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Extension problem of diffeomorphisms of a 3-torus over some 4-manifolds

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Introduction

Let T^3 denote a 3-torus $S^1 \times S^1 \times S^1$. Let $W(\ell, m, n)$ be the 4-manifold obtained from $T^3 \times [0, 1]$ by attaching three 2-handles along the three standard generators S^1 in $T^3 \times 1$ with framing numbers ℓ, m and n (see §1 for the precise definition). Then, $\partial W(\ell, m, n) = \partial_0 W(\ell, m, n) \cup \partial_1 W(\ell, m, n) = T^3 \times 0 \cup H(\ell, m, n)$ and $\pi_1(H(\ell, m, n)) = \langle \alpha, \beta, \gamma;\ \alpha = (\beta^{-1}\gamma\beta\gamma^{-1})^{\ell},\ \beta = (\gamma^{-1}\alpha\gamma\alpha^{-1})^{m},\ \gamma = (\alpha^{-1}\beta\alpha\beta^{-1})^{n} \rangle$. In particular, $H(\ell, m, n)$ is a homology 3-sphere. It is known that $H(0, m, n)$ is diffeomorphic to S^3 and $H(1, 1, 1)$ is the Poincaré homology 3-sphere. We refer the reader to [5], in which Y. Matsumoto proves some facts about $H(\ell, m, n)$ including the claims that the author made before.

We shall prove the following two theorems. Let $SDiff(T^3)$ denote the group of all orientation preserving diffeomorphisms of T^3. For an $f \in SDiff(T^3)$ we consider the matrix $f_* \in SL(3, \boldsymbol{Z})$ which is defined as the induced automorphism f_* of $H_1(T^3)$ with respect to the basis consisting of the classes of three standard generators.

Since T^3 is an irreducible and sufficiently large 3-manifold without boundary, $f_* = g_*$ implies that f and g are mutually isotopic by the theorem of Waldhausen [9].

THEOREM 1. *Let $f \in SDiff(T^3)$. Then, there exists an $F \in SDiff(W(1, 1, 1))$ such that $F|T^3 \times 0 = f$ and $F|H(1, 1, 1) = id$.*

THEOREM 2. *Let $f \in SDiff(T^3)$. Then, there exists an $F \in SDiff(W(0, 0, 0))$ satisfying $F|T^3 \times 0 = f$ and $F|H(0, 0, 0) = id$ if and only if f_* belongs to the subgroup $G = \{(a_{ij}) \in SL(3, \boldsymbol{Z});\ a_{1j} + a_{2j} + a_{3j} \equiv 1 \bmod 2\ (j = 1, 2, 3)\}$.*

REMARK. If we replace $W(0, 0, 0)$ with $W(0, m, n)$, we should replace G with gGg^{-1} where $g = \begin{pmatrix} 1 & m & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

As an application of Theorem 1, we have the following theorems. Take a non-singular algebraic curve C of degree 3 in the complex projective plane P^2. Then, C is diffeomorphic to a 2-torus T^2 and the self-intersection number $[C]^2 = 9$.

Blow up P^2 at 9 distinct points on C. Then, we get an embedded torus T^2 with trivial normal bundle in $P^2\#9(-P^2)$. Remove the interior $T^2\times\mathring{D}^2$ of the regular neighborhood of T^2 and define $N=P^2\#9(-P^2)-T^2\times\mathring{D}^2$.

THEOREM 3. *The manifold* $N\cup T^2\times D^2$ *obtained by reattaching with any diffeomorphism of* T^3 *is diffeomorphic to the original* $P^2\#9(-P^2)$.

THEOREM 4. *The manifold* $N\cup N$ *obtained by attaching with any orientation reversing diffeomorphism of* T^3 *is diffeomorphic to the* K3 *surface.*

To prove Theorem 3 and that the diffeomorphism class of $N\cup N$ in Theorem 4 does not depend on the choice of orientation reversing diffeomorphism of T^3, it suffices to embed $W(-1,\ -1,\ -1)=-W(1,\ 1,\ 1)$ in N so as to be $\partial N=\partial_0 W(-1,\ -1,\ -1)$. This is not difficult and we can prove moreover Proposition 6.1 which says that N is diffeomorphic to $W(-1,\ -1,\ -1)\cup P(E_8;\ -2)$, where $P(E_8;\ -2)$ is the manifold obtained by plumbing according to the graph E_8 weighted by -2. We use the study of elliptic surfaces due to Kodaira for the proof of Proposition 6.1 and the remaining part of Theorem 4.

REMARK. $P^2\#9(-P^2)$ has a structure of an elliptic rational surface and when the reattaching corresponds to a logarithmic transformation, $N\cup T^2\times D^2$ has been known to have a structure of a non-singular rational surface with $b_2=10$ (see Remark 2 in §6); in particular, to be diffeomorphic to $P^2\#9(-P^2)$. The logarithmic transformation of multiplicity m corresponds to the reattaching by f with $f_*=\begin{pmatrix}1&0&0\\0&*&*\\0&*&m\end{pmatrix}\in SL(3,\ \boldsymbol{Z})$.

