)$$

and $\lambda_a = 0$. Consequently we see that

$$(4.26) \quad u_a = kv_a$$

and from (4.16) that

$$(4.27) \quad \xi^1 = \eta_1 = 0.$$

It follows therefore from (4.17) that

$$(4.28) \quad \lambda'^2 = (1 + k^2)\rho^2(1 - \lambda^2),$$

$$(4.29) \quad \eta_a \xi^a = (1 + k^2)\rho^2(1 - \lambda^2).$$

Since the function $1 - \lambda^2$ does not vanish almost everywhere on M , the scalar field ρ depends only on the first coordinate x^1 by virtue of (4.28).

The derivative of (4.28) in x^1 gives the equation

$$(4.30) \quad (1 + k^2)\rho\rho' = \lambda'\lambda''/(1 - \lambda^2) + \lambda\lambda'^3/(1 - \lambda^2)^2.$$

It follows from (4.12) and (4.27) that

$$(4.31) \quad ku_1 + v_1 = 0.$$

Comparing (4.25) with (4.31), we have the equations

$$(4.32) \quad (1 + k^2)\rho u_1 = \lambda' \quad \text{and} \quad (1 + k^2)\rho v_1 = -k\lambda',$$

which shows that the first components of the vectors U and V are dependent on the first coordinate x^1 only. Substituting (4.30) and (4.32) into (4.15), we have the expression

$$(1+k^2)\kappa = \lambda''/(1-\lambda^2) + \lambda\lambda'^2/(1-\lambda^2)^2$$

and κ depends on x^1 only. Consequently, comparing this expression with (4.28), we have

$$(4.33) \quad \kappa/\rho^2 = \lambda''/\lambda'^2 + \lambda/(1-\lambda^2),$$

which shows that the function κ/ρ^2 is dependent only on x^1 . Thus, as seen in the previous paragraph, M is locally a warped product $R \times_{\sigma} \bar{M}$. Substituting (4.33) into (4.24), we can obtain the expression

$$(4.34) \quad \sigma = (1-\lambda^2)^{1/2},$$

and this completes the proof.

From (3.17) and (4.34), the non-trivial components of the Christoffel symbol of M are given by

$$(4.35) \quad \{^a_{c1}\} = -[\lambda\lambda'/(1-\lambda^2)]\delta^a_c, \quad \{^1_{cb}\} = \lambda\lambda'\bar{g}_{cb}, \quad \{^a_{cb}\} = \overline{\{^a_{cb}\}}.$$

By use of these expressions, we can prove the following

LEMMA 4.5. *Under the assumptions of Theorem 4.3, the $(n-1)$ -dimensional manifold \bar{M} is homothetic to a contact metric manifold.*

PROOF. Since $\lambda_a=0$ and $\eta_1=0$, it follows from (4.19) that

$$(4.36) \quad f_b^1 = (\lambda/\lambda')\eta_b,$$

$$(4.37) \quad f_b^a\eta_a = 0.$$

The equation (4.17) is now reduced to

$$(4.38) \quad \eta_a\zeta^a = \lambda'^2.$$

By means of (4.28), (4.33) and (4.36), the equation (4.14) splits into

$$\nabla_1\eta_1 = 0, \quad \nabla_c\eta_1 = [\lambda\lambda'/(1-\lambda^2)]\eta_c,$$

$$(4.39) \quad \nabla_1\eta_b = (\lambda''/\lambda')\eta_b,$$

$$(4.40) \quad \nabla_c\eta_b = [\lambda'^2/(1-\lambda^2)]f_{cb}.$$

The equation (4.39) is expressed as

$$\partial_1\eta_b - \{^a_{1b}\}\eta_a = (\lambda''/\lambda')\eta_b,$$

and, substituting (4.35), we have the equation

$$\partial_1 \eta_b = [\lambda''/\lambda' - \lambda\lambda'/(1-\lambda^2)]\eta_b.$$

The integration of this equation gives

$$(4.41) \quad \eta_b = \lambda'(1-\lambda^2)^{1/2}\bar{\eta}_b,$$

where $\bar{\eta}_b$ are dependent on \bar{M} only and regarded as components of a 1-form of \bar{M} . Since $g_{cb} = (1-\lambda^2)\bar{g}_{cb}$, it follows from (4.41) that

$$\xi^a = [\lambda'/(1-\lambda^2)^{1/2}]\bar{g}^{ab}\bar{\eta}_b.$$

We now put

$$\bar{\xi}^a = \bar{g}^{ab}\bar{\eta}_b.$$

Then $\bar{\xi} = (\bar{\xi}^a)$ is a vector field on \bar{M} and satisfies

$$(4.42) \quad \bar{\eta}_a \bar{\xi}^a = 1$$

by virtue of (4.38). The equation (4.40) is expressed as

$$\partial_c \eta_b - \overline{\{cb\}} \eta_a = [\lambda'^2/(1-\lambda^2)]f_{cb},$$

and, substituting (4.35) and (4.41), we have the equation

$$(4.43) \quad \bar{v}_c \bar{\eta}_b = [\lambda'/(1-\lambda^2)^{3/2}]f_{cb}.$$

We define a tensor field \bar{f} on \bar{M} by

$$(4.44) \quad \bar{v}_c \bar{\eta}_b = \bar{f}_{cb}.$$

Then the tensor field $\bar{f}_b^a = \bar{f}_{bc}\bar{g}^{ca}$ satisfies the relation

$$(4.45) \quad \bar{f}_b^a = [(1-\lambda^2)^{1/2}/\lambda']\bar{f}_b^a.$$

Substituting (4.41) and (4.45) into (4.37), we see that

$$(4.46) \quad \bar{\eta}_a \bar{f}_b^a = \bar{f}_b^a \bar{\xi}^b = 0.$$

The equation (4.20) for $h=a$ and $j=c$ is equivalent to

$$f_c^i f_i^a = -\delta_c^a + [(1-\lambda^2)/\lambda'^2]\eta_c \xi^a,$$

which is reduced to

$$(4.47) \quad [(1-\lambda^2)/\lambda'^2]\bar{f}_c^b \bar{f}_b^a = -\delta_c^a + \bar{\eta}_c \bar{\xi}^a$$

by means of (4.28), (4.36), (4.41) and (4.45). Similarly the equation (4.21) implies that

$$(4.48) \quad [(1 - \lambda^2)/\lambda'^2] \bar{g}_{ac} \bar{f}_b^d \bar{f}_a^c = \bar{g}_{ba} - \bar{\eta}_b \bar{\eta}_a.$$

By the independency of the function λ on \bar{M} , it follows from (4.47) or (4.48) that $(1 - \lambda^2)/\lambda'^2$ is a constant on M , say c^{-2} ($c > 0$). Therefore we easily verify from (4.42), (4.44), (4.46), (4.47) and (4.48) that the totality $(c\bar{f}, c^2\bar{g}, c^{-1}\bar{\xi}, c\bar{\eta})$ constitutes a contact metric structure on the $(n - 1)$ -dimensional manifold homothetic to \bar{M} . This completes the proof.

Combining Lemma 4.4 with Lemma 4.5 and rewriting $c^2\bar{g}_{cb}$ in place of \bar{g}_{cb} on \bar{M} , we can state the following

THEOREM 4.6. *Let M be a Riemannian manifold of dimension $n > 2$ with (f, g, D^2, λ) -structure, where the function λ does not vanish almost everywhere on M . Assume that U and V of a canonical pair are infinitesimal conformal transformations and satisfy the relations*

$$\nabla_j u_i - \nabla_i u_j = 2\tau f_{ji}, \quad \nabla_j v_i - \nabla_i v_j = 2\rho f_{ji}$$

with scalar fields τ and ρ , and ρ does not vanish. Then M is locally a warped product $R \times_{\sigma} \bar{M}$ of a 1-dimensional Euclidean space R and an $(n - 1)$ -dimensional contact metric manifold \bar{M} with the scalar field $\sigma = (1 - \lambda^2)^{1/2}$. Each λ -hyper-surface of M is homothetic to \bar{M} .

We see from (4.28) and (4.47) that $(1 + k^2)\rho^2 = c^2$. Choosing suitably the arc-length x^1 of λ -curves, the solution of (4.28) is given by

$$\lambda = \cos cx^1,$$

and hence σ is expressed as

$$\sigma = \sin cx^1.$$

Therefore both λ and σ are special concircular scalar fields with characteristic constant c^2 . The zero points of σ are those of $1 - \lambda^2$. Thus the following theorem follows from Lemma 3.2 and Remark 3.3.

THEOREM 4.7. *Let M be a Riemannian manifold of dimension $n > 2$ with (f, g, D^2, λ) -structure, where λ does not vanish almost everywhere on M . Assume that U and V of a canonical pair are infinitesimal conformal transformations and satisfy the relations*

$$\nabla_j u_i - \nabla_i u_j = 2\tau f_{ji}, \quad \nabla_j v_i - \nabla_i v_j = 2\rho f_{ji}$$

with scalar fields τ and ρ , and ρ does not vanish. If there is a point of M where $\lambda = \pm 1$, then M is a space of positive constant curvature.

