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On the q-dimension of a space of orderings and q-fans

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Let F be a formally real field, P a preordering and ρ a form over F. We shall say that a pair (ρ, P) is maximal if ρ is P-anisotropic and P is maximal among the preorderings over which ρ is anisotropic. For a given q-cone Q (cf. [7]) we shall define a preordering $P(Q)$ and show that, P being a preordering, (ρ, P) is a maximal pair for some form ρ if and only if P is of finite index and of the form $P=P(Q)$ for some q-cone Q; such a preordering will be called a q-fan in this paper.

The main purpose of this paper is to characterize a q-fan in terms of the q-dimension which is defined in §3, and give a reduction formula on q-dimensions (Theorem 3.6 and Theorem 3.9).

§1. Defintions and preliminaries

Throughout this paper, a field always means a formally real field. We denote by $\dot{F}$ the multiplicative group of F. For a multiplicative subgroup P of $\dot{F}$, P is said to be a preordering of F if P is additively closed and $\dot{F}^2 \subset P$. We denote by $X(F/P)$ the space of all orderings σ with $P(\sigma) \supset P$, where $P(\sigma)$ is the positive cone of σ. A valuation v of F is called a real valuation if its residue field is formally real. The objects: valuation ring, valuation ideal, group of units, residue field and group of values will be denoted by A, M, U, F_v and G respectively. A preordering P of F will be called compatible with a valuation v of F (or v is compatible with P) if $1+M \subset P$. If a preordering P of F is compatible with a valuation v, then $P \cap U$ is a union of cosets of M and $\bar{P}=\varphi(P \cap U)$ is a preordering of F_v, where φ is the canonical surjection: $A \to F_v$.

We shall say that two orderings $\sigma, \tau \in X(F/P)$ are connected in $X(F/P)$ if $\sigma=\tau$ or there exists a fan of index 8 which contains σ and τ, and we denote the relation by $\sigma \sim \tau$. It is known that the relation $\sim$ is an equivalence relation in $X(F/P)$ ([4], Theorem 4.7). Each equivalence class of this relation is called a connected component of $X(F/P)$. We say that a preordering P is connected if $X(F/P)$ is connected. We denote by $\mathrm{gr}(X(F/P))$ the translation group of $X(F/P)$ in the terminology of [4], namely $\mathrm{gr}(X(F/P)) = \{\alpha \in \chi(F/P) | \alpha \cdot X(F/P) = X(F/P)\}$ where $\chi(F/P) = \mathrm{Hom}(\dot{F}/P, \{\pm 1\})$ is the character group of $\dot{F}/P$. For a preordering P of finite index, P is connected if and only if $\dim \dot{F}/P \geqq 3$ and $\dim \mathrm{gr}(X(F/P)) \geqq 1$.

Let v be a valuation compatible with P. We shall define a group isomorphism:

$$\dot{F}/P \longrightarrow G/v(P) \times \dot{F}_v/\bar{P} \cdots\cdots\cdots\cdots (*)$$

as a preliminary step to §3. Let s: $G \to \dot{F}$ be a q-section ([7], §7) with the property that $s(v(P)) \subset P$. We define the group homomorphism f: $\dot{F} \to G \times \dot{F}_v/\dot{F}_v^2$ by $f(x) = (v(x), xs(v(x))^{-1} \bmod M)$. Then by easy calculation, we can see $f^{-1}(v(P) \times \bar{P}) = P$ and we get the group isomorphism $(*)$ ([5], Theorem, p. 186).

PROPOSITION 1.1. *Let P be a preordering of finite index and v be a valuation compatible with P. Then we have*

$$\mathrm{gr}\,(X(F/P)) \cong \mathrm{Hom}\,(G/v(P), \{\pm 1\}) \times \mathrm{gr}\,(X(F_v/\bar{P})).$$

In particular

$$\dim \mathrm{gr}\,(X(F/P)) = \dim G/v(P) + \dim \mathrm{gr}\,(X(F_v/\bar{P}))$$

as Z_2-vector spaces. Moreover if there exists a valuation v of F which is compatible with P and $v(P) \neq G$, then the index of P is 4 or $X(F/P)$ is connected.