In §1 we introduce the precise construction of $W(\ell,\ m,\ n)$ and some facts about $H(\ell,\ m,\ n)$. §2 is devoted to the proof of Theorem 1. In §3 we determine the generators of the group G defined in Theorem 2. With this the proof of Theorem 2 is completed in §4. In §5 we get an embedding of $W(-1,\ -1,\ -1)$ in N and in §6 we prove Proposition 6.1 by studying the structure of an elliptic surface and its general fibre. §§7 and 8 are devoted to the proofs of Theorems 3 and 4 respectively.

§1. Homology 3-sphere $H(\ell,\ m,\ n)$

We consider $T^3=\boldsymbol{R}^3/\boldsymbol{Z}^3$ as the cube whose left-right, front-back and upper-lower sides are identified. Let α, β and γ be the loops corresponding to the coordinate axes with identified end points. And let $0<a<1/2$. Then, $S^1_\alpha=S^1\times(1-a)\times a$, $S^1_\beta=a\times S^1\times(1-a)$ and $S^1_\gamma=(1-a)\times a\times S^1$ are disjoint loops in T^3. They are encircled by the $-$(meridian) loops with base point $(0,\ 0,\ 0)$ of homotopy classes $\beta^{-1}\gamma\beta\gamma^{-1}$, $\gamma^{-1}\alpha\gamma\alpha^{-1}$ and $\alpha^{-1}\beta\alpha\beta^{-1}$.

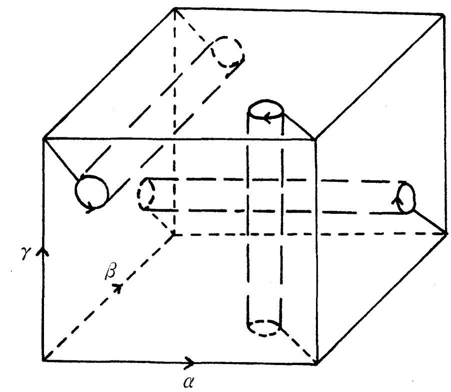


Figure 1

By using van Kampen's theorem we get

$$\begin{aligned}&\pi_1(T^3-S^1_\alpha \cup S^1_\beta \cup S^1_\gamma)\\&=\langle \alpha,\ \beta,\ \gamma;\ [\alpha,\ \beta^{-1}\gamma\beta\gamma^{-1}]=[\beta,\ \gamma^{-1}\alpha\gamma\alpha^{-1}]=[\gamma,\ \alpha^{-1}\beta\alpha\beta^{-1}]=1\rangle .\end{aligned}$$

The framing of the regular neighborhood $N(S^1)$ of S^1 is defined to be an isotopy class of a diffeomorphism $h\colon S^1\times D^2 \to N(S^1)$. To fix the notation we draw the parallel axes $h(S^1\times 0 \cup S^1\times pt)$ with $pt\in \partial D^2$ in the cube and consider #(right-handed screws) − #(left-handed screws) as a complete numerical invariant of the framing, called the framing number. (Ex. stands for the $(+1)$-framing). Three 2-handles $D^2\times D^2$ are attached to $T^3\times[0,1]$ along $N(S^1_\alpha)$, $N(S^1_\beta)$ and $N(S^1_\gamma)$ in $T^3\times 1$ with the framing numbers ℓ, m and n respectively. $W(\ell, m, n)=T^3\times[0,1]\cup(D^2\times D^2\cup D^2\times D^2\cup D^2\times D^2)$ denotes the resulting surgery trace. Then, we have $\alpha\cdot(\beta^{-1}\gamma\beta\gamma^{-1})^{-\ell}=\beta\cdot(\gamma^{-1}\alpha\gamma\alpha^{-1})^{-m}=\gamma\cdot(\alpha^{-1}\beta\alpha\beta^{-1})^{-n}=1$ in $H(\ell, m, n)=\partial_1 W(\ell, m, n)$. Hence, $\pi_1(H(\ell, m, n))=\langle\alpha,\ \beta,\ \gamma;\ \alpha=(\beta^{-1}\gamma\beta\gamma^{-1})^{\ell},\ \beta=(\gamma^{-1}\alpha\gamma\alpha^{-1})^{m},\ \gamma=(\alpha^{-1}\beta\alpha\beta^{-1})^{n}\rangle$. In particular, $H(\ell, m, n)$ is a homology 3-sphere.

As mentioned in the introduction we refer the reader to [5] for the related results. For example $\mu(H(\ell, m, n))=\ell mn \bmod 2$.

REMARK. If we attach a 0-framed 2-handle to $W(\ell, m, n)$ along $N(S^1_\alpha)$ in $T^3\times 0$, we have an embedded 2-sphere as the union of $S^1_\alpha\times[0,1]$ and the axes of two 2-handles attached along $N(S^1_\alpha)$. Then, the self-intersection number of the 2-sphere is equal to the framing number ℓ, when the orientation of $W(\ell, m, n)$ is given by the 4-ple of vectors (v_1, v_2, v_3, v_4) where $v_1=-\overrightarrow{10}$ in $T^3\times[0,1]$ and (v_2, v_3, v_4) gives the orientation of $T^3\times 1$. As a boundary of $W(\ell, m, n)$, $T^3\times 0$ inherits the orientation opposite to that of $T^3\times 1$.