If M is complete, then $\sigma = \sin cx^1$ has stationary points corresponding to

$x^1=0$ and $x^1=\pi/c$. Therefore, by use of a theorem due to Y. Tashiro [21, 22], We can state

THEOREM 4.8. *Let M be a complete Riemannian manifold of dimension $n > 2$ with (f, g, D^2, λ) -structure, where λ does not vanish almost everywhere on M . Assume that U and V of a canonical pair are infinitesimal conformal transformations and satisfy the relations*

$$\nabla_j u_i - \nabla_i u_j = 2\tau f_{ji}, \quad \nabla_j v_i - \nabla_i v_j = 2\rho f_{ji},$$

with the scalar fields τ and ρ , and ρ does not vanish. Then the manifold M is a sphere.

5. Submanifolds with induced metric compound structures of rank 2

Let M be a differentiable manifold of dimension $n \geq 2$ and ι an immersion of M into an m -dimensional almost Hermitian manifold \tilde{M} ($m - n \geq 2$). As stated in Paragraph 1, the n vectors B_i and $m - n$ mutually orthogonal unit normal vectors C_q span the tangent space $T_x(\tilde{M})$ of \tilde{M} at any point x of M . A metric compound structure (f, g, v, f^\perp) is naturally induced on M from the almost Hermitian structure (G, J) of \tilde{M} . We assume that the structure is of rank 2. Then, by Theorem 2.2, the submanifold M admits an (f, g, D^2, λ) -structure and the normal bundle $N(M)$ of M an $(f^\perp, g^\perp, D_N^2, \lambda)$ -structure.

Since the components v_q^h of v are expressed in the form

$$v_q^h = \alpha_q u^h + \beta_q v^h,$$

the transforms (1.4) and (1.5) are reduced to

$$(5.1) \quad J_\lambda^\kappa B_i^\lambda = f_i^h B_h^\kappa + u_i \alpha_p C_p^\kappa + v_i \beta_p C_p^\kappa$$

and

$$(5.2) \quad J_\lambda^\kappa C_q^\lambda = -\alpha_q u^h B_h^\kappa - \beta_q v^h B_h^\kappa + f_{qp} C_p^\kappa.$$

For any vector field X belonging to the orthogonal complement D^{n-2} of D^2 , it follows from (5.1) that

$$J_\lambda^\kappa X^i B_i^\lambda = f_i^h X^i B_h^\kappa.$$

For the vector fields $U=(u^h)$ and $V=(v^h)$ as a canonical pair of the (f, g, D^2, λ) -structure on M , we have the expressions

$$J_\lambda^\kappa u^i B_i^\lambda = -\lambda v^h B_h^\kappa + (1 - \lambda^2) \alpha_p C_p^\kappa,$$

$$J_\lambda^\kappa v^i B_i^\lambda = \lambda u^h B_h^\kappa + (1 - \lambda^2) \beta_p C_p^\kappa$$

by virtue of (2.11), (2.12) and (2.14). From these expressions, we see that D^2 is anti-invariant and hence M is the so-called CR -submanifold of \tilde{M} (see [1] and [2], as to CR -submanifolds) if and only if λ vanishes identically. Thus we have the following

LEMMA 5.1. *Let M be a submanifold with (f, g, D^2, λ) -structure in an almost Hermitian manifold. Then M is a CR -submanifold if and only if the function λ vanishes identically.*

On a submanifold M of an almost Hermitian manifold \tilde{M} , the van der Waerden-Bortolotti covariant differentiation ∇_j is defined by

$$\nabla_j B_i^\kappa = \partial_j B_i^\kappa + \Gamma_{\mu\lambda}^\kappa B_j^\mu B_i^\lambda - B_h^\kappa \{j^h\},$$

where $\Gamma_{\mu\lambda}^\kappa$ and $\{j^h\}$ are the Christoffel symbols of \tilde{M} and M respectively. Since $\nabla_j B_i^\kappa$ is normal to M for fixed i and j , we have the equation of Gauss

$$(5.3) \quad \nabla_j B_i^\kappa = h_{jip} C_p^\kappa,$$

where h_{jip} is the second fundamental tensor of the immersion ι . The equation of Weingarten is given by

$$(5.4) \quad \nabla_j C_q^\kappa = -h_j^i{}_q B_i^\kappa + l_{jqp} C_p^\kappa,$$

where we have put

$$\nabla_j C_q^\kappa = \partial_j C_q^\kappa + \Gamma_{\mu\lambda}^\kappa B_j^\mu C_q^\lambda, \quad h_j^i{}_q = g^{ih} h_{jqh},$$

and l_{jqp} is the third fundamental tensor of the immersion ι and called the *induced normal connection of M* . A normal vector field $N = \gamma_p C_p^\kappa$ is called a *normal section* on M and a subbundle of the normal bundle $N(M)$ spanned by two linearly independent normal vector fields a *normal 2-section* on M . The tensor $h_{jip} \gamma_p$ is called the *second fundamental tensor belonging to the normal section N* . The covariant differentiation ∇^\perp with respect to the normal connection is defined by

$$\nabla_j^\perp \gamma_p = \partial_j \gamma_p + \gamma_q l_{jqp},$$

and we have

$$\nabla_j N^\kappa = -h_j^i{}_p \gamma_p B_i^\kappa + (\nabla_j^\perp \gamma_p) C_p^\kappa.$$

The normal vector field

$$H = H_p C_p^\kappa, \quad H_p = (1/n) g^{ji} h_{jip},$$

is called the *mean curvature vector field* of M . If, for a normal vector field N , the relation

$$h_{jip}\gamma_p = \rho g_{ji}$$

is satisfied with a function ρ on M , then N is called an *umbilical section* on M , or M is said to be *umbilical with respect to N* . If N is a unit normal vector field, then the function $\rho = h_i^i \gamma_p / n$ is called the *mean curvature belonging to N* . If $\rho = 0$ identically, then N is called a *geodesic section* on M , or M is said to be *geodesic with respect to N* . If M is umbilical (resp. geodesic) with respect to every local normal section on M , then M is said to be *totally umbilical* (resp. *geodesic*). The normal section N is said to be *parallel in the normal bundle* if $\nabla_j N^\kappa$ is tangent to M , that is, $\nabla_j \gamma_p = 0$. Moreover N is said to be *concurrent* along M if N is an umbilical (not geodesic) section and parallel in the normal bundle. The submanifold M is said to be *pseudo-umbilical* if M is umbilical with respect to the mean curvature vector field H .

Now we assume that the ambient manifold \tilde{M} is Kaehlerian. Differentiating (5.1) covariantly along M and making use of (5.1) to (5.4), we have

$$\begin{aligned} & - (u^h h_{jip} \alpha_p + v^h h_{jip} \beta_p) B_h^\kappa + h_{jq} f_{qp} C_p^\kappa \\ & = (f_i^h h_{jhp} + \alpha_p \nabla_j u_i + \beta_p \nabla_j v_i + u_i \partial_j \alpha_p + v_i \partial_j \beta_p + u_i \alpha_q l_{jqp} + v_i \beta_q l_{jqp}) C_p^\kappa \\ & \quad + (\nabla_j f_i^h - u_i h_j^h \alpha_p - v_i h_j^h \beta_p) B_h^\kappa \end{aligned}$$

and, comparing the tangential and normal components, the equations

$$(5.5) \quad \nabla_j f_i^h = u_i h_j^h \alpha_p - u^h h_{jip} \alpha_p + v_i h_j^h \beta_p - v^h h_{jip} \beta_p,$$

$$(5.6) \quad \alpha_p \nabla_j u_i + \beta_p \nabla_j v_i = h_{jq} f_{qp} - f_i^h h_{jhp} - u_i \nabla_j \alpha_p - v_i \nabla_j \beta_p.$$

Contracting (5.6) with α_p and β_p and making use of (2.2) and (2.16), we have

$$(5.7) \quad \nabla_j u_i = -\lambda h_{jip} \beta_p - f_i^h h_{jhp} \alpha_p + v_i \beta_p \nabla_j \alpha_p$$

and

$$(5.8) \quad \nabla_j v_i = \lambda h_{jip} \alpha_p - f_i^h h_{jhp} \beta_p - u_i \beta_p \nabla_j \alpha_p$$

respectively. Substituting (5.7) and (5.8) into (5.6), we also have

$$(5.9) \quad \begin{aligned} h_{jq} f_{qp} - f_i^h h_{jhp} - u_i \nabla_j \alpha_p - v_i \nabla_j \beta_p & = \lambda h_{jq} (\alpha_q \beta_p - \alpha_p \beta_q) \\ & \quad - f_i^h h_{jq} (\alpha_q \alpha_p + \beta_q \beta_p) + (\beta_q \nabla_j \alpha_q) (\alpha_p v_i - \beta_p u_i). \end{aligned}$$