PROOF. The group isomorphism $\dot{F}/P \cong G/v(P) \times \dot{F}_v/\bar{P}$ naturally induces a group isomorphism $\chi(\dot{F}/P) \cong \mathrm{Hom}\,(G/v(P), \{\pm 1\}) \times \chi(\dot{F}_v/\bar{P})$. Considering $X(F/P)$ and $X(F_v/\bar{P})$ as subsets of $\chi(\dot{F}/P)$ and $\chi(\dot{F}_v/\bar{P})$ respectively, we get a natural bijection: $X(F/P) \cong \mathrm{Hom}\,(G/v(P), \{\pm 1\}) \times X(F_v/\bar{P})$. Then it follows immediately that

$$\mathrm{gr}\,(X(F/P)) \cong \mathrm{Hom}\,(G/v(P), \{\pm 1\}) \times \mathrm{gr}\,(X(F_v/\bar{P}).$$

If $v(P) \neq G$, then $\dim \mathrm{gr}\,(X(F/P)) \geqq 1$ and this implies that $X(F/P)$ is connected or the index of P is 4. Q. E. D.

For two forms f and g over F, we write $f \cong g \pmod P$ if $\dim f = \dim g$ and for any $\sigma \in X(F/P)$, $sgn_\sigma(f) = sgn_\sigma(g)$ where $sgn_\sigma(f)$ and $sgn_\sigma(g)$ are the signatures at σ of f and g respectively. If $f \cong xg \pmod P$ for some $x \in \dot{F}$, then we say that the forms f and g are P-similar. We now recall the definitions of the residue class forms of a form $\rho = \langle a_1, \dots, a_n \rangle$ and the sets of valuations $\Omega(P)$, $\Omega(P, a_1, \dots, a_n)$ which were introduced in [1]. Let v be a valuation of F which is compatible with a preordering P. If $v(a_i) \equiv v(a_j) \pmod{v(P)}$ for any i, j, then it is clear that there exist $x \in \dot{F}$ and $t_i \in P$ $(i = 1, \dots, n)$ such that $v(xt_ia_i) = 0$ for any i. Let $\bar{\rho} = \langle (xt_1a_1)^-, \dots, (xt_na_n)^- \rangle$ be the form over F_v, where the bar means the residue class modulo M. We shall show that $\bar{\rho}$ is unique up to $\bar{P}$-similarity. Assume that $x' \in \dot{F}$ and $t'_i \in P$ $(i = 1, \dots, n)$ satisfy the same conditions. We put $\bar{\rho}' = \langle (x't'_1a_1)^-, \dots, (x't'_na_n)^- \rangle$ and $\alpha = (xt_1a_1)\,(x't'_1a_1)^{-1}$. Then α is a unit of A

and we have $\alpha(x't'_i a_i)=(t_1 t_1'^{-1} t'_i t_i^{-1})(x t_i a_i)$, $t_1 t_1'^{-1} t'_i t_i^{-1} \in P \cap U$ for every i. These relations imply $\bar{\rho} \cong \bar{\alpha} \cdot \bar{\rho}'$ (mod $\bar{P}$) and the conclusion follows. For a form $\rho = \langle a_1,\dots, a_n\rangle$, the equivalence relation in $\{a_1,\dots, a_n\}$ defined by $v(a_i) \equiv v(a_j)$ (mod $v(P)$) gives rise to a partition of this set. Let $\rho = \rho_1 \perp \cdots \perp \rho_t$ be the decomposition of ρ with respect to this partition; that is, t is the number of classes and each ρ_i satisfies the condition mentioned above. The forms $\bar{\rho}_i$ $(i=1,\dots, t)$ of F_v are called the residue class forms of ρ. As for $\Omega(P)$, it is the set of valuations which are compatible with at least one ordering $\sigma \in X(F/P)$, and $\Omega(P, a_1,\dots, a_n) = \{v \in \Omega(P) | v(a_i) \not\equiv v(a_j) \text{ (mod } v(P)) \text{ for some } i \text{ and } j\}$. For $v \in \Omega(P)$, there exists the least preordering which is compatible with v and contains P. We denote it by P^v.

