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On the distribution of a statistic in multivariate inverse regression analysis

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§1. Introduction

In a multivariate inverse regression problem we are interested in making inference about an unknown $q \times 1$ vector $\boldsymbol{x} = (x_1, \ldots, x_q)'$ from an observed $p \times 1$ response vector $\boldsymbol{y} = (y_1, \ldots, y_p)'$. Brown [3] has summarized various aspects of the problem. We assume that $\boldsymbol{y}$ is random, $\boldsymbol{x}$ is fixed and

$$
\begin{aligned}
(1.1) \qquad \boldsymbol{y} &= \boldsymbol{a} + B'\boldsymbol{x} + \boldsymbol{e} \\
&= \Theta' \begin{bmatrix} 1 \\ \boldsymbol{x} \end{bmatrix} + \boldsymbol{e}
\end{aligned}
$$

where $\Theta' = [\boldsymbol{a}, B']$: $p \times (1+q)$ is the matrix of unknown parameters and $\boldsymbol{e}$ is an error vector having a multivariate normal distribution $N_p[\boldsymbol{0}, \Sigma]$. Further, suppose that the N informative observations on $\boldsymbol{y}$ and $\boldsymbol{x}$ have been given. When $p \geq q$, it is possible to obtain a natural point estimate for $\boldsymbol{x}$, and to construct a confidence region for $\boldsymbol{x}$, based on a statistic, which is a quadratic form of the estimate. For an application of the confidence region it is required to give the upper percentage point of the statistic.

The purpose of this paper is to study the distribution of the statistic mentioned above. We shall derive an asymptotic expansion for the distribution function of the statistic up to the order N^{-2} and hence for the upper percentage point of the statistic. In Section 3 we treat the distribution problem in the situation where Θ is known and Σ is unknown. We note that the distribution of the statistic in this case is essentially the same as one of a statistic in growth curve model. The distribution has been studied by Rao [6] and Gleser and Olkin [4]. The numerical accuracy of our asymptotic approximations is checked by comparing with exact results of Gleser and Olkin [4]. In Section 4 we treat the distribution problem in the situation where Θ and Σ are unknown. In this case a reduction of the distribution problem is given. By using the reduction and perturbation method we shall obtain the asymptotic expansion of the distribution function of the statistic. Some formulas used in deriving the asymptotic expansions are summarized in Section 5.

§2. A distribution problem

Suppose that the N independent observations on $\boldsymbol{y}$ and $\boldsymbol{x}$ following the model (1.1) have been given, and let these observations be denoted by

$$Y = [\boldsymbol{y}_1, \dots, \boldsymbol{y}_N]', \quad X = [\boldsymbol{x}_1, \dots, \boldsymbol{x}_N]'.$$

Then the observations satisfy

$$Y = [\boldsymbol{j}_N, X]\Theta + E \tag{2.1}$$

where $\boldsymbol{j}_N = (1, \dots, 1)'$ and E is an $N \times p$ error matrix whose rows are independently distributed as $N_p[\boldsymbol{0}, \Sigma]$. The observation $\boldsymbol{y}$ and $\boldsymbol{x}$ for a new object satisfy the model (1.1). We assume that $\boldsymbol{y}$ is observed, but $\boldsymbol{x}$ is unknown. Since $\boldsymbol{x}$ is a fixed variate we may without loss of generality assume that

$$X'\boldsymbol{j}_N = \boldsymbol{0}. \tag{2.2}$$

Further, we put on the usual assumptions;

$$\operatorname{rank}(X) = q \quad \text{and} \quad n = N - q - 1 \geq p. \tag{2.3}$$

If Θ and Σ are known, from (1.1) we may estimate $\boldsymbol{x}$ by

$$\hat{\boldsymbol{x}}_0 = (B\Sigma^{-1}B')^{-1}B\Sigma^{-1}(\boldsymbol{y} - \boldsymbol{a}) \tag{2.4}$$

which is obtained by the maximum likelihood method based on (1.1) or by minimizing

$$(\boldsymbol{y} - \boldsymbol{a} - B'\boldsymbol{x})'\Sigma^{-1}(\boldsymbol{y} - \boldsymbol{a} - B'\boldsymbol{x})$$

with respect to $\boldsymbol{x}$. Since $\hat{\boldsymbol{x}}$ is distributed as $N_p[\boldsymbol{x}, (B\Sigma^{-1}B')^{-1}]$, this suggests to use a confidence region for $\boldsymbol{x}$, based on

$$Q_0 = (\hat{\boldsymbol{x}}_0 - \boldsymbol{x})'B\Sigma^{-1}B'(\hat{\boldsymbol{x}}_0 - \boldsymbol{x}). \tag{2.5}$$