§2. Proof of Theorem 1

Note at first the following facts. Assume that f and g of $SDiff(T^3)$ extend respectively to G and F of $SDiff(W(1, 1, 1))$ which restrict to the identity on $H(1, 1, 1)$. Then, $f\circ g$ extends to $F\circ G$ with $F\circ G|H(1, 1, 1)=id$ and f^{-1} extends to F^{-1} with $F^{-1}|H(1, 1, 1)=id$. Also, if f is isotopic to f' as diffeomorphisms of T^3, then f' can extend to F' as the union of the isotopy between f' and f with F and this F' satisfies $F'|H(1, 1, 1)=id$. So, we have only to prove Theorem 1 for the generators of $SL(3, \boldsymbol{Z})=\pi_0(SDiff(T^3))$. Of course, by the operation of matrixes on $\boldsymbol{R}^3$, $SL(3, \boldsymbol{Z})$ is naturally a subgroup of $SDiff(T^3)$.

A unimodular matrix with just one non-zero entry (i.e. ± 1) in each row and column is called a permutation matrix. And a permutation matrix is called restricted if at least one diagonal entry is 1. Set $Q=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $P_1=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. Then, $SL(3, \boldsymbol{Z})$ is generated by the restricted permutation matrixes and the matrix Q. In fact, by the Euclidean division algorithm we see that $SL(2, \boldsymbol{Z})$ is generated by Q and P_1, and $SL(3, \boldsymbol{Z})$ is generated by $SL(2, \boldsymbol{Z})$ and the restricted permutation matrixes.

Recall that the surgery trace $W(1, 1, 1)$ is the union of $T^3\times[0, 1]$ and three 2-handles attached along $N(S^1_\alpha)$, $N(S^1_\beta)$ and $N(S^1_\gamma)$. Let $f\in SDiff(T^3)$ be defined by a non-identity restricted permutation matrix P. Then, f makes one axis fixed and the other two axes permuted or orientation reversed. The framing numbers of the regular neighborhoods of three axes are all $+1$ and f may reverse the orientation of the axes but preserves the orientation of T^3. So, $f|\cup N(\text{axes})$ preserves the framing. Hence, f can extend naturally over the corresponding 2-handles, i.e., there is an element F of $SDiff(W(1, 1, 1))$ such that $F|T^3\times 0=f$. Moreover, we can assume that F restricts to the identity on one of the 2-handles.

LEMMA 2.1. *Let $F\in SDiff(W(1, 1, 1))$. Suppose that F restricts to the identity on one of the 2-handles of $W(1, 1, 1)$. Then, $g=F|H(1, 1, 1)$ is isotopic to the identity.*

PROOF. Take the axis $S^1_{\bar{a}}$ in $H(1, 1, 1)$ of the dual 2-handle of $W(1, 1, 1)$. Since $F|$the 2-handle$=id$, $g|N(S^1_{\bar{a}})=id$. Recall that $H(1, 1, 1)-\mathring{N}(S^1_{\bar{a}})$ is diffeomorphic to $S^3-\mathring{N}$(trefoil knot); this is because $T^3-\mathring{N}(S^1_\alpha\cup S^1_\beta\cup S^1_\gamma)=S^3-\mathring{N}$(Borromean rings) [4] and the surgery along two components reduces the remaining component to the trefoil knot (see [5] for example). So, $h=g|H(1, 1, 1)-\mathring{N}(S^1_{\bar{a}})$ is considered to be a diffeomorphism of $S^3-\mathring{N}$(trefoil knot) onto itself which restricts to the identity on the boundary. It is known that the outerautomorphism group of $\pi_1(S^3-\text{trefoil knot})=\langle a, b;\, a^2=b^3\rangle$ is of order 2; the non-trivial element is represented by $a\to a^{-1}$ and $b\to b^{-1}$ (see Schreier

[8]). Since the meridian ab^{-1} remains unchanged by h, h induces the identity on π_1. Note that the non-trivial knot complement is an irreducible, ∂-irreducible and sufficiently large manifold which is not homeomorphic to a line bundle. By the theorem of Waldhausen [9, Cor. 7, 5], h is isotopic to the identity relative to the boundary. Hence, g is isotopic to the identity. q.e.d.

This completes the proof for the restricted permutation matrixes, because F can be reconstructed as the union of the original F and the isotopy between g and the identity.