Proposition 1.2. ([1], Theorem 3.3) *Let $\rho = \langle a_1,\dots, a_n\rangle$ be a form such that ρ is P-anisotropic and σ-indefinite for any $\sigma \in X(F/P)$. Then there exists a valuation $v \in \Omega(P, a,\dots, a_n)$ such that ρ is P^v-anisotropic.*

§2. q-cones and q-fans

In [6], Prestel introduced the notion of q-cones and pre-q-cones which generalize that of orderings and preorderings respectively. A subset Q of $\dot{F}$, will be called a pre-q-cone if it satisfies the following conditions:

$$(1)\quad Q+Q \subset Q \quad (2)\quad F^2 \cdot Q \subset Q \quad (3)\quad Q \cap -Q = \phi \quad (4)\quad 1 \in Q.$$

For a pre-q-cone Q, if $Q \cup -Q = \dot{F}$, then Q will be called a q-cone of F. (In [7], a pre-q-cone Q contains the element $0 \in F$ and does not necessarily contain $1 \in Q$. In this paper we assume $0 \notin Q$ and $1 \in Q$ for convenience.) It is easily shown that if Q is a pre-q-cone, then $S_F \cdot Q \subset Q$ where $S_F = D_F(\infty) = \Sigma \dot{F}^2$.

Definition and Proposition 2.1. *For a pre-q-cone Q of F, we define $P(Q) = \{x \in Q | xQ \subset Q\}$. Then $P(Q)$ is a preordering of F. For a preordering P of F, if there exists a q-cone Q such that $P(Q)=P$, then P will be called a q-fan.*

The proof is easy and omitted.

Definition 2.2. Let ρ be a form and P be a preordering of F over which ρ is anisotropic. If ρ is P'-isotropic for any preordering $P' \supsetneqq P$, then we say that the pair (ρ, P) is *a maximal pair.*

By [1], Corollary 3.4, if (ρ, P) is a maximal pair, then P has a finite index.

Lemma 2.3. *Let P be a preordering and Q be a pre-q-cone of F. Then the following statements hold.*

(1) *$P(Q) \supset P$ if and only if Q is a union of cosets of P.*

(2) *If $P(Q) \supset P$, then there exists a q-cone $Q_1 \supset Q$ such that $P(Q_1) \supset P$.*

PROOF. The assertion (1) follows immediately from the definition. As for the assertion (2), let $M=\{Q'|Q'$ is a pre-q-cone which contains Q and is a union of cosets of $P\}$. Then M is an inductive set with respect to the inclusion relation, and by Zorn's Lemma, there exists a maximal element Q_1 of M. It is easy to show that Q_1 is a required one. Q. E. D.

THEOREM 2.4. *Let P be a preordering of F which is of finite index. Then the following statements are equivalent:*

(1) *P is a q-fan.*

(2) *There exists a form ρ such that (ρ, P) is a maximal pair.*

PROOF. (1)⇒(2): Let Q be a q-cone of F such that $P(Q)=P$. By Lemma 2.3, (1), there exist $a_1,\dots, a_n \in \dot{F}$ such that $Q=a_1P \cup \cdots \cup a_nP$. We put $\rho=\langle a_1,\dots, a_n\rangle$; then it is clear that ρ is P-anisotropic. Let P' be a preordering of F which contains P properly and take an element $x \in P'-P$. Then we have $xQ \not\subset Q$ and so there exists $\alpha \in Q$ such that $x\alpha \notin Q$. This implies $-x\alpha \in Q$ and ρ is P'-isotropic.

(2)⇒(1): We put $Q'=D(\rho/P)$, where $D(\rho/P)$ is the set $\{b \in \dot{F}|\rho$ represents b over $P\}$. Then it follows from the maximality of P that Q' is a pre-q-cone and $P(Q')=P$. By Lemma 2.3, (2), there exists a q-cone Q such that $P(Q)\supset P$. It is clear that the form ρ is $P(Q)$-anisotropic and the maximality of P shows that $P(Q)=P$. Q. E. D.