The confidence region for $\boldsymbol{x}$ of confidence $1-\alpha$ is given by the ellipsoid $\{x | Q_0 \leq \chi_q^2(\alpha)\}$, where $\chi_q^2(\alpha)$ is the upper α point of a χ^2-distribution with q degrees of freedom. If Θ and Σ are unknown, it is natural to replace them by their estimates. The usual estimates of Θ and Σ based on (2.1) are given by

$$\hat{\boldsymbol{a}} = \bar{\boldsymbol{y}} = \frac{1}{N}\sum_{j=1}^N \boldsymbol{y}_j, \quad \hat{B} = (X'X)^{-1}X'Y,$$

$$\hat{\Sigma} = S = \frac{1}{n}Y'\left\{I_N - \frac{1}{N}\boldsymbol{j}_N\boldsymbol{j}_N' - X(X'X)^{-1}X'\right\}Y$$

$$= \frac{1}{n}\sum_{j=1}^N (\boldsymbol{y}_j - \bar{\boldsymbol{y}} - \hat{B}'\boldsymbol{x}_j)(\boldsymbol{y}_j - \bar{\boldsymbol{y}} - \hat{B}'\boldsymbol{x}_j)'.$$

We use the following statistics according as Θ is known or not:

(i) The case when Θ is known and Σ is unknown;

$$Q_1 = (\hat{x}_1 - x)' B S^{-1} B' (\hat{x}_1 - x) \tag{2.6}$$

where $\hat{x}_1 = (BS^{-1}B')^{-1} BS^{-1}(y - a)$.

(ii) The case when Θ and Σ are unknown;

$$Q_2 = (\hat{x}_2 - x)' \hat{B} S^{-1} \hat{B}' (\hat{x}_2 - x) \tag{2.7}$$

where $\hat{x}_2 = (\hat{B} S^{-1} \hat{B}')^{-1}(y - \bar{y})$.

Our main interest concerns with the case (ii). The case (i) is a step for obtaining the distribution of Q_2. It may be noted that in the case (i) the statistic obtained from Q_1 by changing S to

$$\tilde{S} = \frac{1}{N}(Y - j_N a' - XB)'(Y - j_N a' - XB)$$

should be used. However, we note that the distribution of the statistic can be obtained from the distribution of Q_1 by changing n to N.

To obtain the actual confidence regions we need to evaluate the value of $U_i(\alpha)$ $(i = 1, 2)$ such that

$$\Pr\{Q_i \le U_i(\alpha)\} = 1 - \alpha.$$

Wood [8] has proposed this type of confidence regions, but he does not solve the distribution problem. It is easily checked that the asymptotic value of $U_i(\alpha)$ as $N \to \infty$ is $\chi_q^2(\alpha)$. To obtain a more accurate approximation for $U_i(\alpha)$, we shall derive their asymptotic expansions which are given in Sections 3 and 4. It may be noted that the distribution of Q_1 is much simpler than that of Q_2.

§3. The case when Θ is known and Σ is unknown

In this section we shall obtain an asymptotic expansion of the distribution of Q_1 in (2.6), which gives an asymptotic expansion of $U_1(\alpha)$. Since nS and $y - a$ are independently distributed as a Wishart distribution $W_p(\Sigma, n)$ and a normal distribution $N_p[B'x, \Sigma]$ respectively, we note that the distribution of $\hat{x}_1$ is essentially the same as that of the estimate of regression coefficients in a general multivariate linear model. From this fact and Lemma A.2 in Gleser and Olkin [4], we have the following:

THEOREM 1. *Let* $r = (y - a - B'\hat{x}_1)' S^{-1} (y - a - B'\hat{x}_1)$. *Then it holds that*

(i) $\dfrac{n-p+1}{q} \dfrac{Q_1/n}{1 + r/n} \sim F_{q, n-p+1}$,

(ii) $\dfrac{n-p+q+1}{p-q}\dfrac{r}{n} \sim F_{p-q,n-p+q+1}$ *if* $p>q$, *and* $r=0$ *if* $p=q$.