Contracting (5.7) with u^i or (5.8) with v^i , we obtain

$$(5.10) \quad \lambda_j = u^i h_{jip} \beta_p - v^i h_{jip} \alpha_p$$

by use of $u^i \nabla_j u_i = v^i \nabla_j v_i = -\lambda \lambda_j$. Differentiating (5.2) covariantly along M and making use of (5.1) to (5.4), we have

$$\begin{aligned}
 & - (f^i h_j^h \alpha_q + u^i \alpha_p l_{jqp} + v^i \beta_p l_{jqp}) B_i^\kappa - (u^i h_{jq} \alpha_p + v^i h_{jq} \beta_p - f_{pr} l_{jr q}) C_p^\kappa \\
 & = - (u^i \partial_j \alpha_q + v^i \partial_j \beta_p + \alpha_q \nabla_j u^i + \beta_q \nabla_j v^i + h_j^i \rho f_{qp}) B_i^\kappa \\
 & \quad + (\partial_j f_{qp} - u^i h_{jip} \alpha_q - v^i h_{jip} \beta_p + f_{qr} l_{jr p}) C_p^\kappa,
 \end{aligned}$$

from which we obtain the equation (5.6) again and

$$(5.11) \quad \nabla_j^\perp f_{qp} = u^i (h_{jip} \alpha_q - h_{jq} \alpha_p) + v^i (h_{jip} \beta_q - h_{jq} \beta_p).$$

If $\nabla_j^\perp f_{qp} = 0$ identically, then f^\perp is said to be *parallel in the normal bundle*. If both the vector fields α and β of the $(f^\perp, g^\perp, D_N^\perp, \lambda)$ -structure are umbilical sections on M and one of them is not a geodesic section, then the subbundle D_N^\perp is called an *umbilical 2-section* on M . If both α and β are geodesic sections on M , then D_N^\perp is called a *geodesic 2-section* on M . If both α and β are parallel in the normal bundle (resp. concurrent along M), then D_N^\perp is called a *parallel* (resp. *concurrent*) *2-section* on M . The following lemma is easily seen and justifies the above terminologies of 2-sections.

LEMMA 5.2. *Let γ be any vector field in the subbundle D_N^\perp .*

- (1) *If both the orthonormal vector fields α and β are umbilical sections on M , then so is γ .*
- (2) *If both the orthonormal vector fields α and β are parallel in the normal bundle, then $\nabla^\perp \gamma$ belongs to D_N^\perp .*

We shall denote by τ and ρ the mean curvatures belonging to α and β respectively, that is,

$$\tau = H_p \alpha_p \quad \text{and} \quad \rho = H_p \beta_p.$$

The sum of the squared mean curvatures belonging to two orthonormal vector fields in D_N^\perp is independent of the choice of the vector fields in D_N^\perp . We shall call it the *sum of two squared mean curvatures of D_N^\perp* and denote it by $v^2 = \tau^2 + \rho^2$.

We prove the following

LEMMA 5.3. *Let M be a submanifold with (f, g, D^2, λ) -structure in a Kaehlerian manifold. If the codimension of M is equal to 2 or the subbundle D_N^\perp is a parallel 2-section on M , then the mean curvature vector field H of M is given by*

$$(5.12) \quad H = \tau \alpha + \rho \beta.$$

PROOF. If D_N^\perp is a parallel 2-section on M , then it follows from (5.9) that

$$(5.13) \quad h_{jq} f_{qp} - f_i^h h_{jhp} = \lambda h_{jq} (\alpha_q \beta_p - \alpha_p \beta_q) - f_i^h h_{jq} (\alpha_q \alpha_p + \beta_q \beta_p)$$

and, contracting this equation with g^{ji} ,

$$H_q f_{qp} = \lambda(\rho\alpha_p - \tau\beta_p)$$

because $f^{ji}h_{jip}=0$. Transvecting this equation with f_{pr} and making use of (2.16) and (2.17), we have the relation (5.12). In the case where the codimension is equal to 2, it is obvious. This completes the proof.

By virtue of Lemmas 5.1 and 5.3, we can prove

THEOREM 5.4. *Let M be a submanifold with (f, g, D^2, λ) -structure in a Kaehlerian manifold. If the subbundle D_N^2 is a concurrent 2-section on M , then M is either a CR-submanifold or a pseudo-umbilical submanifold.*

PROOF. If the function λ vanishes identically, then M is a CR-submanifold by Lemma 5.1. Assume that λ does not vanish. Since D_N^2 is a concurrent 2-section on M , that is,

$$(5.14) \quad h_{jq}\alpha_q = \tau g_{ji}, \quad h_{jq}\beta_q = \rho g_{ji} \quad \text{and} \quad \nabla_j^\perp \alpha_p = \nabla_j^\perp \beta_p = 0,$$

it follows from Lemma 5.3 that

$$h_{jip}H_p = v^2 g_{ji},$$

and hence M is pseudo-umbilical.

We can state the following

THEOREM 5.5. *Let M be a submanifold with (f, g, D^2, λ) -structure in a Kaehlerian manifold. If the subbundle D_N^2 is a concurrent 2-section on M and the tensor field f^\perp is parallel in the normal bundle, then M is either a CR-submanifold or minimal.*

PROOF. If D_N^2 is a concurrent 2-section on M , it follows from (5.13) and (5.14) that

$$h_{jq}f_{qp} - f_i^h h_{jhp} = -\lambda(\rho\alpha_p - \tau\beta_p)g_{ji} + (\tau\alpha_p + \rho\beta_p)f_{ji}.$$

Contracting this equation with u^i , we obtain

$$(5.15) \quad u^i h_{jq}f_{qp} + \lambda v^i h_{jip} = -\lambda(\rho\alpha_p - \tau\beta_p)u_j + \lambda(\tau\alpha_p + \rho\beta_p)v_j.$$

On the other hand, if the tensor field f^\perp is parallel in the normal bundle, the equation (5.11) is reduced to

$$u^i h_{jip}\alpha_q - u^i h_{jiq}\alpha_p + v^i h_{jip}\beta_q - v^i h_{jiq}\beta_p = 0.$$

Contracting this equation with α_q and β_q , we have

$$(5.16) \quad u^i h_{jip} = \tau(\alpha_p u_j + \beta_p v_j)$$

and

$$(5.17) \quad v^i h_{jip} = \rho(\alpha_p u_j + \beta_p v_j)$$

respectively. Substituting (5.16) and (5.17) into (5.15), we obtain

$$\lambda(\rho\alpha_p - \tau\beta_p)u_j = 0.$$

Since α and β are linearly independent, the above equation shows that $\tau = \rho = 0$ if λ does not vanish. By means of Lemma 5.2, we see that $H = 0$ and hence the submanifold M is minimal. The remaining part of the statements follows from Lemma 5.1.

6. Submanifolds with induced (f, g, D^2, λ) -structure

In this paragraph we consider submanifolds of dimension $n > 2$ having umbilical 2-sections in an m -dimensional Kaehlerian manifold \tilde{M} ($m - n \geq 2$).

If the subbundle D_N^2 of the normal bundle $N(M)$ is an umbilical 2-section on a submanifold M of \tilde{M} , then the equations (5.5), (5.7), (5.8) and (5.10) are reduced to

$$(6.1) \quad \nabla_k f_{ji} = \tau(u_j g_{ki} - u_i g_{kj}) + \rho(v_j g_{ki} - v_i g_{kj}),$$

$$(6.2) \quad \nabla_j u_i = \tau f_{ji} - \lambda \rho g_{ji} + l_j v_i,$$

$$(6.3) \quad \nabla_j v_i = \rho f_{ji} + \lambda \tau g_{ji} - l_j u_i$$

and

$$(6.4) \quad \lambda_i = \rho u_i - \tau v_i$$

respectively, where τ and ρ are the mean curvatures belonging to the vector fields α and β respectively and we have put

$$(6.5) \quad l_j = \beta_p \nabla_j^\perp \alpha_p.$$

We prove the following

LEMMA 6.1. *Let M be a submanifold of dimension $n > 2$ with induced (f, g, D^2, λ) -structure in a Kaehlerian manifold. If the subbundle D_N^2 is an umbilical 2-section on M , then we have*

$$(6.6) \quad \rho_j + \tau l_j = A u_j, \quad \tau_j - \rho l_j = -A v_j,$$

$$(6.7) \quad \nabla_j \lambda_i = A(u_j u_i + v_j v_i) - \lambda v^2 g_{ji},$$

where v^2 is the sum of two squared mean curvatures of D_N^2 and

$$(6.8) \quad A = u^i(\rho_i + \tau l_i)/(1 - \lambda^2) = -v^i(\tau_i - \rho l_i)/(1 - \lambda^2).$$