COROLLARY 2.5. *For a form ρ and a preordering P of F, the following statements are equivalent:*

(1) *ρ is P-anisotropic.*

(2) *There exists a q-fan P' of finite index such that $P'\supset P$ and ρ is P'-anisotropic.*

EXAMPLE 2.6. Let P be a preordering of finite index. If P is an ordering, then clearly P is a q-fan. Moreover a non-trivial fan P is a q-fan. In fact let $\{1, a_2,\dots, a_n\}$ be a complete system of representatives of the positive cone of some ordering $\sigma \in X(F/P)$, i.e. $P(\sigma)=P \cup a_2P \cup \cdots \cup a_nP$. We put $\rho=\langle 1=a_1, a_2,\dots, a_{n-1}, -a_n\rangle$. Since P is a fan, ρ is P-anisotropic, and we can readily see that $Q=D(\rho/P)$ is a q-cone and $P(Q)\supset P$. Conversely take an element $x \in Q-P$. We have only to show $xQ \not\subset Q$. To do this, we may assume that $x=a_2$ or $x=-a_n$. Since $P \cup a_2P \cup \cdots \cup a_nP$ is an ordering, we have $a_2a_n \in a_jP$ for some $j (j\neq n)$. Then $a_2(-a_n)P=-a_jP \not\subset Q$, which implies $xQ \not\subset Q$.

The following proposition is shown implicitly in the proof of [1], Corollary 3.4.

PROPOSITION 2.7. *Let v be a valuation which is compatible with a pre-*

ordering P. Let $\rho=\langle a_1,\dots, a_n\rangle$ be a form such that (ρ, P) is a maximal pair. Then the following statements hold.

(1) *The value group G is generated by $v(a_i)$ $(i=1,\dots, n)$ and $v(P)$.*

(2) *Let $\bar{\rho}_i$ $(i=1,\dots, t)$ be the residue class forms of ρ and $\bar{P}_i$ $(i=1,\dots, t)$ be preorderings of F_v such that $\bar{P}_i \supset \bar{P}$ and $(\bar{\rho}_i, \bar{P}_i)$ are maximal pairs. (Since each $\bar{\rho}_i$ is $\bar{P}$-anisotropic by* [1], Proposition 3.1, *Zorn's Lemma guarantees the existence of $\bar{P}_i$.) Then we have $\bar{P}=\bigcap \bar{P}_i$ $(i=1,\dots, t)$.*

THEOREM 2.8. *Let P be a preordering of finite index. If P is a q-fan, then P is connected. In particular, if ρ is P-anisotropic, then there exists a connected preordering $P'(P'\supset P)$ of finite index such that ρ is P'-anisotropic.*

PROOF. By Theorem 2.4, there exists a form $\rho=\langle a_1,\dots, a_n\rangle$ such that (ρ, P) is a maximal pair. When P is an ordering, the assertion is clear. Therefore we may assume that P is not an ordering. Then for any $\sigma\in X(F/P)$, ρ is σ-indefinite by the maximality of P. So it follows from Proposition 1.2 that there exists a valuation $v\in\Omega(P, a_1,\dots, a_n)$ such that ρ is P^v-anisotropic. Hence $P^v=P$ by the maximality of P and so P is compatible with v. There exist a_i and a_j such that $v(a_i)\not\equiv v(a_j) \pmod{v(P)}$, and we can see that $v(P)\neq G$. It follows from Proposition 1.1 that $\dim \mathrm{gr}(X(F/P))\geqq 1$. Since P is a q-fan, P is not an intersection of two orderings, and so P is connected. Q. E. D.

DEFINITION 2.9. For a preordering P of F, we define $Y(F/P)=\{Q\colon$ a q-cone of $F|P(Q)\supset P\}$. Naturally the set $X(F/P)$ can be identified with a subset of $Y(F/P)$.

Let P be a preordering of finite index and $X_1,\dots, X_n$ be the connected components of $X(F/P)$. We put $P_i=X_i^{\perp}$, i.e. P_i is the preordering of X_i. Then we have $P=\bigcap P_i$ and it is the decomposition of P into connected components (cf. [3], §2).

COROLLARY 2.10. *Notation being as above, we have $Y(F/P)=\bigcup Y(F/P_i)$ (disjoint union).*

PROOF. It is clear that $Y(F/P_i)\subset Y(F/P)$ for any i. Let Q be an element of $Y(F/P)$. By Theorem 2.8, $P(Q)$ is connected and this implies $P(Q)\supset P_i$ for some i, and $Q\in Y(F/P_i)$. Thus $Y(F/P)=\bigcup Y(F/P_i)$. Next we shall show that $Y(F/P_i)\cap Y(F/P_j)=\phi$ for any $i\neq j$. Assume on the contrary that there exists a q-cone $Q\in Y(F/P_i)\cap Y(F/P_j)$. Then $P(Q)$ contains P_i and P_j; since $P(Q)$ is a preordering, this implies that $X_i=X(F/P_i)$ and $X_j=X(F/P_j)$ have a common ordering, a contradiction. Q. E. D.