Now let f be the element of $SDiff(T^3)$ defined by Q. Then, $f_*([\alpha], [\beta], [\gamma]) = ([\alpha], [\beta], [\alpha+\gamma])$ and we can assume that $f|N(S^1_\alpha) \cup N(S^1_\beta) = id$. We consider that two 2-handles of $W(1, 1, 1)$ are attached along $N(S^1_\alpha)$ and $N(S^1_\beta)$. The 3rd 2-handle is originally attached along $N(S^1_\gamma)$ but we can deform it, for example, by sliding it over the handles attached along $N(S^1_\alpha)$. The resulting surgery trace is still diffeomorphic to $W(1, 1, 1)$ but the new 3rd 2-handle is considered to be attached along the regular neighborhood of a connected sum $S^1_{\alpha+\gamma}$ of S^1_γ with a parallel axis $(S^1_\alpha)'$ on $\partial N(S^1_\alpha)$ by a band in $T^3 - \mathring{N}(S^1_\alpha \cup S^1_\beta)$. If we take a band as in the following figure, we can see that $S^1_{\alpha+\gamma}$ is isotopic to $f(S^1_\gamma)$ in $T^3 - S^1_\alpha \cup S^1_\beta$.

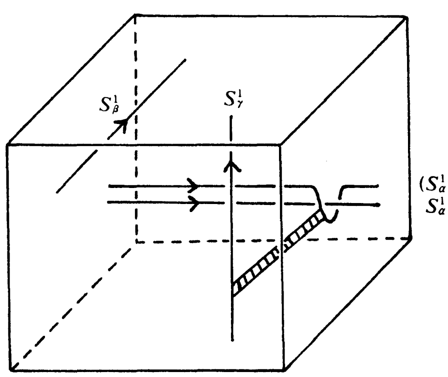


Figure 2

Let f.n.(S^1) stand for the framing number of $N(S^1)$. With the above band sum

$$\text{f.n.}(S^1_{\alpha+\gamma}) = \text{f.n.}(S^1_\gamma) + \text{f.n.}(S^1_\alpha) - 1 = 1,$$

because the figure of the parallel axis of $S^1_{\alpha+\gamma}$ has an extra . So, f can extend to a diffeomorphism F of $W(1, 1, 1)$ which restricts to the identity on the handle attached along $N(S^1_\alpha)$. By Lemma 2.1 this completes the proof of Theorem 1.

§ 3. Generators for the groups of stochastic matrixes modulo 2

Let $G_0=\{(a_{ij})\in SL(3, \boldsymbol{Z});\ a_{i1}+a_{i2}+a_{i3}\equiv 1 \bmod 2\ (i=1, 2, 3)$ and $a_{1j}+a_{2j}+a_{3j}\equiv 1 \bmod 2\ (j=1, 2, 3)\}$. Set $R=\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $S=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

PROPOSITION 3.1. *G_0 is generated by the restricted permutation matrixes and R.*

To prove this we use the following elementary lemma.

LEMMA 3.2. *If $|b|>|a|$ is satisfied for non-zero integers a and b, then* $|b-2(\operatorname{sign}(ab))a|<|b|$.

PROOF. This is because $-|b|<-|a|+(|b|-|a|)=|b|-2|a|<|b|$ and $||b|-|2a||=|b-2(\operatorname{sign}(ab))a|$.

PROOF OF PROPOSITION 3.1. Let $g=(a_{ij})$ be an element of G_0. Note that $G_0 \bmod 2$ is generated by the images of restricted permutation matrixes. So, by multiplying restricted permutation matrixes we can assume that $a_{ij}\equiv\delta_{ij} \bmod 2$. By multiplying $R^{\pm1}$ from left the matrix (a_{ij}) is transformed into (a'_{ij}) with $a'_{1j}=a_{1j}\pm 2a_{3j}$ $(j=1, 2, 3)$ and $a'_{ij}=a_{ij}$ (otherwise). Similarly the multiplication of $P_1R^{\pm1}P_1^{-1}$ from left transforms (a_{ij}) into (a''_{ij}) with $a''_{3j}=a_{3j}\mp 2a_{1j}$ $(j=1, 2, 3)$ and $a''_{ij}=a_{ij}$ (otherwise). Here, P_1 is the restricted permutation matrix defined in §1. With this fact and Lemma 3.2, we can decrease $|a_{11}|$ and $|a_{31}|$ by even integers unless $|a_{31}|=0$. So, the reduced (a_{ij}) satisfies $a_{31}=0$. Consider the multiplications of $P_2R^{\pm1}P_2^{-1}$ and $P_3P_2R^{\pm1}P_2^{-1}P_3^{-1}$ from left for the restricted permutation matrixes $P_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ and $P_3=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, by Lemma 3.2 we can decrease $|a_{11}|$ and $|a_{21}|$ further by even integers without changing $a_{31}=0$ unless $|a_{21}|=0$. So, we get a reduced matrix satisfying $a_{21}=a_{31}=0$. Then, $a_{11}=\pm 1$. And if $a_{11}=-1$, we multiply a restricted permutation matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; we may assume $a_{11}=1$ and $a_{21}=a_{31}=0$. By multiplying $R^{\pm1}$ and $P_2R^{\pm1}P_2^{-1}$ from right we can reduce further so that $a_{12}=a_{13}=0$. We can reduce also the remaning terms in the same way and finally get the identity matrix from the original (a_{ij}) by multiplying the restricted permutation matrixes and $R^{\pm1}$. q.e.d.