PROOF. It follows from (6.1), (6.2) and (6.3) that

$$(6.9) \quad \nabla_k f_{ji} + \nabla_j f_{ik} + \nabla_i f_{kj} = 0,$$

$$(6.10) \quad \nabla_j u_i - \nabla_i u_j = 2\tau f_{ji} + l_j v_i - l_i v_j,$$

$$(6.11) \quad \nabla_j v_i - \nabla_i v_j = 2\rho f_{ji} - l_j u_i + l_i u_j.$$

Differentiating (6.10) covariantly, we have

$$\nabla_k \nabla_j u_i - \nabla_j \nabla_k u_i = 2\tau_k f_{ji} + 2\tau \nabla_k f_{ji} + (\nabla_k l_j) v_i - (\nabla_k l_i) v_j + l_j \nabla_k v_i - l_i \nabla_k v_j.$$

If we take the cyclic sum of this equation with respect to i, j and k and making use of (6.9), (6.11) and Bianchi's identity, then we have

$$\begin{aligned} & 2(\tau_k - \rho l_k) f_{ji} + 2(\tau_j - \rho l_j) f_{ik} + 2(\tau_i - \rho l_i) f_{kj} \\ & + (\nabla_k l_j - \nabla_j l_k) v_i + (\nabla_j l_i - \nabla_i l_j) v_k + (\nabla_i l_k - \nabla_k l_i) v_j = 0. \end{aligned}$$

By contraction of this equation with f^{ji} , we obtain

$$\begin{aligned} & (n-4+2\lambda^2)(\tau_k - \rho l_k) + 2u^i(\tau_i - \rho l_i)u_k + 2v^i(\tau_i - \rho l_i)v_k \\ & - \lambda u^i(\nabla_k l_i - \nabla_i l_k) + v_k f^{ji} \nabla_j l_i = 0 \end{aligned}$$

Contracting the last equation with u^k and v^k , we have

$$(6.12) \quad u^i(\tau_i - \rho l_i) = 0$$

and

$$(6.13) \quad (n-2)v^i(\tau_i - \rho l_i) = \lambda(\nabla_j l_i - \nabla_i l_j)v^j u^i - (1-\lambda^2)f^{ji} \nabla_j l_i$$

respectively. Similarly it follows from the covariant differentiation of the equation (6.11) that

$$(6.14) \quad v^i(\rho_i + \tau l_i) = 0,$$

$$(6.15) \quad (n-2)u^i(\rho_i + \tau l_i) = -\lambda(\nabla_j l_i - \nabla_i l_j)v^j u^i + (1-\lambda^2)f^{ji} \nabla_j l_i.$$

Comparing (6.13) with (6.15), we have

$$(6.16) \quad v^i(\tau_i - \rho l_i) + u^i(\rho_i + \tau l_i) = 0.$$

On the other hand, differentiating (6.4) covariantly and making use of (6.2) and (6.3), we have

$$(6.17) \quad \nabla_j \lambda_i = (\rho_j + \tau l_j)u_i - (\tau_j - \rho l_j)v_i - \lambda v^2 g_{ji},$$

which implies the equation

$$(\rho_j + \tau l_j)u_i - (\rho_i + \tau l_i)u_j = (\tau_j - \rho l_j)v_i - (\tau_i - \rho l_i)v_j.$$

By virtue of (6.12), (6.14) and (6.16), we can obtain the equation (6.6) from the contractions of the above equation with u^i and v^i . Substituting (6.6) into (6.17), we obtain the equation (6.7). This completes the proof.

The gradient vector field of the function λ will be denoted by A . We also define a vector field ξ on M by

$$(6.18) \quad \xi^h = \tau u^h + \rho v^h$$

and denote by η the associated 1-form of ξ . Then we can state the following

THEOREM 6.2. *Let M be a submanifold of dimension $n > 2$ with induced (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . Assume that the subbundle D_N^2 is an umbilical 2-section on M . Then the vector fields $v^{-1}A$ and $v^{-1}\xi$ form a canonical pair of the structure, and A and ξ satisfy the the equations*

$$(6.19) \quad \nabla_k f_{ji} = \eta_j g_{ki} - \eta_i g_{kj},$$

$$(6.20) \quad \nabla_j \lambda_i = \psi(\lambda_j \lambda_i + \eta_j \eta_i) - \lambda v^2 g_{ji},$$

$$(6.21) \quad \nabla_j \eta_i = \psi(\lambda_j \eta_i - \eta_j \lambda_i) + v^2 f_{ji},$$

where $\psi = A/v^2$.

PROOF. From (2.8), (2.9), (2.11), (2.12), (2.14), (6.4) and (6.18), we have the relations

$$(6.22) \quad f_j^i f_i^h = -\delta_j^h + v^{-2}(\lambda_j \lambda^h + \eta_j \xi^h),$$

$$(6.23) \quad f_j^i \lambda_i = \lambda \eta_j, \quad f_j^i \eta_i = -\lambda \lambda_j,$$

$$(6.24) \quad \lambda_i \xi^i = \eta_i \lambda^i = 0, \quad \lambda_i \lambda^i = \eta_i \xi^i = v^2(1 - \lambda^2),$$

$$(6.25) \quad g_{kh} f_j^k f_i^h = g_{ji} - v^{-2}(\lambda_j \lambda_i + \eta_j \eta_i),$$

and hence $v^{-1}A$ and $v^{-1}\xi$ are a canonical pair of the structure. See also the proof of Theorem 4.3.

It follows from (2.9) and (6.25) that

$$\lambda_j \lambda_i + \eta_j \eta_i = v^2(u_j u_i + v_j v_i).$$

Comparing this equation with (6.7), we have the equation (6.20). Differentiating (6.18) covariantly and making use of (6.2) and (6.3), we can obtain the equation (6.21). The equation (6.19) follows from (6.1) and (6.18). This completes the proof.

The equation (6.21) shows that the vector field ξ is a Killing one on M .

It follows from (6.6) that

$$\tau\tau_j + \rho\rho_j = A\lambda_j,$$

or equivalently

$$(6.26) \quad \nabla_j v^2 = 2v^2\psi\lambda_j.$$

From (6.20) and (6.24) we obtain the equation

$$\lambda^j \nabla_j \lambda_i = [A(1 - \lambda^2) - \lambda v^2]\lambda_i,$$

which shows that trajectories of the gradient vector field A of λ are geodesic arcs in M . As seen in Paragraph 3, the submanifold M is locally diffeomorphic to the product $R \times \bar{M}$ of a 1-dimensional manifold R and an $(n-1)$ -dimensional manifold \bar{M} . We can choose an adapted coordinate system (x^1, x^a) for the function λ in a neighborhood W , with respect to which components of the metric tensor g of M are equal to

$$g_{11} = 1, \quad g_{b1} = g_{1a} = 0$$

and the first coordinate x^1 is the arc-length of λ -curves.

In terms of such a coordinate system (x^1, x^a) , it follows from (6.24) that $\eta_1 = \xi^1 = \lambda_a = 0$ and

$$(6.27) \quad v^2 = \lambda'^2 / (1 - \lambda^2).$$

Taking account of (6.26) and (6.27), we have

$$(6.28) \quad \psi = \lambda'' / \lambda'^2 + \lambda / (1 - \lambda^2).$$

The equations (6.27) and (6.28) show that v^2 and ψ are functions of the first coordinate x^1 only. From the first equation of (6.23) we obtain

$$(6.29) \quad f_{b^1} = (\lambda / \lambda') \eta_b.$$

Putting $j=b$ and $i=1$ in (6.21) and making use of (6.27) to (6.29), we have the equation

$$\nabla_b \eta_1 = -(\lambda'' / \lambda') \eta_b.$$

Since the component $\{^1_{b1}\}$ of the Christoffel symbol and η_1 vanish identically, this equation is reduced to

$$\{^a_{b1}\} \eta_a = (\lambda'' / \lambda') \eta_b.$$

Putting $j=1$ and $i=a$ in (6.21) again, we also obtain

$$\partial_1 \eta_a - \{^e_{1a}\} \eta_e = (\lambda'' / \lambda') \eta_a.$$

By these two equations we have

$$\partial_1 \eta_a = 2(\lambda''/\lambda')\eta_a$$

and we may put

$$(6.30) \quad \eta_a = \lambda'^2 \bar{\eta}_a,$$

where $\bar{\eta}_a$ depend only on x^a and define a 1-form on \bar{M} .