§3. Valuations and q-fans

For a preordering P, there exists a finest valuation v compatible with P. Its valuation ring A is generated by A_Q^σ, $\sigma \in X(F/P)$, where A_Q^σ is the finest valuation ring compatible with $\sigma \in X(F/P)$, i.e. $A_Q^\sigma = \{a \in F \mid b - a \in P(\sigma)$ and $b + a \in P(\sigma)$ for some $b \in Q\}$ and Q is the field of rational numbers. We shall call v the finest valuation of P and A the finest valuation ring of P. The set of valuations compatible with P forms a chain.

LEMMA 3.1. *Let v_1, v_2 be valuations compatible with a preordering P, and A_1, A_2 be the valuation rings of v_1, v_2 respectively. If $A_1 \subset A_2$, then* $\dim G_1/v_1(P) \geqq \dim G_2/v_2(P)$.

PROOF. It is easy to see that $v_1^{-1}(v_1(P)) = P \cdot U_1$ and $v_2^{-1}(v_2(P)) = p \cdot U_2$, where U_1 and U_2 are the groups of units of A_1 and A_2 respectively. Then, since $U_1 \subset U_2$ and $\dot{F}/PU_i \cong G_i/v_i(P)$ $(i=1, 2)$, we have $\dim G_1/v_1(P) \geqq \dim G_2/v_2(P)$. Q. E. D.

LEMMA 3.2. *Let P be a preordering of finite index and v be the finest valuation compatible with P. If P is connected, then* $\dim G/v(P) \geqq 1$.

PROOF. First we note $\dim \mathrm{gr}(X(F/P)) \geqq 1$ and $\dim \dot{F}/P \neq 2$ since P is connected. We take $\tau \in \mathrm{gr}(X(F/P))$, $\tau \neq 1$. We write $X(F/P) = \{\sigma_1, \ldots, \sigma_k, \tau\sigma_1, \ldots, \tau\sigma_k\}$. Then we have $k \geqq 2$ since $\dim \dot{F}/P \neq 2$. We let P_i be the preordering of a 4 fan $\{\sigma_1, \sigma_i, \tau\sigma_1, \tau\sigma_i\}$ $(i=2, \ldots, k)$. By [2], Theorem 2.7, there exists a valuation v_i such that v_i is compatible with P_i and $\bar{P}_i$ is trivial (i.e. the index of $\bar{P}_i$ equals 2 or 4), for any $i=2, \ldots, k$. For the value group G_i of v_i, we have $v_i(P_i) \neq G_i$ by Proposition 1.1. The valuation ring A_i of the valuation v_i contains the finest valuation ring $A_Q^{\sigma_1}$ compatible with σ_1; hence the set $\{A_i\}$ forms a chain. We may assume that A_2 is the maximal one. Then the valuation v_2 is compatible for any ordering of $X(F/P)$, so v_2 is compatible with P. Then the valuation ring A of v is contained in A_2 and hence $\dim G/v(P) \geqq \dim G_2/v_2(P) \geqq 1$ by Lemma 3.1.

Q. E. D.

LEMMA 3.3. *Let v be the finest valuation of a preordering P. Then any valuation of F_v compatible with $\bar{P}$ is trivial.*

PROOF. Let $\bar{v}$ be a valuation of F_v compatible with $\bar{P}$, A and $\bar{A}$ be the valuation rings of v and $\bar{v}$ respectively and $\varphi: A \to F_v$ be the canonical surjection. Then $A' = \varphi^{-1}(\bar{A})$ is a valuation ring of F, and it is clear that the valuation v' corresponding to A' is compatible with P. Since v is the finest valuation of P and $A' \subset A$, it follows that $A' = A$ and $\bar{v}$ is trivial. Q. E. D.