Let $G=\{(a_{ij})\in SL(3, \boldsymbol{Z});\ a_{1j}+a_{2j}+a_{3j}\equiv 1 \bmod 2\ (j=1, 2, 3)\}$. Then, G_0 is a subgroup of G of index 4. In fact, $G/G_0\cong G \bmod 2/G_0 \bmod 2$ with $|G \bmod 2|=24$ and $|G_0 \bmod 2|=6$. Moreover, $G \bmod 2$ is generated by $G_0 \bmod 2$

and the image of S. This implies the following proposition.

PROPOSITION 3.3. *G is generated by the restricted permutation matrixes together with R and S.*

§4. Proof of Theorem 2

To prove the 'if'-part it suffices to show that the diffeomorphisms defined by the generators of G extend to diffeomorphisms of $W(0, 0, 0)$ as in the proof of Theorem 1. In fact, since any orientation preserving diffeomorphism of $H(0, 0, 0)=S^3$ is isotopic to the identity, we can modify the diffeomorphism of $W(0, 0, 0)$ so as to restrict to the identity on $H(0, 0, 0)$ without changing it near $T^3\times 0$.

In case $f\in SDiff(T^3)$ is defined by the operation of a restricted permutation matrix, the extension is easily constructed as in §2.

Let now f be defined by the matrix R. Then, $f_*([\alpha], [\beta], [\gamma])=([\alpha], [\beta], [2\alpha+\gamma])$. We consider the handle adding (or sliding) defined by the following band sum $S^1_{2\alpha+\gamma}$ of S^1_γ with the parallel axes $(S^1_\alpha)'$ and $(S^1_\alpha)''$ on $\partial N(S^1_\alpha)$.

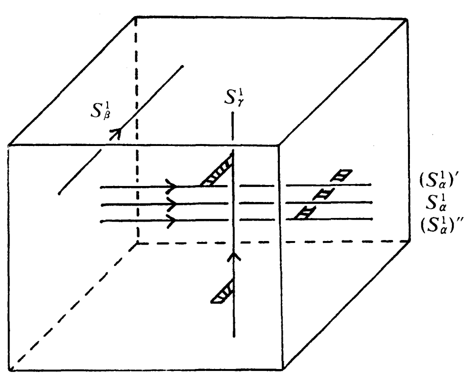


Figure 3

Observe that $f|N(S^1_\alpha)\cup N(S^1_\beta)=id$ and $S^1_{2\alpha+\gamma}$ is isotopic to $f(S^1_\gamma)$ in $T^3-S^1_\alpha\cup S^1_\beta$. Moreover,

$$\text{f.n.}(S^1_{2\alpha+\gamma})=\text{f.n.}(S^1_\gamma)+1-1=0.$$

So, f extends to a diffeomorphism of $W(0, 0, 0)$.

If f is defined by S, then $f_*([\alpha], [\beta], [\gamma])=([\alpha], [\beta], [\alpha+\beta+\gamma])$. We consider the band sum $S^1_{\alpha+\beta+\gamma}$ of S^1_γ with the parallel axes $(S^1_\alpha)'$ and $(S^1_\beta)'$ on $\partial N(S^1_\alpha)$ and $\partial N(S^1_\beta)$ respectively as in the following figure.

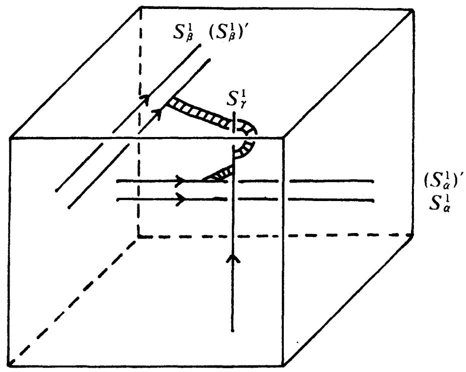


Figure 4

Then, $f|N(S^1_\alpha)\cup N(S^1_\beta)=id$ and $S^1_{\alpha+\beta+\gamma}$ is isotopic to $f(S^1_\gamma)$ in $T^3-S^1_\alpha\cup S^1_\beta$. Moreover, the framing number is zero as the sum of and . So, f extends to a diffeomorphism of $W(0, 0, 0)$. Hence we complete the proof of the 'if'-part.

There is a natural identification $id\colon \partial T^2\times D^2\to\partial_0 W(0, 0, 0)$ such that $(id)^{-1}(S^1)$ with 0-framing bounds a framed 2-disk in $T^2\times D^2$. For an element f of $SDiff(T^3)$ we consider a manifold $M=W(0, 0, 0)\cup_f T^2\times D^2$ obtained by an attaching diffeomorphism $id\circ f\colon \partial T^2\times D^2\to\partial_0 W(0, 0, 0)$. Assume that f extends to an $F\in SDiff(W(0, 0, 0)$. Then, M is diffeomorphic to $W(0, 0, 0)\cup_{id} T^2\times D^2=S^2\times S^2-\mathring{D}^4$. In particular, M is of even type. But, if f_* is not contained in G, there is a column with a_{1*}, a_{2*} and a_{3*} satisfying $a_{1*}+a_{2*}+a_{3*}=$even. We may assume $*=3$, because $PGP^{-1}=G$ for any restricted permutation matrix P. Then, observe that $f(S^1_\gamma)$ is realized by some band sum $a_{13}S^1_\alpha+a_{23}S^1_\beta+a_{33}S^1_\gamma$ and its framing number modulo 2 is equal to $(a_{13}-1)+(a_{23}-1)+(a_{33}-1)\bmod 2$. In this case M has a homology class of odd self-intersection number, that is, M is of odd type. This is a contradiction, which completes the proof of the 'only if'-part.