Since ξ is a Killing vector field on M , we have

$$\mathcal{L}_\xi g_{ji} = \xi^h \partial_h g_{ji} + (\partial_j \xi^h) g_{hi} + (\partial_i \xi^h) g_{jh} = 0,$$

where \mathcal{L}_ξ is the operator of Lie differentiation with respect to the vector field ξ . Putting $j=b$ and $i=1$ in the above equation, we obtain $(\partial_1 \xi^a) g_{ba} = 0$, which implies that $\partial_1 \xi^a = 0$. Therefore the vector ξ is independent of x^1 . Since $\xi^1 = 0$, ξ is regarded as a vector field on \bar{M} . Moreover we see from (6.24), (6.27) and (6.30) that

$$(6.31) \quad \bar{\eta}_a \xi^a = 1.$$

Putting $j=b$ and $i=a$ in (6.20) and making use of (6.27), (6.28) and (6.30), we have the equation

$$V_b \lambda_a = - [\lambda \lambda'^2 / (1 - \lambda^2)] g_{ba} + [\lambda'^2 \lambda'' + \lambda \lambda'^4 / (1 - \lambda^2)] \bar{\eta}_b \bar{\eta}_a.$$

Since the left term of the equation is written as

$$V_b \lambda_a = - \{ \begin{smallmatrix} 1 \\ ba \end{smallmatrix} \} \lambda' = (1/2) \lambda' \partial_1 g_{ba}$$

with respect to the adapted coordinate system, we have the linear differential equation

$$\partial_1 g_{ba} + [2\lambda \lambda' / (1 - \lambda^2)] g_{ba} = 2[\lambda' \lambda'' + \lambda \lambda'^3 / (1 - \lambda^2)] \bar{\eta}_b \bar{\eta}_a$$

in each components g_{ba} of the metric tensor g of M . The solutions of these equations are given by

$$(6.32) \quad g_{ba} = (1 - \lambda^2) g_{ba}^* + \lambda'^2 \bar{\eta}_b \bar{\eta}_a,$$

where g_{ba}^* depend on x^a only. Since ξ is a unit vector field on \bar{M} and $\eta_b = g_{ba} \xi^a$, it follows from (6.30) and (6.32) that

$$(6.33) \quad g_{ba}^* \xi^a = 0.$$

Now we put

$$(6.34) \quad \bar{g}_{ba} = g_{ba}^* + \bar{\eta}_b \bar{\eta}_a.$$

Then we can easily verify the positive definiteness of \bar{g} . Let the $(n-1)$ -dimensional manifold \bar{M} be a Riemannian one endowed with the metric tensor \bar{g} . From (6.32) and (6.34) we have the relations

$$(6.35) \quad g_{ba} = (1 - \lambda^2)\bar{g}_{ba} - (1 - \lambda^2 - \lambda'^2)\bar{\eta}_b\bar{\eta}_a.$$

The contravariant components of \bar{g} are then given by

$$g^{ba} = (1 - \lambda^2)^{-1}\bar{g}^{ba} - [(1 - \lambda^2)^{-1} - (\lambda')^{-2}]\xi^b\xi^a.$$

Now we denote by $\bar{\xi}$ the restriction of ξ on a λ -hypersurface $\bar{M}(x)$ through an ordinary point x of λ , which is regarded as a vector field on \bar{M} . Then we have

$$\bar{g}^{ba}\bar{\eta}_a = \bar{\xi}^b.$$

Since ξ is a Killing vector field on M , we have

$$\mathcal{L}_\xi \eta_i = \xi^j \partial_j \eta_i + (\partial_i \xi^j) \eta_j = 0.$$

Putting $i = a$ in this equation, we obtain the equation

$$\xi^e \partial_e \bar{\eta}_a + \bar{\eta}_e \partial_a \xi^e = 0.$$

Let $\bar{\nabla}$ be the Riemannian connection with respect to \bar{g} of \bar{M} . Then it follows from the above equation that

$$(6.36) \quad \xi^e \bar{\nabla}_e \bar{\eta}_a = \bar{\eta}_e \bar{\nabla}_a \xi^e = 0$$

because $\bar{\xi}$ is a unit vector field on \bar{M} . By use of (6.36), the Christoffel symbol $\{\overset{h}{j}_i\}$ of M splits into the components

$$(6.37) \quad \begin{cases} \{\overset{1}{11}\} = \{\overset{1}{c1}\} = \{\overset{1}{11}\} = 0, \\ \{\overset{a}{c1}\} = -[\lambda\lambda'/(1-\lambda^2)]\delta_c^a + [\lambda\lambda'/(1-\lambda^2) + \lambda''/\lambda']\bar{\eta}_c\bar{\xi}^a, \\ \{\overset{1}{cb}\} = \lambda\lambda'\bar{g}_{cb} - (\lambda\lambda' + \lambda'\lambda'')\bar{\eta}_c\bar{\eta}_b, \\ \{\overset{a}{cb}\} = \{\overset{a}{cb}\} + [\lambda'^2/(1-\lambda^2) - 1](\bar{\eta}_c\bar{\nabla}_b \bar{\xi}^a + \bar{\eta}_b\bar{\nabla}_c \bar{\xi}^a). \end{cases}$$

From (6.19), we see that $\nabla_1 f_b^a = 0$. The second equation of (6.23) implies that $f_e^a \xi^e = \eta_e f_b^e = 0$. Therefore we can verify that $\partial_1 f_b^a = 0$. This shows that f_b^a do not depend on the first coordinate x^1 and hence define a tensor field of type $(1, 1)$ on \bar{M} , which will be denoted by $\bar{f} = (\bar{f}_b^a)$. Then we have the equation

$$(6.38) \quad \bar{f}_e^a \bar{\xi}^e = \bar{\eta}_e \bar{f}_b^e = 0.$$

By means of (6.35), the covariant components $\bar{f}_{ba} = \bar{g}_{ea} \bar{f}_b^e$ on \bar{M} are related to those of f on M by

$$f_{ba} = (1 - \lambda^2)\bar{f}_{ba}.$$

It follows from the equations (6.22) and (6.25) that the tensor field \bar{f} satisfies

$$(6.39) \quad \bar{f}_b^e \bar{f}_e^a = -\delta_b^a + \bar{\eta}_b \bar{\xi}^a$$

or equivalently

$$(6.40) \quad \bar{g}_{ac} \bar{f}_b^d \bar{f}_a^c = \bar{g}_{ba} - \bar{\eta}_b \bar{\eta}_a.$$

Putting $j=b$ and $i=a$ in (6.21) and making use of the expressions (6.37), we have

$$(6.41) \quad \bar{V}_b \bar{\xi}^a = \bar{f}_b^a \quad \text{and} \quad \bar{V}_b \bar{\eta}_a = \bar{f}_{ba}.$$

Putting $k=c$, $j=b$ and $i=a$ in (6.19), we have

$$\nabla_c f_{ba} = \lambda'^2(1-\lambda^2)(\bar{\eta}_b \bar{g}_{ca} - \bar{\eta}_a \bar{g}_{cb}).$$

On the other hand, by taking account of the expression

$$\nabla_c f_{ba} = \partial_c f_{ba} - \{c b\}^e f_{ea} - \{c a\}^e f_{be} - \{c b\}^1 f_{1a} - \{c a\}^1 f_{b1}$$

and the equations (6.29), (6.30) and (6.37), we obtain

$$\nabla_c f_{ba} = (1-\lambda^2) \bar{V}_c \bar{f}_{ba} + (1-\lambda^2)(\lambda'^2-1)(\bar{\eta}_b \bar{g}_{ca} - \bar{\eta}_a \bar{g}_{cb}).$$

Consequently these equations give

$$(6.42) \quad \bar{V}_c \bar{f}_{ba} = \bar{\eta}_b \bar{g}_{ca} - \bar{\eta}_a \bar{g}_{cb}.$$

Hence the equations (6.38) to (6.42) show that the totality $(\bar{f}, \bar{g}, \bar{\xi}, \bar{\eta})$ constitutes a Sasakian structure on \bar{M} .