THEOREM 3.4. *Let P be a preordering of finite index and v be the finest* *valuation of P. If P is connected and is not a fan, then* $\dim G/v(P)=\dim \mathrm{gr}\,(X(F/P))$. *In particular, the induced preordering* $\bar{P}$ *of* F_v *is not connected.*

PROOF. Assume on the contarary that $\dim G/v(P)\neq\dim \mathrm{gr}\,(X(F/P))$. Then we have $\dim G/v(P)<\dim \mathrm{gr}\,(X(F/P))$ and $\dim \mathrm{gr}\,(X(F_v/\bar{P}))\geqq 1$ by Proposition 1.1. From [2], Example 2.6, $\bar{P}$ is not a fan, so $\dim \dot{F}_v/\bar{P}\neq 2$ and $\bar{P}$ is connected. Then it follows from Lemma 3.2 that $\bar{v}(\bar{P})\neq\bar{G}$, where $\bar{v}$ is the finest valuation of $\bar{P}$ and $\bar{G}$ is its value group. This contradicts Lemma 3.3. Q. E. D.

DEFINITION 3.5. Let P be a preordering of finite index. Then P can be written as $P=P_1\cap\cdots\cap P_n$, where P_i is a q-fan for any $i=1,\dots, n$. We call the least number of n *the q-dimension* of P and denote it by q-dim (P).

THEOREM 3.6. *Let P be a connected preordering of finite index and v be a valuation which is compatible with P. Then the following statements are equivalent.*

(1) *P is a q-fan.*

(2) q-dim $(\bar{P})\leqq 2^r$, $r=\dim G/v(P)$.

PROOF. (1)⇒(2): Let (ρ, P) be a maximal pair, and $\bar{\rho}_i\,(i=1,\dots, t)$ be residue class forms of ρ. Then it follows from $t\leqq 2^r$ that q-dim $(\bar{P})\leqq 2^r$ by Proposition 2.7, (2).

(2)⇒(1): If $\bar{P}$ is an ordering of F_v, then P is a fan ([2], Example 2.6) and the assertion follows from Example 2.6. If $r=0$, then $\bar{P}$ is a q-fan and there exists a q-cone $\bar{Q}$ of F_v such that $P(\bar{Q})=\bar{P}$. We write $\bar{Q}=\bar{a}_1\bar{P}\cup\cdots\cup\bar{a}_m\bar{P}$, $a_i\in U$, $\bar{a}_i= a_i \bmod M$ $(i=1,\dots, m)$. It is clear that the form $\rho=\langle a_1,\dots, a_m\rangle$ is P-anisotropic. Since $\dot{F}/P\cong\dot{F}_v/\bar{P}$, we see that $Q=D(\rho/P)=a_1P\cup\cdots\cup a_mP$ and Q is a q-cone of F. Then it follows immediately that $P(Q)=P$, and so P is a q-fan. Next we consider the case $r\geqq 1$ and $\bar{P}$ is not an ordering. We can write $\bar{P}=\bar{P}_1\cap\cdots\cap\bar{P}_s$, $2\leqq s\leqq 2^r$, such that $\bar{P}_i\neq\bar{P}_j$ for any $i\neq j$ and each $\bar{P}_i$ is a q-fan. (If $\bar{P}$ is a q-fan, then we write $\bar{P}=\bar{P}\cap P(\tau)$, where $P(\tau)$ is the positive cone of some ordering $\tau\in X(F_v/\bar{P})$.) Let $\bar{Q}_i\,(i=1,\dots, s)$ be q-cones of F_v such that $P(\bar{Q}_i)=\bar{P}_i$. We write $\bar{Q}_i=\bar{a}_{i1}\bar{P}\cup\cdots\cup \bar{a}_{im}\bar{P}$ $(i=1,\dots, s)$, where $a_{ij}\in U$, $\bar{a}_{ij}=a_{ij} \bmod M$ and $2m$ is the index of $\bar{P}$. Let $s\colon G\to\dot{F}$ be a q-section with $s(v(P))\subset P$ and $\alpha_1,\dots,\alpha_r$ be elements of G such that the set $\{\bar{\alpha}_1,\dots,\bar{\alpha}_r\}$ is a basis of the $\boldsymbol{Z}_2$-vector space $G/v(P)$. Let A be the set $\{\varepsilon_1\alpha_1+\cdots+\varepsilon_r\alpha_r;\ \varepsilon_i=0, 1\}$ consisting of 2^r elements of G. Let y_i $(i=1,\dots, 2^r)$ be elements of F such that $y_1=1$ and $A=\{v(y_i);\ i=1,\dots, 2^r\}$. We put $\rho= \Sigma y_i\langle a_{i1},\dots, a_{im}\rangle$ $(\mathrm{i}=1,\dots, 2^r$, where $a_{ij}=a_{sj}$ for $i\geqq s$.) Since the residue class forms of ρ are $\bar{P}$-anisotropic, ρ is P-anisotropic ([1], Proposition 3.1). Also $2\cdot\dim\rho$ equals the index of P by the group isomorphism $(*)$ in §1. Since $\{y_ia_{ij};\ i=1,\dots, 2^r, j=1,\dots, m\}$ are the complete system of representatives of $Q=D(\rho/P)$ over P, Q is a q-cone. It is clear that $P(Q)\supset P$ and we shall show the reverse inclusion. It suffices to show that $f(P(Q)) \subset v(P) \times \bar{P}$, where f is the group homomorphism defined in §1.