§5. Embedding of $W(-1, -1, -1)$ in N

We shall show the following lemma with the minimum knowledge about algebraic curves.

LEMMA 5.1. *$W(-1, -1, -1)=-W(1, 1, 1)$ is realizable as a submanifold in N (defined in the introduction) such that $\partial N=\partial_0 W(-1, -1, -1)$.*

PROOF. Let D^4 be a small disk neighborhood of $(0, 0, 1)$ in P^2. Since the diffeomorphism class of (P^2, C) is independent of a particular choice of the non-singular curve C of degree 3 in P^2, we may take $C=\{(x, y, z)\in P^2;$

$x^2z+y^3+\varepsilon z^3=0\}$ with sufficiently small $\varepsilon>0$. Then, due to Milnor [6] $C \cap \partial D^4$ is a right-handed (2, 3) torus knot k (the orientation in [6] is opposite to ours) and $C \cap D^4$ is isotopic to the minimal Seifert surface S of k in ∂D^4. So, we may consider that our T^2 is a union of S with a 2-disk outside D^4. Take two circles γ_1 and γ_2 on S as in the following figure.

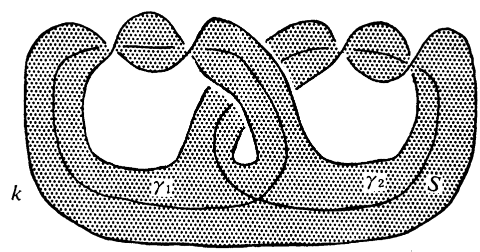


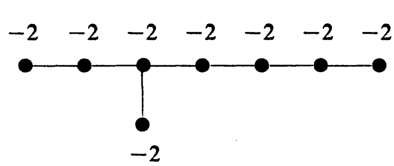
Figure 5

Then, we can find smoothly embedded 2-disks D_1 and D_2 in D^4 such that $D_i \cap S = \partial D_i = \gamma_i$ $(i=1, 2)$ and $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 =$ one point. In $P^2\#9(-P^2)$ there is also a smoothly embedded 2-sphere which intersects with T^2 transversally at one point. Therefore, on $\partial N(T^2)=T^3$ there are three disjoint circles along which we can attach 2-handles embedded in N. The framing numbers of the 2-handles are all -1; in fact, for the first two the framing number is equal to the linking number of the parallel circles on S in ∂D^4 and for the last one it is equal to the self-intersection number of the 2-sphere (see Remark in §1). This means that $W(-1, -1, -1) \subset N$ with $\partial_0 W(-1, -1, -1) = \partial N(T^2) = \partial N$.

§6. A structure of N via an elliptic surface

We shall prove a stronger result (=Proposition 6.1) than Lemma 5.1 by using the facts about algebraic curves and elliptic surfaces.

PROPOSITION 6.1. *N is diffeomorphic to the union $W(-1, -1, -1) \cup P(E_8; -2)$, where $P(E_8; -2)$ is the manifold obtained by plumbing according to the graph E_8 weighted -2.*



We use the construction of the basic member $\psi: V \to \Delta$ of the family $\mathscr{F}(\mathscr{J}, G)$ of elliptic surfaces over a non-singular curve Δ due to Kodaira [2, §8]. In our case $\Delta=P^1$ and $\Delta'=\Delta-\{a_1, a_2\}$ for some two points a_1 and a_2 in P^1; the functional invariant $\mathscr{J}: \Delta-\{a_1, a_2\}\to P^1$ is the constant function $\mathscr{J}=0$; and the homological invariant G is determined by the monodromy $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ around a_1 and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ around a_2. Then, we have an elliptic surface $\psi: V\to P^1$ with only two singular fibres, one is of type II* on a_1 and another of type II on a_2. This is constructed by blowing up and down the quotient of the product bundle $E\times P^1$ by the monodromy group of order 6, where E is the elliptic curve $\boldsymbol{C}/(\boldsymbol{Z}+\boldsymbol{Z}\omega)$ with $\omega^2+\omega+1=0$. Proposition 6.1 will be proved by the following two lemmas.