Two alternative expressions for $\Pr\{Q_1 \leqq u\}$ have been obtained by Rao [6] in terms of the hypergeometric function, and Gleser and Olkin [4] in terms of the solution of a certain integral equation. However, in either form it is difficult to obtain an explicit expression for $U_1(\alpha)$.

From Theorem 1 we can write

$$\Pr\{Q_1 \leq u\} = \mathrm{E}_r[\Pr\{qF_{q,n-p+1} \leq u+du \mid r\}] \tag{3.1}$$

where $F_{f,k}$ is a random variable having an F-distribution with f and k degrees of freedom and $d=\left(1+\frac{1}{n}r\right)^{-1}\left\{1-\frac{1}{n}(p-1)\right\}-1$. It is possible (e.g., see Siotani [7]) to expand the distribution of $qF_{q,n-p+1}$ in terms of χ^2-distributions. Letting $m=n-p-1$ and $T=\left(1+\frac{2}{m}\right)^{-1}qF_{q,m+2}$, we have

$$qF_{q,n-p+1} = \left(1+\frac{2}{m}\right)T.$$

Since the limiting distribution of T is χ_q^2 and $\mathrm{E}[T]=q$, we consider the asymptotic expansion of the distribution of T with respect to m. By modifying the asymptotic formula for $qF_{q,m+2}$ with respect to $m+2$, we can expand (3.1) as

$$\begin{aligned} \Pr\{Q_1 \leq u\} = \mathrm{E}_r\Big[& G_q(\tilde{u}+d\tilde{u}) \\ & + \frac{q(q+2)}{4m}\{G_q(\tilde{u}+d\tilde{u}) - 2G_{q+2}(\tilde{u}+d\tilde{u}) + G_{q+4}(\tilde{u}+d\tilde{u})\} \\ & + \frac{q(q+2)}{96m^2}\textstyle\sum_{j=0}^4 h_j G_{q+2j}(\tilde{u}+d\tilde{u})\Big] + O(m^{-3}) \end{aligned} \tag{3.2}$$

where $\tilde{u}=\left(1+\frac{2}{m}\right)^{-1}u$, $G_k(u)$ is the distribution function of a χ^2-variate with k degrees of freedom and the coefficients h_j are given by

$$\begin{aligned} & h_0 = (q-2)(3q+4), \quad h_1 = -12q(q+2), \quad h_2 = 6(q+2)(3q+8), \\ & h_3 = -4(q+2)(3q+10), \quad h_4 = 3(q+4)(q+6). \end{aligned} \tag{3.3}$$

The expectation with respect to r can be asymptotically evaluated by using Lemma 2 in Section 5. After simplifications, we obtain the following:

THEOREM 2. *It holds that*

$$\Pr\{Q_1 \leq u\} = G_q(u) + g_q(u)\left\{\frac{1}{m}a_1(u) + \frac{1}{m^2}a_2(u)\right\} + O(m^{-3})$$

where $g_k(u)=G_k'(u)$,

$$a_1(u) = -\frac{1}{2}\{(4p-3q)u+u^2\},$$

$$a_2(u) = \left\{(q+1)p^2 - \frac{1}{2}(3q^2-q-4)p + \frac{1}{48}(27q^3-56q^2-72q+8)\right\}u$$
$$-\left\{p^2 - \frac{1}{2}(4q+1)p + \frac{1}{48}(45q^2+2q-28)\right\}u^2$$
$$+\frac{1}{48}(-24p+21q+10)u^3 - \frac{1}{16}u^4.$$

TABLE 1
Comparison of approximations to $U_1(\alpha)$ for $\alpha=0.05$, 0.01

			Up to the order $(m=n-p-1)$			
p	q	n	O(1)	$O(m^{-1})$	$O(m^{-2})$	Exact
4	1	20	3.841	5.998	6.259	6.237
			6.635	10.977	11.898	11.918
		35	3.814	4.920	4.985	4.982
			6.635	8.806	9.036	9.040
	2	20	5.991	9.185	9.530	9.489
			9.210	15.108	16.302	16.297
		35	5.991	7.588	7.675	7.668
			9.210	12.159	12.458	12.455
8	2	20	5.991	14.704	17.720	17.386
			9.210	23.951	31.291	31.548
		35	5.991	9.678	10.217	10.193
			9.210	15.447	16.761	16.792
	4	20	9.488	22.205	25.849	25.176
			13.277	33.359	42.357	42.021
		35	9.488	14.868	15.520	15.467
			13.277	21.773	23.384	23.374
	6	20	12.592	27.811	31.072	30.347
			16.812	40.358	49.149	48.538
		35	12.592	19.031	19.614	19.552
			16.812	26.774	28.347	28.313

The upper figures are for $\alpha=0.05$ and the lower figures are for $\alpha=0.01$.