Step 1. First we shall show that $v(P(Q)) \subset v(P)$. Assume on the contrary that there exists $\alpha \in P(Q)$ such that $v(\alpha) \notin v(P)$. Then we can write $v(\alpha) = v(y_k) + b$ for some $y_k \neq 1$ and $b \in v(P)$. It follows from the facts $v(\alpha a_{1i}) = v(\alpha) \in v(y_k) + v(P)$ and $\alpha a_{1i} \in Q$ that $\alpha a_{1i} \in y_k a_{k1} P \cup \cdots \cup y_k a_{km} P$ $(i = 1, \ldots, m)$. It is now easy to show that by a suitable renumbering of $\{a_{ki}\}$, we may assume that $\alpha a_{1i} \in y_k a_{ki} P$ for any $i = 1, \ldots, m$. So we can write $\alpha a_{1i} = y_k a_{ki} p_i$, $p_i \in P$ $(i = 1, \ldots, m)$. The residue class forms of $\langle \alpha a_{11}, \ldots, \alpha a_{1m} \rangle$ and $\langle y_k a_{k1}, \ldots, y_k a_{km} \rangle$ are $\langle \bar{a}_{11}, \ldots, \bar{a}_{1m} \rangle$ and $\langle \bar{a}_{k1}, \ldots, \bar{a}_{km} \rangle$ respectively and they are $\bar{P}$-similar by the argument in §1. This shows that $\bar{Q}_1 = \beta \bar{Q}_k$ for some $\beta \in \dot{F}_v$ and so $\bar{P}_1 = \bar{P}_k$. This is a contradiction.

Step 2. Next we shall show that $\alpha s(v(\alpha))^{-1} \bmod M \in \bar{P}$ for any $\alpha \in P(Q)$. Let k be an integer with $1 \leqq k \leqq s$. Since $v(\alpha) \in v(P)$, we have $\alpha y_k a_{ki} \in y_k a_{k1} P \cup \cdots \cup y_k a_{km} P$ $(i = 1, \ldots, m)$. Similarly to Step 1, we can find a bijection $\sigma: \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that $\alpha a_{ki} \in a_{k\sigma(i)} P$ $(i = 1, \ldots, m)$. So we can write $\alpha a_{ki} = a_{k\sigma(i)} p_i$ for some $p_i \in P$. Then since $s(v(\alpha))^{-1} \in P$ and $v(p_i) = v(\alpha)$, we have $p_i s(v(\alpha))^{-1} \in P \cap U$; therefore $\beta \bar{a}_{ki} = \bar{a}_{k\sigma(i)} p'_i$, where $\beta = \alpha s(v(\alpha))^{-1} \bmod M$ and $p'_i = p_i s(v(\alpha))^{-1} \bmod M \in \bar{P}$. This shows that $\beta \bar{Q}_k = \bar{Q}_k$, and so $\beta \in \bar{P}_k = P(\bar{Q}_k)$. Thus we have $\beta = \alpha s(v(\alpha))^{-1} \in \cap \bar{P}_k = \bar{P}$. Q. E. D.