The restrictions f_b^a, g_{ba}, ξ^a and η_b of f, g, ξ and η of M on each λ -hypersurface $\bar{M}(x)$ form a tensor field, an induced metric tensor, a vector field and a 1-form of $\bar{M}(x)$ respectively. From the equations (6.22) to (6.25), (6.27) and (6.29), we have

$$f_c^b f_b^a = -\delta_c^a + (1/\lambda'^2) \eta_c \xi^a, \quad \eta_a f_b^a = f_b^a \xi^b = 0, \\ \eta_a \xi^a = \lambda'^2, \quad g_{ac} f_b^d f_a^c = g_{ba} - (1/\lambda'^2) \eta_b \eta_a$$

with respect to the adapted coordinate system (x^1, x^a) . If we denote by $\hat{\nabla}$ the covariant differentiation with respect to the induced metric g_{ba} , then it follows from (6.19) to (6.21) that

$$\hat{\nabla}_b \eta_a = [\lambda'^2/(1-\lambda^2)] f_{ba}, \quad \hat{\nabla}_c f_{ba} = [1/(1-\lambda^2)] (\eta_b g_{ca} - \eta_a g_{cb})$$

on $\bar{M}(x)$, by use of (6.27) and (6.29). Since λ and λ' are constants on each $\bar{M}(x)$, we consider a vector field $\hat{\xi}^a$ and a 1-form $\hat{\eta}_b$ defined by $\hat{\xi}^a = \lambda' \xi^a$ and $\hat{\eta}_b = \lambda' \eta_b$

on $\bar{M}(x)$. Then we see that the totality $(f, g, \hat{\xi}, \hat{\eta})$ constitutes an almost contact metric structure on $\bar{M}(x)$. The last two equations are rewritten as

$$\hat{V}_b \hat{\eta}_a = [\lambda'/(1-\lambda^2)]f_{ba}, \quad \hat{V}_c f_{ba} = [\lambda'/(1-\lambda^2)](\hat{\eta}_b g_{ca} - \hat{\eta}_a g_{cb}).$$

The normality of an almost contact metric structure (f, g, ξ, η) is defined by the Nijenhuis tensor of the structure, that is,

$$N_{cb}{}^a = f_c{}^d(V_d f_b{}^a - V_b f_d{}^a) - f_b{}^d(V_d f_c{}^a - V_c f_d{}^a) + \eta_c V_b \xi^a - \eta_b V_c \xi^a = 0,$$

see [13, 20]. By means of the above equations, it can be verified that the almost contact metric structure $(f, g, \hat{\xi}, \hat{\eta})$ on $\bar{M}(x)$ is normal.

To speak in general, we suppose that an almost contact metric structure (f, g, ξ, η) on a manifold satisfies the relations

$$(6.43) \quad V_b \eta_a = (p/q)f_{ba}, \quad V_c f_{ba} = (p/q)(\eta_b g_{ca} - \eta_a g_{cb}),$$

$p(>0)$ and $q(\neq 0)$ being constants. If we define a structure $(\bar{f}, \bar{g}, \bar{\xi}, \bar{\eta})$ by

$$(6.44) \quad \bar{\xi}^a = (1/q)\xi^a, \quad \bar{\eta}_b = q\eta_b, \quad \bar{f}_b{}^a = f_b{}^a, \quad \bar{g}_{ba} = pg_{ba} + (q^2 - p)\eta_b \eta_a,$$

then we see that the structure $(\bar{f}, \bar{g}, \bar{\xi}, \bar{\eta})$ is Sasakian. Such a deformation from a normal almost contact metric structure to a Sasakian one defined by (6.44) will be said to be *AS-homothetic*. An *AS-homothetic* deformation is reduced to a *D-homothetic* one introduced by S. Tanno [18] if and only if $p=q$.

Summing up our arguments stated above, we see that each λ -hypersurface $\bar{M}(x)$ is *AS-homothetic* to \bar{M} . Thus we have

THEOREM 6.3. *Let M be a submanifold of dimension $n > 2$ with (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . If the subbundle D_N^2 of the normal bundle is an umbilical 2-section on M , then each λ -hypersurface is *AS-homothetic* to a Sasakian manifold \bar{M} .*

By a straightforward computation from (6.39), the components of the curvature tensor $R_{kji}{}^h$ of M are given by

$$(6.45) \quad \left\{ \begin{aligned} R_{111}{}^1 &= R_{111}{}^a = R_{11b}{}^a = 0, \\ R_{dcb}{}^1 &= \left(\frac{\lambda\lambda'^3}{1-\lambda^2} + \lambda'\lambda'' \right) (\bar{\eta}_d \bar{f}_{cb} - \bar{\eta}_c \bar{f}_{db} - 2\bar{\eta}_b \bar{f}_{dc}), \\ R_{1cb}{}^a &= \left(\frac{\lambda''}{\lambda'(1-\lambda^2)} + \frac{\lambda\lambda'}{(1-\lambda^2)^2} \right) [\lambda'^2(2\bar{\eta}_c \bar{f}_b{}^a + \bar{\eta}_b \bar{f}_c{}^a) - (1-\lambda^2)\bar{f}_{cb} \bar{\xi}^a], \\ R_{d1b}{}^1 &= - \left(\lambda'^2 + \lambda\lambda'' + \frac{\lambda^2 \lambda'^2}{1-\lambda^2} \right) \bar{g}_{db} + \left(\lambda'^2 + \lambda\lambda'' + \frac{\lambda^2 \lambda'^2}{1-\lambda^2} + \lambda'\lambda'' \right) \bar{\eta}_d \bar{\eta}_b, \end{aligned} \right.$$

$$\begin{aligned}
 R_{1c1}{}^a &= - \left(\frac{\lambda'^2}{1-\lambda^2} + \frac{\lambda\lambda''}{1-\lambda^2} + \frac{\lambda^2\lambda'^2}{(1-\lambda^2)^2} \right) \delta_c^a + \left(\frac{\lambda'^2}{1-\lambda^2} + \frac{\lambda\lambda''}{1-\lambda^2} + \frac{\lambda^2\lambda'^2}{(1-\lambda^2)^2} + \frac{\lambda''}{\lambda'} \right) \bar{\eta}_c \bar{\xi}^a, \\
 R_{dcb}{}^a &= \bar{R}_{dcb}{}^a - \frac{\lambda^2\lambda'^2}{1-\lambda^2} (\delta_d^a \bar{g}_{cb} - \delta_c^b \bar{g}_{da}) \\
 &\quad + \left(\frac{\lambda'^4}{(1-\lambda^2)^2} + \frac{\lambda^2\lambda'^2}{1-\lambda^2} + \frac{\lambda\lambda'^2\lambda''}{1-\lambda^2} - 1 \right) (\delta_d^a \bar{\eta}_c \bar{\eta}_b - \delta_c^a \bar{\eta}_d \bar{\eta}_b) \\
 &\quad - \left(\frac{\lambda^2\lambda'^2}{1-\lambda^2} + \frac{\lambda'^2}{1-\lambda^2} + \lambda\lambda'' - 1 \right) (\bar{g}_{ab} \bar{\eta}_c \bar{\xi}^a - \bar{g}_{cb} \bar{\eta}_d \bar{\xi}^a) \\
 &\quad + \left(\frac{\lambda'^2}{1-\lambda^2} - 1 \right) (2\bar{f}_{dc} \bar{f}_b{}^a + \bar{f}_{ab} \bar{f}_c{}^a - \bar{f}_{cb} \bar{f}_d{}^a),
 \end{aligned}$$

where $\bar{R}_{dcb}{}^a$ are components of the curvature tensor of \bar{M} .

If the submanifold M is a space of constant curvature k , that is,

$$(6.46) \quad R_{kji}{}^h = k(\delta_k^h g_{ji} - \delta_j^h g_{ki}),$$

then, comparing the second equation of (6.45) with (6.46), we have

$$(6.47) \quad \lambda'^2/(1-\lambda^2) = v^2 = k,$$

which shows that k is positive. Since $\lambda'' = -k\lambda$ from (6.47), the last equation of (6.45) is written as

$$(6.48) \quad \bar{R}_{dcb}{}^a = k(\delta_d^a \bar{g}_{cb} - \delta_c^a \bar{g}_{db}) - (k-1)(\delta_d^a \bar{\eta}_c \bar{\eta}_b - \delta_c^a \bar{\eta}_d \bar{\eta}_b + \bar{g}_{cb} \bar{\eta}_d \bar{\xi}^a - \bar{g}_{db} \bar{\eta}_c \bar{\xi}^a - \bar{f}_d{}^a \bar{f}_{cb} + \bar{f}_c{}^a \bar{f}_{db} + 2\bar{f}_{dc} \bar{f}_b{}^a)$$

by taking account of (6.35) and (6.46). The equation (6.48) shows that the manifold \bar{M} is a Sasakian space form, that is, a space of constant \bar{f} -sectional curvature.

Conversely, if (6.47) and (6.48) are satisfied, then we can verify that the submanifold M is a space of constant curvature k . Thus we have

THEOREM 6.4. *Let M be a submanifold of dimension $n > 2$ with (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . Assume that the subbundle D_N^2 is an umbilical 2-section on M . Then M is a space of constant curvature if and only if the manifold \bar{M} AS-homothetic to each λ -hypersurface is a Sasakian space form and the sum of two squared mean curvatures of D_N^2 is a constant.*

If the subbundle D_N^2 of the normal bundle $N(M)$ is a concurrent 2-section on M , then we have $l_j = 0$ by (6.5). Therefore the assumptions of Lemma 4.1 are satisfied by the equations (6.2) and (6.3) and hence we have $\tau = k\rho$, k being a constant. From this fact and the equation (6.6), we see that τ and ρ are con-

stants. Thus the following is immediately obtained from Theorem 6.4.