Applying the general inverse expansion formula by Hill and Davis [5], we can express the upper α point of Q_1 as

$$(3.4)\qquad U_1(\alpha) = u_\alpha + \frac{1}{2m}\{(4p-3p)u_\alpha + u_\alpha^2\}$$
$$+ \frac{1}{24m^2}[\{24p^2 - 12(7q+4)p + 55q^2 + 36q - 4\}u_\alpha$$
$$+ (36p-35q-14)u_\alpha^2 + 4u_\alpha^3] + O(m^{-3}),$$

where $u_\alpha = \chi_q^2(\alpha)$. The accuracy of the approximation (3.4) can be checked by comparing the exact values in certain cases due to Gleser and Olkin [4]. Some numerical comparisons are given in Talbe 1. From these numerical results we can see that the approximations based on the limiting distribution are poor for the cases of $n=20$, 35, but the approximations up to the order m^{-2} are still excellent for the same cases. Further, when we use the approximations up to the order m^{-2}, the resulting confidence regions are expected to be conservative.

§4. The case when Θ and Σ are unknown

In this section we treat the distribution of Q_2 in (2.7). First we see that the distribution of Q_2 is closely related to that of

$$(4.1)\qquad Q = (\hat{\boldsymbol{x}}-\boldsymbol{x})'\hat{B}S^{-1}\hat{B}'(\hat{\boldsymbol{x}}-\boldsymbol{x})$$
$$= \boldsymbol{s}'S^{-1}\hat{B}'(\hat{B}S^{-1}\hat{B}')^{-1}\hat{B}S^{-1}\boldsymbol{s}$$

where $\boldsymbol{s}=\boldsymbol{y}-\boldsymbol{a}-\hat{B}'\boldsymbol{x}$ and

$$(4.2)\qquad \hat{\boldsymbol{x}} = (\hat{B}S^{-1}\hat{B}')^{-1}\hat{B}S^{-1}(\boldsymbol{y}-\boldsymbol{a}).$$

Since the distribution function of Q depends on n, B, Σ, X and $\boldsymbol{x}$, we may write

$$\Pr\{Q \leq u\} = F(u;\, n, B, \Sigma, X, \boldsymbol{x}).$$

We can write

$$(4.3)\qquad \left(1+\frac{1}{N}\right)^{-1} Q_2 = \boldsymbol{s}^{*\prime}S^{-1}\hat{B}'(\hat{B}S^{-1}\hat{B}')^{-1}BS^{-1}\boldsymbol{s}^*$$

where $\boldsymbol{s}^*=(1+1/N)^{-1/2}(\boldsymbol{y}-\bar{\boldsymbol{y}}-\hat{B}'\boldsymbol{x})$. The conditional distributions of $\boldsymbol{s}$ and $\boldsymbol{s}^*$ given $\hat{B}$ are $N_p[(B-\hat{B})'\boldsymbol{x}, \Sigma]$ and $N_p[(B-\hat{B})'\boldsymbol{x}^*, \Sigma]$, respectively, where

$$(4.4)\qquad \boldsymbol{x}^* = \left(1+\frac{1}{N}\right)^{-1/2}\boldsymbol{x}.$$

From this property, (4.1) and (4.3) we have

(4.5) $$\Pr\left\{\left(1+\frac{1}{N}\right)^{-1}Q_2 \leq u\right\} = F(u;\, n,\, B,\, \Sigma,\, X,\, \boldsymbol{x}^*).$$

Therefore, the distribution function of $(1+1/N)^{-1}Q_2$ and its percentage point are obtained from those of Q by changing $\boldsymbol{x}$ to $\boldsymbol{x}^*$. So, in the following we shall consider the distribution of Q.

Let

$$\bar{B} = \hat{B}\,\Sigma^{-1/2}, \quad B_0 = B\,\Sigma^{-1/2}.$$

Then it is easy to see that the elements of

(4.6) $$M = (X'X)^{1/2}(\bar{B}-B_0)$$

are independently distributed as $N(0,\, 1)$. Let

$$R = (\boldsymbol{y}-\boldsymbol{a}-\hat{B}'\hat{\boldsymbol{x}})'S^{-1}(\boldsymbol{y}-\boldsymbol{a}-\hat{B}'\hat{\boldsymbol{x}})$$

which corresponds to the statistic r in the case of Θ being known.