COROLLARY 3.7. *Let P be a connected preordering of finite index and v be the finest valuation of P. Then the following statements are equivalent*:

(1) *P is a q-fan.*

(2) q-dim $(\bar{P}) \leqq 2^r$, *where* $r = \dim \mathrm{gr}(X(F/P))$.

PROOF. The assertion follows immediately from Theorem 3.4 and Theorem 3.6. Q. E. D.

Let v be a valuation of F compatible with a preordering P of F. Let T and S be preorderings of F and F_v respectively such that $T \supset P$ and $S \supset \bar{P}$. We say that T is the lifting of S if $X(F/T)$ consists of all orderings which lift the orderings of $X(F_v/S)$, i.e. $X(F/T) = \{\sigma \in X(F/P) \mid \bar{\sigma} \in X(F_v/S)\}$. If T is the lifting of S then it is clear that $\bar{T} = S$.

LEMMA 3.8. *Notation being as above, we have* $|X(F/T)| = 2^r \times |X(F_v/S)|$ and $\dim G/v(T) = r$ $(r = \dim G/v(P))$, namely $v(T) = v(P)$.

PROOF. It is clear that for an ordering $\tau \in X(F_v/\bar{P})$, there exists exactly 2^r orderings $\sigma_i \in X(F/P)$ which lift the ordering τ. So we have $|X(F/T)| = 2^r \times |X(F_v/S)|$. This shows that $\dim G/v(T) = r$ since $X(F/T) \cong \mathrm{Hom}\,(G/v(T), \{\pm 1\}) \times X(F_v/S)$. Q. E. D.

THEOREM 3.9. *Let P be a connected preordering of finite index and v be the* *finest valuation of* P. *Then* q-dim (P) *is the least integer* s *satisfying* $s \geqq$ q-dim $(\bar{P})/2^r$, *where* $r = \dim \mathrm{gr}(X/P)$.

PROOF. We put $m =$ q-dim $(\bar{P})$ and write $\bar{P} = \bar{P}_1 \cap \cdots \cap \bar{P}_m$ where each $\bar{P}_i$ is a q-fan. By Corollary 3.7, we may assume $2^r < m$. We put $S_i = \bar{P}_{2^r(i-1)+1} \cap \cdots \cap \bar{P}_{2^r i}$ $(i = 1, \dots, s-1)$ and $S_s = \bar{P}_{2^r(s-1)+1} \cap \cdots \cap \bar{P}_m$. We let T_i be the liftings of S_i $(i = 1, \dots, s)$ and $T = \cap T_i$ $(i = 1, \dots, s)$. By Theorem 3.6 and Lemma 3.8, each T_i is a q-fan. Since we have the isomorphisms $\dot{F}/P \cong G/v(P) \times \dot{F}_v/\bar{P}$ and $\dot{F}/T \cong G/v(T) \times \dot{F}_v/\bar{T}$, it follows froms $\bar{T} = \cap \bar{T}_i = \bar{P}$ and $v(T) = v(P)$ that $T = P$. Hence q-dim $(P) \leqq s$. Conversely we write $P = P_1 \cap \cdots \cap P_n$ where each P_i is a q-fan. Then by Theorem 3.6, q-dim $(\bar{P}_i) \leqq 2^t \leqq 2^r$ $(t = \dim G/v(P_i))$; therefore q-dim $(\bar{P}) \leqq 2^r$q-dim (P) since $\bar{P} = \bar{P}_1 \cap \cdots \cap \bar{P}_n$. Q. E. D.

REMARK 3.9. (1) The converse of Theorem 2.8 is not valid. In fact let K be a field with exactly three orderings and $F = K((x))$ and $P = S_F = \Sigma \dot{F}^2$. Then $X(F/P)$ is connected and $\dim \mathrm{gr}(X(F/P)) = 1$. Since $F_v \cong K$ has exactly three orderings, we have q-dim $(\bar{P}) = 3$ and by Corollary 3.7, P is not a q-fan.

(2) In Example 2.6, we showed that a non-trivial fan is a q-fan. The converse is false. In fact we put $L = K((x))((y))$ where K is the field given in (1) and $P = S_L = \Sigma \dot{L}^2$. Then $X(L/P)$ is not a fan. However since $\dim \mathrm{gr}(X(L/P)) = 2$, we can see that P is a q-fan by Corollary 3.7.

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