COROLLARY 6.5. *Let M be a submanifold of dimension $n > 2$ with (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . Assume that the subbundle D_N^2 of the normal bundle is a concurrent 2-section on M . Then M is a space of constant curvature if and only if the manifold \bar{M} AS-homothetic to each λ -hypersurface is a Sasakian space form.*

In the case where the submanifold M is of codimension 2 in \bar{M} , the assumptions of Theorem 6.4 show that M is a non-minimal totally umbilical submanifold. The squared mean curvature $\|H\|^2$ of M is expressed as $\|H\|^2 = v^2$ by Lemma 5.3. Therefore the sum of two squared mean curvatures of D_N^2 is a constant if the mean curvature vector field H of M is parallel in the normal bundle. By means of Theorem 6.4, we have the following

COROLLARY 6.6. *Let M be an $n (> 2)$ -dimensional non-minimal totally umbilical submanifold of codimension 2 in a Kaehlerian manifold. Then M is a space of constant curvature if and only if the manifold \bar{M} AS-homothetic to each λ -hypersurface is a Sasakian space form and the mean curvature vector field H is parallel in the normal bundle.*

In the remaining of this Paragraph, we assume that the sum of two squared mean curvatures of D_N^2, v^2 , is a constant, say c^2 , in addition to the assumptions of Theorem 6.3. Then the equation (6.26) gives $\psi = 0$ and hence the equation (6.20) is reduced to

$$(6.49) \quad \nabla_j \lambda_i = -c^2 \lambda g_{ji},$$

which shows that λ is a special concircular scalar field with characteristic constant c^2 . As we have already seen in Paragraph 3, the submanifold M is locally a warped product $R \times_{\sigma} \bar{M}$ of a 1-dimensional Euclidean space R and $(n-1)$ -dimensional Riemannian manifold \bar{M} , to which each λ -hypersurface $\bar{M}(x)$ is homothetic. With respect to an adapted coordinate system (x^1, x^a) for λ , it follows from (3.4), (3.5) and (6.49) that

$$(6.50) \quad \sigma = \lambda',$$

and the metric form of M is given by

$$g_{ji} dx^j dx^i = (dx^1)^2 + \lambda'^2 \bar{g}_{cb} dx^c dx^b,$$

where \bar{g}_{cb} is the metric tensor of \bar{M} . It follows from (6.24) that $\eta_1 = \xi^1 = 0, \lambda^a = 0$ and $\lambda'^2 / (1 - \lambda^2) = c^2$. From (3.17) and (6.50), the non-trivial components of the Christoffel symbol $\{\overset{h}{j}_i\}$ are expressed by

$$(6.51) \quad \{^a_{c1}\} = (\lambda''/\lambda')\delta^a_c, \quad \{\bar{1}_{cb}\} = -\lambda'\lambda''\bar{g}_{cb}, \quad \{^a_{cb}\} = \overline{\{^a_{cb}\}}.$$

It is obvious that the equations (6.29) and (6.30) are satisfied. Putting $k=1$, $j=b$ and $i=a$ in (6.19) and using (6.51), we obtain

$$(6.52) \quad f_{ba} = \lambda'^2 \bar{f}_{ba},$$

where \bar{f}_{ba} depend on \bar{M} only and form a 2-form on \bar{M} . Moreover we see from (6.19) and (6.21) that f_b^a and ξ^a are independent of x^1 , which will be denoted by \bar{f}_b^a and $\bar{\xi}^a$ respectively. Making use of (6.22) to (6.25), (6.29) and (6.30), we can verify that the totality $(\bar{f}, \bar{g}, \bar{\xi}, \bar{\eta})$ constitutes an almost contact metric structure on \bar{M} . By means of (6.21), (6.30), (6.51) and (6.52), we have

$$\bar{V}_b \bar{\eta}_a = c^2 \bar{f}_{ba}.$$

Putting $k=c$, $j=b$ and $i=a$ in (6.19) again, we can obtain

$$\bar{V}_c \bar{f}_{ba} = c^2 (\bar{\eta}_b \bar{g}_{ca} - \bar{\eta}_a \bar{g}_{cb}).$$

Taking account of the last two equations, we can verify that \bar{M} is a normal almost contact metric manifold. If $c^2=1$, then we see that \bar{M} is a Sasakian manifold. Thus we can state the following

THEOREM 6.7. *Let M be a submanifold of dimension $n>2$ with (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . If the subbundle D^2_N of the normal bundle is an umbilical 2-section on M , and if the sum of two squared mean curvatures of D^2_N is a constant, then M is locally a warped product $R \times_{\lambda} \bar{M}$ of a 1-dimensional Euclidean space R and an $(n-1)$ -dimensional normal almost contact metric manifold \bar{M} , and each λ -hypersurface is homothetic to \bar{M} . In the case where the sum of two squared mean curvatures of D^2_N is equal to 1, \bar{M} is a Sasakian manifold.*

If the subbundle D^2_N of the normal bundle is a concurrent 2-section on M , then $l_j=0$ and τ and ρ are constants. Moreover it follows from (6.2) and (6.3) that the assumptions of Theorem 4.6 are satisfied. By Theorem 4.6, the function σ is equal to $(1-\lambda^2)^{1/2}$. Therefore we have $c^2=1$ by (6.50). Combining this fact with Theorem 6.7, we can state the following

THEOREM 6.8. *Let M be a submanifold of dimension $n>2$ with (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . If the subbundle D^2_N is a concurrent 2-section on M , then M is locally a warped product $R \times_{\lambda} \bar{M}$ of a 1-dimensional Euclidean space R and an $(n-1)$ -dimensional Sasakian manifold \bar{M} , and each λ -hypersurface is homothetic to \bar{M} .*

The solution of $\lambda'^2 = c^2(1 - \lambda^2)$ along a λ -curve is given by

$$\lambda = \sin cx^1.$$

Therefore, if M is complete, λ has stationary points corresponding to $x^1 = 0$ and $x^1 = \pi/c$. Thus, by use of a theorem due to Y. Tashiro ([21, 22]), we can state

THEOREM 6.9. *Let M be a complete submanifold of dimension $n > 2$ with (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . If the subbundle D_N^2 of the normal bundle is an umbilical 2-section on M , and if the sum of two squared mean curvatures of D_N^2 is a constant, then M is a sphere.*

By an argument similar to Corollary 6.6, we have immediately, from Theorem 6.9, the following corollary first due to M. Okumura [12].

COROLLARY 6.10. *Let M be an $n (> 2)$ -dimensional complete non-minimal totally umbilical submanifold of codimension 2 in a Kaehlerian manifold. If the mean curvature vector field of M is parallel in the normal bundle, then M is a sphere.*

7. Submanifolds with induced normal (f, g, D^2, λ) -structures

In this Paragraph we consider a submanifold M of dimension $n > 2$ with induced normal (f, g, D^2, λ) -structure in an m -dimensional Kaehlerian manifold \tilde{M} ($m - n \geq 2$).

If we put

$$h_{jip}\alpha_p = H_{ji}, \quad h_{jip}\beta_p = K_{ji} \quad \text{and} \quad \beta_p \nabla_j^{\perp} \alpha_p = l_j,$$

then the equations (5.5), (5.7), (5.8) and (5.10) are written as

$$(7.1) \quad \nabla_k f_{ji} = u_j H_{ki} - u_i H_{kj} + v_j K_{ki} - v_i K_{kj},$$

$$(7.2) \quad \nabla_j u_i = -\lambda K_{ji} - f_i^h H_{jh} + l_j v_i,$$

$$(7.3) \quad \nabla_j v_i = \lambda H_{ji} - f_i^h K_{jh} - l_j u_i,$$

$$(7.4) \quad \lambda_j = u^i K_{ji} - v^i H_{ji}$$

on the submanifold M . We shall prove the following

LEMMA 7.1. *Let M be a submanifold with induced (f, g, D^2, λ) -structure in a Kaehlerian manifold. Then the distribution D^2 is involutive if and only if the following is satisfied:*

$$(7.5) \quad f_j^h v^i H_{ih} + \lambda u^i H_{ji} - (f_j^h u^i K_{ih} - \lambda v^i K_{ji}) + Au_j + Bv_j = 0$$

with scalar fields A and B on M .

PROOF. If D^2 is involutive, then we have the equation

$$[U, V]^h = \bar{A}u^h + \bar{B}v^h,$$

where \bar{A} and \bar{B} are scalar fields on M . Putting $A = \bar{A} + u^i l_i$ and $B = \bar{B} + v^i i_i$, this equation is reduced to (7.5) by virtue of (7.2) and (7.3). The converse is trivial.