THEOREM 3. *The conditional distribution of* $\{(n-p+1)/(nq)\}Q/(1+R/N)$ *given* $\hat{B}$ *and* R *is a noncentral F-distribution with* q *and* $n-p-1$ *degrees of freedom and noncentrality parameter*

$$\lambda = \left(1+\frac{1}{n}R\right)^{-1}\boldsymbol{x}'(\bar{B}-B_0)\bar{B}'(\bar{B}\bar{B}')^{-1}\bar{B}(\bar{B}-B_0)'\boldsymbol{x}.$$

Further the conditional distribution of $[(n-p+q+1)/\{n(p-q)\}]R$ *given* $\hat{B}$ *is a noncentral F-distribution with* $p-q$ *and* $n-(p+q)+1$ *degress of freedom and noncentrality*

$$\mu = \boldsymbol{x}'(\bar{B}-B_0)\{I_p-\bar{B}'(\bar{B}\bar{B}')^{-1}\bar{B}\}(\bar{B}-B_0)'\boldsymbol{x}.$$

PROOF. This theorem will be proved by the same technique as in the proof of Theorem 1. In order to find the expressions for noncentrality parameters λ and μ, we sketch the outline. Assume that $\bar{B}$ of rank q is given. This assumption produces no effect to S and $\boldsymbol{y}$ since they are independent of $\bar{B}$. Following Gleser and Olkin [4] we use a nonsingular $q\times q$ matrix T and an orthogonal matrix Γ of order p such that

$$\bar{B} = T[I_q,\, 0]\Gamma' = T\Gamma_1'$$

where $\Gamma=[\Gamma_1,\, \Gamma_2]$ and Γ_1 is a $p\times q$ matrix. Then

$$W = n\Gamma'\,\Sigma^{-1/2}\,S\,\Sigma^{-1/2}\,\Gamma = \begin{pmatrix} W_{11} & W_{21} \\ W_{12} & W_{22} \end{pmatrix} \sim W_p(I_p,\, n),$$

$$\boldsymbol{z} = \Gamma'\,\Sigma^{-1/2}\,(\boldsymbol{y}-\boldsymbol{a}) = \begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{pmatrix} \sim N_p[\boldsymbol{\zeta},\, I_p]$$

where $W_{11}: q \times q$, $z_1: q \times 1$ and $\zeta = \Gamma' B_0' x = (\zeta_1', \zeta_2')'$, $\zeta_1: q \times 1$. Further

$$Q = n b' W_{11.2}^{-1} b, \quad R = n v' v = n z_2' W_{22}^{-1} z_2$$

where $W_{11.2} = W_{11} - W_{12} W_{22}^{-1} W_{21}$, $b = z_1 - T' x - L v$, $L = W_{12} W_{22}^{-1/2}$ and $v = W_{22}^{-1/2} z_2$. It is seen that the conditional distribution of b given v is $N_q[\beta, (1 + v'v) I_q]$, where $\beta = \Gamma_1'(B_0 - \bar{B}) x$. Hence the theorem follows by noting $(1 + v'v)^{-1} \beta' \beta = \lambda$ and $\zeta_2' \zeta_2 = \mu$ and by using the result (Anderson [1], p. 319) on Hotelling's T^2 statistic.

From Theorem 3 we have

$$\begin{aligned} (4.7) \qquad \Pr\{Q \le u\} &= \mathrm{E}_{R,\hat{B}}[\Pr\{q F_{q,n-p+1}(\lambda) \le u + Du \mid R, \hat{B}\}] \\ &= \mathrm{E}_{R,\hat{B}}\left[\Pr\left\{\left(1 + \frac{2}{m}\right)^{-1} q F_{q,m+2}(\lambda) \le \tilde{u} + D\tilde{u} \mid R, \hat{B}\right\}\right] \end{aligned}$$

where $m = n - p - 1$, $\tilde{u} = \left(1 + \frac{2}{m}\right)^{-1} u$, $D = \left(1 + \frac{1}{n} R\right)^{-1} \left\{1 - \frac{1}{n}(p-1)\right\} - 1$ and $F_{f,k}(\lambda)$ is a random variable having a noncentral F-distribution with f and k degrees of freedom and noncentrality parameter λ. In general, the distribution of Q_2 depends on $X'X$, which depends on n. We assume that

$$(4.8) \qquad \frac{1}{m} X'X = H = O(1) \text{ and its limit is nonsingular.}$$