LEMMA 6.2. *N is diffeomorphic to $V-\mathring{N}$(a general fibre), where $\mathring{N}$(a general fibre) is the interior of the regular neighborhood of $\psi^{-1}(a)$ with $a\in P^1-\{a_1, a_2\}$.*

LEMMA 6.3. *$V-\mathring{N}$(a general fibre) is diffeomorphic to $W(-1, -1, -1)\cup P(E_8; -2)$.*

PROOF OF LEMMA 6.2. By the construction we see that $\pi_1(V)=0$ and Euler number $\chi(V)=\chi(\psi^{-1}(a_1))+\chi(\psi^{-1}(a_2))=12$. The formula of M. Noether $12(p_g+1)=\chi+c_1^2$ implies $p_g=0$ because $c_1^2=K^2=0$ for the elliptic surfaces. In fact, due to Kodaira [2, Ths. 12.1 and 12.3] the canonical line bundle K of V is induced from the complex line bundle $\mathfrak{k}-\mathfrak{f}$ over Δ, where $\mathfrak{k}$ is a cotangent line bundle of Δ and $c(\mathfrak{f})=-p_g-1$. Here, we note that complex line bundles over Δ are determined by their Chern number $c: H^1(\Delta; C^\times)\to H^2(\Delta; \boldsymbol{Z})\cong \boldsymbol{Z}$. In our case $c(\mathfrak{k})=-2$ and $c(\mathfrak{f})=-1$. This implies also that $K=-[F]$ where F is an irreducible positive divisor defined by a general fibre of the elliptic surface V. So, $P_2=\dim H^0(V; \mathcal{O}(2K))=0$. Hence, V is a rational surface (see [7]), because the irregularity $q=0$ which follows from $\pi_1(V)=0$. This means that, if we blow down V 9 times, $\sigma^{-9}(V)$ is biholomorphic to P^2. Now we remark that the basic member ψ admits a holomorphic section $s: P^1\to V$ and $s(P^1)$ is a non-singular rational curve with $(s(P^1))^2=c(\mathfrak{f})=-1$. Since $\psi^{-1}(a_1)$ consists of 9 rational irreducible curves whose intersection form is of the graph E_9 weighted -2, we find the generators of $H_2(V; \boldsymbol{Z})$ as follows.

By blowing down the 9 exceptional curves (i.e. non-singular rational curves with self-intersection number $=-1$) successively as above, the general fibre reduces to a non-singular curve C of genus 1 in P^2. By Chow's theorem [1], C is an algebraic curve. Since a non-singular algebraic curve of degree d in P^2 has genus $(d-1)(d-2)/2$, C is a non-singular algebraic curve of degree 3. Describe this process in the inverse direction; we blow up P^2 at a point on C and get a curve C_1 in $\sigma(P^2)$ and inductively we blow up $\sigma^i(P^2)$ at a point on C_i and get a curve C_{i+1} in $\sigma(\sigma^i(P^2))=\sigma^{i+1}(P^2)$; and finally $C_9=F\subset\sigma^9(P^2)=V$. The diffeo-

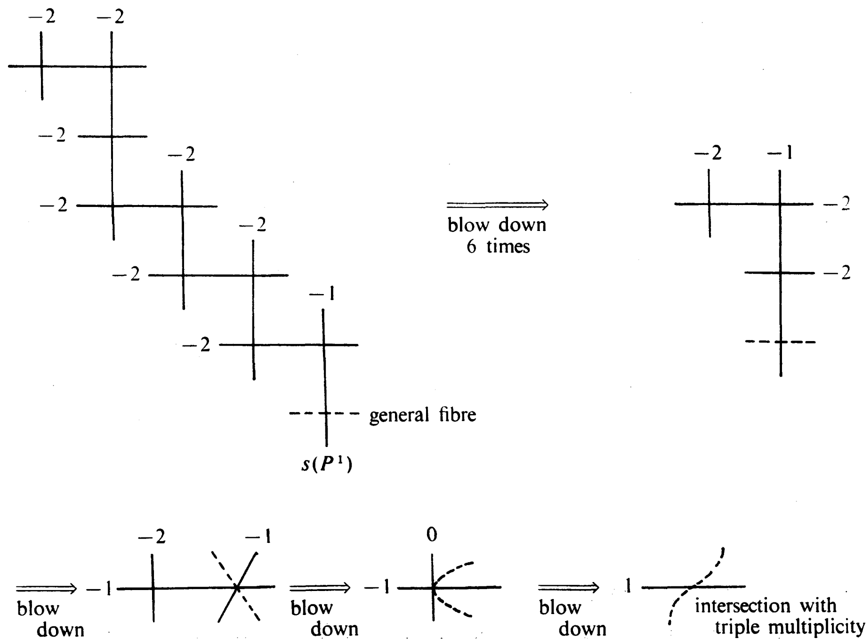


Figure 6

morphism class of $(\sigma^9(P^2), C_9)$ is independent of the choice of the point on C_i at which we blow up $\sigma^i(P^2)$. Hence, for our T^2 in $P^2\#9(-P^2)$ defined in the introduction we have a diffeomorphism $g\colon V\to P^2\#9(-P^2)$ satisfying $g(F)=T^2$. Remove the interior of the regular neighborhood and we get a diffeomorphism of $V-\mathring{N}(F)$ onto N. q.e.d.