If we substitute (7.1) to (7.3) into (2.19), then we have

$$(7.6) \quad S_{ji}{}^h = (f_j{}^k H_k{}^h - f_k{}^h H_j{}^k)u_i - (f_i{}^k H_k{}^h - f_k{}^h H_i{}^k)u_j + (f_j{}^k K_k{}^h - f_k{}^h K_j{}^k)v_i \\ - (f_i{}^k K_k{}^h - f_k{}^h K_i{}^k)v_j + L_i(u_j v^h - v_j u^h) - L_j(u_i v^h - v_i u^h),$$

where we have put $L_i = \alpha_q \beta_p l_{iqp}$. The normal connection restricted on the subbundle D_N^2 of the normal bundle is said to be flat if $L_i = 0$ identically. We prove the following

LEMMA 7.2. *Let M be a submanifold with normal (f, g, D^2, λ) -structure in a Kaehlerian manifold. If the distribution D^2 is involutive and the normal connection restricted on the subbundle D_N^2 is flat, then we have the relations*

$$(7.7) \quad (1 - \lambda^2)u^i H_{ji} = u^i u^h H_{ih} u_j + u^i v^h H_{ih} v_j,$$

$$(7.8) \quad (1 - \lambda^2)v^i H_{ji} = u^i v^h H_{ih} u_j + v^i v^h H_{ih} v_j,$$

$$(7.9) \quad (1 - \lambda^2)u^i K_{ji} = u^i u^h K_{ih} u_j + u^i v^h K_{ih} v_j,$$

$$(7.10) \quad (1 - \lambda^2)v^i K_{ji} = u^i v^h K_{ih} u_j + v^i v^h K_{ih} v_j,$$

$$(7.11) \quad u^j u^i H_{ji} - v^j v^i H_{ji} = -2u^j v^i K_{ji}, \quad u^j u^i K_{ji} - v^j v^i K_{ji} = 2u^j v^i H_{ji}.$$

PROOF. Under the assumptions of the lemma, it follows from (7.6) that

$$(7.12) \quad (f_j{}^k H_k{}^h - f_k{}^h H_j{}^k)u_i - (f_i{}^k H_k{}^h - f_k{}^h H_i{}^k)u_j + (f_j{}^k K_k{}^h - f_k{}^h K_j{}^k)v_i \\ - (f_i{}^k K_k{}^h - f_k{}^h K_i{}^k)v_j = 0.$$

Contracting this equation with u^h and substituting (7.5), we have

$$(7.13) \quad (f_j{}^k u^h H_{kh} - \lambda v^k H_{kj})u_i - (f_i{}^k u^h H_{kh} - \lambda v^k H_{ki})u_j \\ + (f_j{}^k v^h H_{kh} + \lambda u^k H_{kj})v_i - (f_i{}^k v^h H_{kh} + \lambda u^k H_{ki})v_j + A(u_j v_i - v_j u_i) = 0,$$

and, contracting this equation with u^i ,

$$(1 - \lambda^2)(f_j{}^k u^h H_{kh} - \lambda v^k H_{kj}) + 2\lambda u^i v^h H_{ih} u_j - \lambda(u^i u^h H_{ih} - v^i v^h H_{ih})v_j \\ - A(1 - \lambda^2)v_j = 0.$$

Contracting this equation with v^j , we have $A=0$ and

$$(7.14) \quad (1-\lambda^2)(f_j^k u^h H_{kh} - \lambda v^k H_{kj}) = -2\lambda u^i v^h H_{ih} u_j + \lambda(u^i u^h H_{ih} - v^i v^h H_{ih})v_j.$$

Substituting (7.14) into (7.13), we can obtain the equation

$$(7.15) \quad (1-\lambda^2)(f_j^k v^h H_{kh} + \lambda u^k H_{kj}) = \lambda(u^i u^h H_{ih} - v^i v^h H_{ih})u_j + 2\lambda u^i v^h H_{ih} v_j.$$

If we substitute (7.14) into the contraction of (7.15) with f_j^i , then we have the relations (7.7) and (7.8). Similarly, from the contraction of (7.12) with v^h , we see that $B=0$ and obtain the relations (7.9) and (7.10). Since $A=B=0$, we also have the relation (7.11) by taking account of the equations (7.5) and (7.7) to (7.10). This completes the proof.

We also prove the following

LEMMA 7.3. *In addition to the assumptions of Lemma 7.2, we assume that one of two orthonormal vector fields in D_N^2 is an umbilical (not geodesic) section on M . Then the subbundle D_N^2 is an umbilical 2-section on M .*

PROOF. If α is an umbilical section on M , that is,

$$(7.16) \quad H_{ji} = \tau g_{ji},$$

τ being the mean curvature belonging to α , then it follows from (7.11) that

$$u^j u^i K_{ji} = v^j v^i K_{ji}, \quad u^j v^i K_{ji} = 0.$$

From (7.9), (7.10) and (7.16), we obtain

$$(7.17) \quad u^i K_{ji} = \rho u_j \quad \text{and} \quad v^i K_{ji} = \tau v_j,$$

where $\rho = u^j u^i K_{ji}/(1-\lambda^2)$. By virtue of (7.12), (7.16) and (7.17), we have the equation

$$(7.18) \quad f_j^k K_k^h = f_k^h K_j^k.$$

It follows from (7.4), (7.16) and (7.17) that

$$\lambda_j = \rho u_j - \tau v_j.$$

Differentiating this equation covariantly and making use of (7.2) and (7.3), we obtain

$$\nabla_j \lambda_i = \rho_j u_i - \tau_j v_i + \tau \rho f_{ji} - \lambda \rho K_{ji} + \tau f_i^h K_{jh} - \lambda \tau^2 g_{ji}.$$

If we take the skew-symmetric parts of this equation and make use of (7.18), then we have

$$(7.19) \quad \rho_j u_i - \rho_i u_j - \tau_j v_i + \tau_i v_j + 2\tau \rho f_{ji} + 2\tau f_i^h K_{jh} = 0,$$

and, contracting this equation with u^i and making use of (7.17),

$$(7.20) \quad \rho_i = au_i + bv_i,$$

where $a = u^i \rho_i / (1 - \lambda^2)$ and $b = -u^i \tau_i / (1 - \lambda^2)$. Similarly, from (7.19), we can obtain

$$(7.21) \quad \tau_i = -bu_i + cv_i,$$

where $c = v^i \tau_i / (1 - \lambda^2)$. Moreover, substituting (7.20) and (7.21) into (7.19), we have the equation

$$f_j{}^h K_{ih} = \rho f_{ji}.$$

Consequently, contracting this equation with $f_k{}^j$ and making use of (7.17), we have

$$(7.22) \quad K_{ji} = \rho g_{ji}$$

and this completes the proof.

Combining Lemma 7.3 and Theorem 6.3, we can state

THEOREM 7.4. *Let M be a submanifold of dimension $n > 2$ with normal (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . Assume that the distribution D^2 is involutive and the normal connection restricted on the subbundle D_N^2 is flat. If one of two orthonormal vector fields in D_N^2 is an umbilical (not geodesic) section on M , then each λ -hypersurface is AS-homothetic to a Sasakian manifold \bar{M} .*

The following theorem is a combination of Lemma 7.3 with Theorem 6.4.

THEOREM 7.5. *Let M be a submanifold of dimension $n > 2$ with normal (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . Assume that the distribution D^2 is involutive, the normal connection restricted on the subbundle D_N^2 is flat, and one of two orthonormal vector fields in D_N^2 is an umbilical (not geodesic) section on M . Then M is a space of constant curvature if and only if the manifold \bar{M} AS-homothetic to each λ -hypersurface is a Sasakian space form and the sum of two squared mean curvatures of D_N^2 is a constant.*

Under the assumptions of Lemma 7.2, if one of two orthonormal vector fields α and β in D_N^2 is concurrent along M , then we see that $l_j = 0$ and the relations (7.16) and (7.22) are valid, that is, the subbundle D_N^2 is an umbilical 2-section on M . It follows from (7.2) and (7.3) that the vector fields U and V are infinitesimal conformal transformations and satisfy the relations (4.1) and (4.2). By Lemmas 4.1 and 6.1, both the mean curvatures belonging to α and β are constants. There-

fore, by Theorems 4.6 and 6.7, we see that the submanifold M is locally a warped product $R \times_{\rho} \bar{M}$ of a 1-dimensional Euclidean space R and an $(n-1)$ -dimensional Sasakian manifold \bar{M} , to which each λ -hypersurface is homothetic. Thus we state

THEOREM 7.6. *Let M be a submanifold of dimension $n > 2$ with normal (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . Assume that the distribution D^2 is involutive and the normal connection restricted on the subbundle D_N^2 is flat. If one of two orthonormal vector fields in D_N^2 is concurrent along M , then M is locally a warped product $R \times_{\lambda} \bar{M}$ of a 1-dimensional Euclidean space R and an $(n-1)$ -dimensional Sasakian manifold \bar{M} .*

The following theorem follows from Theorem 4.8 or 6.9.

THEOREM 7.7. *Let M be a complete submanifold of dimension $n > 2$ with normal (f, g, D^2, λ) -structure in a Kaehlerian manifold, where λ does not vanish almost everywhere on M . Assume that the distribution D^2 is involutive and the normal connection restricted on the subbundle D_N^2 is flat. If one of two orthonormal vector fields in D_N^2 is concurrent along M , then M is a sphere.*

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