This assumption implies $\bar{B} - B_0 = m^{-1/2} H^{-1/2} M$ and hence $m\lambda$ has an asymptotic distribution. Considering an asymptotic expansion of the distribution of $(1 + 2/m)^{-1} q F_{q,m+2}(\lambda)$ with respect to m, we can write (4.7) as

$$\begin{aligned} (4.9) \quad \Pr\{Q \le u\} = \mathrm{E}_{R,\hat{B}}\Big[& G_q(\tilde{u} + D\tilde{u}, \lambda) \\ & + \frac{q(q+2)}{4m}\{G_q(\tilde{u} + D\tilde{u}, \lambda) - 2G_{q+2}(\tilde{u} + D\tilde{u}, \lambda) + G_{q+4}(\tilde{u} + D\tilde{u}, \lambda)\} \\ & + \frac{q(q+2)}{96m^2} \textstyle\sum_{j=0}^4 h_j G_{q+2j}(\tilde{u} + D\tilde{u}, \lambda) \\ & + \frac{q+2}{2m} \lambda \{G_{q+2}(\tilde{u} + D\tilde{u}, \lambda) - 2G_{q+4}(\tilde{u} + D\tilde{u}, \lambda) + G_{q+6}(\tilde{u} + D\tilde{u}, \lambda)\}\Big] \\ & + O(m^{-3}) \end{aligned}$$

where h_j are defined in (3.3), and $G_k(u, \lambda)$ is the distribution function of a noncentral χ^2-distribution with k degrees of freedom and noncentrality parameter λ. The expectation to R and $\hat{B}$ can be asymptotically evaluated by perturbation method. By using Lemma 2 in Section 5 we have the following theorem:

THEOREM 4. *Under the assumption in* (4.8), *it holds that*

$$\Pr\{Q \le u\} = G_q(u) + g_q(u)\Big[\frac{1}{m}\{a_1(u)+b_1(u)\} + \frac{1}{m^2}\{a_2(u)+b_2(u)\}\Big] + O(m^{-3})$$

where $a_j(u)$ *are the same as in Theorem* 3,

$$b_1(u) = -(x'H^{-1}x)u,$$

$$b_2(u) = \frac{1}{4}(x'H^{-1}x)\{q(4p-3q)u - 2(2p-2q-1)u^2 - u^3\} + \frac{1}{4}(x'H^{-1}x)^2\{(q+2)u - u^2\} - \{x'(HB\Sigma^{-1}B'H)^{-1}x\}\frac{1}{q}(p-q)(p-q-1)u.$$

From Theorem 4 we have an asymptotic expansion for the upper α point for Q, which is given by

$$U(\alpha) = U_1(\alpha) + \frac{1}{m}(x'H^{-1}x)u_\alpha + \frac{1}{2m^2}(x'H^{-1}x)\{(4p-3q)u_\alpha + u_\alpha^2\} + \frac{1}{qm^2}\{x'(HB\Sigma^{-1}B'H)^{-1}x\}(p-q)(p-q-1)u_\alpha + O(m^{-3}), \tag{4.10}$$

where $U_1(\alpha)$ is approximated in (3.4) up to the order m^{-2}. We note that the upper α point of $(1+1/N)^{-1}Q_2$ is obtained from $U(\alpha)$ by changing x to x^*. The formula (4.10) includes unknown parameters x, B and Σ in terms of the orders m^{-1} and m^{-2}. So, in practice we should replace them by their estimates. When $p=q$ or $q+1$, the unknown parameters B and Σ disappear from this formula. The statistic Q_2 is also useful in testing the hypothesis H_0: $x=x_0$, where x_0 is a given vector. In this case the $U(\alpha)$, replaced x by $x_0^*=(1+1/N)^{-1/2}x_0$, may be expected to be a good approximation for the critical point of $(1+1/N)^{-1}Q_2$.