Proof of Lemma 6.3. Remark that the regular neighborhood of the singular fibre is given by blowing up and down of the oribit space of the equivariant regular neighborhood of the central fibre in the product T^2-bundle with the operation of monodromy. So, we can infer that the preimage of a small disk neighborhood of the image of a singular fibre by the projection ψ is a regular neighborhood of the singular fibre. On the other hand the regular neighborhood of the singular fibre of type II* is $P(E_9; -2)$ and that of type II (i.e. of rational curve with one cusp singularity) is the regular neighborhood of the union of a general fibre F and two vanishing cycles corresponding to D_1 and D_2 in the proof of Lemma 5.1. We know now that the closure Z of the complement is a T^2-bundle over $S^1\times I$ and hence there is a diffeomorphism $h\colon Z\to\partial P(E_9)\times I$. Therefore, $V-\mathring{N}(F)=W(-1, -1, \infty)\cup Z\cup P(E_9; -2)$, which is diffeomorphic to $W(-1, -1, \infty)\cup P(E_9; -2)$. Here, ∞ in $W(-1, -1, \infty)$ means that the handle along $N(S^1_y)$ does not attached.

Note that $P(E_9; -2)$ is made of $P(E_8; -2)$ with one 2-handle attached and the axis of the dual handle is the intersection $P(E_9;\ -2) \cap s(P^1)$. Since we may choose h so that $h(s(P^1) \cap Z) = (\partial P(E_9;\ -2) \cap s(P^1)) \times I$, this dual 2-handle is considered to be a 2-handle attached to $W(-1, -1, \infty)$ along $N(S^1_\gamma)$. Its framing number is equal to the self-intersection number of $s(P^1)$. Therefore, $V - \mathring{N}(F)$ is diffeomorphic to $W(-1, -1, -1) \cup P(E_8;\ -2)$. q.e.d.

REMARK 1. The orientation of the manifolds is consistent with the observation that $H(1, 1, 1) = \Sigma(2, 3, 5) = \partial P(E_8)$ and $\partial_1 W(-1, -1, -1) = H(-1, -1, -1) = -H(1, 1, 1)$.

REMARK 2. If a general fibre of the elliptic surface V is replaced with a multiple fibre with multiplicity m by a logarithmic transformation $L_b(b \neq a, a_1, a_2)$, then $K = -[F] + (m-1)[F_b] = -[F_b]$ (see [3, p. 772]) which implies $P_2 = 0$. Hence, $L_b(V)$ is a rational surface with $\chi = 12$ because $\pi_1 = 0$. In particular, $L_b(V)$ is diffeomorphic to $P^2 \# 9(-P^2)$.

§7. Proof of Theorem 3

Let $f \in SDiff(T^3)$. Then, by Theorem 1 there exists an $F \in SDiff(W(-1, -1, -1))$ such that $F|T^3 \times 0 = f$ and $F|H(-1, -1, -1) = id$. We can make union of the diffeomorphisms $id|\overline{N - W(-1, -1, -1)}$ and F which induces a diffeomorphism of $N \cup_d T^2 \times D^2$ onto $N \cup_f T^2 \times D^2$. Note that there exists an orientation reversing diffeomorphism of T^3 which extends to a diffeomorphism of $T^2 \times D^2$, for example, $J \in GL(3, \mathbf{Z})$ with $J([\alpha], [\beta], [\gamma]) = ([\alpha], [\beta], -[\gamma])$. Hence, $N \cup_J T^2 \times D^2$ is diffeomorphic to $N \cup_{id} T^2 \times D^2$. Therefore, also for any orientation reversing diffeomorphism of f of T^3, $N \cup_f T^2 \times D^2$ is diffeomorphic to $N \cup_{id} T^2 \times D^2 = P^2 \# 9(-P^2)$ by Theorem 1.

§8. Proof of Theorem 4

Take the rational elliptic surface $\psi\colon V \to P^1$ given in the proof of Proposition 6.1. We may assume that a_1 and a_2 are contained in the interior of the unit disk D of $\mathbf{C} \subset P^1$. Recall that, near ∂D, ψ is a product bundle with a fixed elliptic curve E as fibre. Take a biholomorphic map $j\colon \mathring{D} \to P^1 - D$ defined by $j(z) = 1/z$. Then, the operation of the matrix J in §7 is isotopic to $id_{T^2} \times j|\partial D$. So, by Lemma 6.2 $N \cup_J N$ admits a structure of an elliptic surface W with only four singular fibres, two of which are of type II* and the others of type II. We see $\pi_1(W) = 0$ and $\chi(W) = 24$. Therefore, $p_g = 1$ from the formula of M. Noether, and $K = 0$ because $c(\mathfrak{k}) = -2$ and $c(\mathfrak{f}) = -p_g - 1 = -2$ (see [2, Ths. 12.1 and 12.3]). Hence, W is a K3 surface. By Theorem 1 $N \cup_f N$ is diffeomorphic to this manifold for any orientation reversing diffeomorphism f because $f \circ J \in SDiff(T^3)$.

REMARK. Any K3 surfaces are mutually deformable [3] and hence diffeomorphic to each other.

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