§5. Some formulas

This section gives some formulas used for the evaluation of the expectations in (3.2) and (4.9). Recall $\bar{B}$ satisfies

$$\bar{B} = B_0 + m^{-1/2}H^{-1/2}M$$

where $m=n-p-1$, $H=O(1)$ and all the elements of M are independently distributed as $N(0, 1)$. Then by the perturbation method on λ, μ and D, we have

LEMMA 1.

$$\mathrm{E}[\mu] = \frac{1}{m}(p-q)x'H^{-1}x + O(m^{-2}), \quad \mathrm{E}[\mu^\ell] = O(m^{-\ell}) \qquad \textit{for} \quad \ell \geqq 1,$$

$$\mathrm{E}[\lambda] = \frac{1}{m}px'H^{-1}x + \frac{1}{m^2}(p-q)\{-px'H^{-1}x + (p-q-1)x'(HB\Sigma^{-1}B'H)^{-1}x\} + O(m^{-3}),$$

$$\mathrm{E}[\lambda^2] = \frac{1}{m^2}q(q+2)(x'H^{-1}x)^2 + O(m^{-3}), \quad \mathrm{E}[\lambda^\ell] = O(m^{-\ell}) \qquad \textit{for} \quad \ell \geqq 1,$$

$$\mathrm{E}[D] = -\frac{1}{m}(2p-q-1) + \frac{1}{m^2}\{3p^2-(2q-1)p-q-1\} - \frac{1}{m^2}(p-q)x'H^{-1}x + O(m^{-3}),$$

$$\mathrm{E}[D^2] = \frac{1}{m^2}\{4p^2-2(2q+1)p+q^2+1\} + O(m^{-3}),$$

$$\mathrm{E}[D^\ell] = O(m^{-\ell}) \qquad \textit{for} \quad \ell \geqq 1,$$

$$\mathrm{E}[D\lambda] = -\frac{1}{m^2}q(2p-q-1)x'H^{-1}x + O(m^{-3}).$$

LEMMA 2. *Let d and D be the random variables appeared in Sections* 3 *and* 4, *respectively. Then*

$$\mathrm{E}[G_k(u+du)] = G_k(u) + g_k(u)\left\{\frac{1}{m}c_1(u) + \frac{1}{m^2}c_2(u)\right\} + O(m^{-3}),$$

$$\mathrm{E}[G_k(u+Du,\lambda)] = G_k(u) + g_k(u)\left[\frac{1}{m}\{c_1(u)+C_1(u)\} + \frac{1}{m^2}\{c_2(u)+C_2(u)\}\right] + O(m^{-3})$$

where $c_1(u) = -(2p-q-1)u$, $C_1(u) = -\frac{q}{k}(x'H^{-1}x)u$,

$$c_2(u) = \frac{1}{4}[k\{4p^2-2(2q+1)p+q^2+1\} + 2(2p^2+4p-q^2-2q-3)]u - \frac{1}{4}\{4p^2-2(2q+1)p+q^2+1\}u^2,$$

$$C_2(u) = \frac{1}{2k}(x'H^{-1}x)[\{2(q-k)(p-q)+qk(2p-q-1)\}u - q(2p-q-1)u^2] + \frac{q(q+2)}{4k(k+2)}(x'H'^{-1}x)^2\{(k+2)u-u^2\} - \frac{1}{k}\{x'(HB\Sigma^{-1}B'H)^{-1}x\}(p-q)(p-q-1)u.$$

PROOF. The first formula is obtained from the second by letting $\boldsymbol{x}=\boldsymbol{0}$. Considering the Taylor series expansion with respect to $u+Du$ about u and noting that $m\lambda$ and mD have limiting distributions, we have

$$\mathrm{E}\,[G_k(u+Du,\lambda)] = \mathrm{E}\left[\sum_{j=0}^{2}\frac{(Du)^j}{j!}\frac{\partial^j}{\partial u^j}G_k(u,\lambda)\right] + O(m^{-3})$$

$$= \mathrm{E}\left[\sum_{\ell=0}^{2} e^{-\lambda/2}\frac{(\lambda/2)^\ell}{\ell!}G_{k+2\ell}(u) + Du\sum_{\ell=0}^{1} e^{-\lambda/2}\frac{(\lambda/2)^\ell}{\ell!}g_{k+2\ell}(u)\right.$$

$$\left. + \frac{1}{2}D^2u^2g_k'(u)\right] + O(m^{-3}).$$

The validity of these reductions can be done by the same method as in Anderson [2]. Using Lemma 1 and noting $ug_k(u)=\frac{k}{2}\{G_k(u)-G_{k+2}(u)\}$, we can get the second result.

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