HIROSHIMA MATH. J.
14 (1984), 333–358

Joins of weakly ascendant subalgebras of Lie algebras

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(Received November 25, 1983)

Introduction

Amayo [2] proved that several classes of finitely generated Lie algebras are ascendantly coalescent, where a class $\mathfrak{X}$ of Lie algebras is ascendantly coalescent if in any Lie algebra the join of any pair of ascendant $\mathfrak{X}$-subalgebras is always an ascendant $\mathfrak{X}$-subalgebra. On the other hand, Tôgô [10] introduced the concept of weakly ascendant subalgebras of Lie algebras generalizing that of ascendant subalgebras. It might be hopeless to search classes $\mathfrak{X}$ such that in any Lie algebra the join of any pair of weakly ascendant $\mathfrak{X}$-subalgebras is always a weakly ascendant $\mathfrak{X}$-subalgebra, for there exists a Lie algebra in which the join of a certain pair of 1-dimensional weak subideals is not a weakly ascendant subalgebra and is non-abelian simple (cf. [4, Example 5.1]). However, in the recent papers [5] and [6] the author presented various classes of Lie algebras in which the join of any pair, or any family, of weak subideals (resp. subideals) is always a weak subideal (resp. a subideal). In this paper we shall investigate the class $\mathfrak{L}$(wasc) (resp. $\mathfrak{L}$(asc)) of Lie algebras in which the join of any pair of weakly ascendant subalgebras (resp. ascendant subalgebras) is always a weakly ascendant subalgebra (resp. an ascendant subalgebra), and the class $\mathfrak{L}^{\infty}$(wasc) (resp. $\mathfrak{L}^{\infty}$(asc)) of Lie algebras in which the join of any family of weakly ascendant subalgebras (resp. ascendant subalgebras) is always a weakly ascendant subalgebra (resp. an ascendant subalgebra).

Section 2 is devoted to investigating general properties of weakly ascendant subalgebras of Lie algebras. We shall show as generalizations of [2, Theorem 2.5] and [10, Theorem 4] that if H wasc L then $H/H_L \in \mathrm{L}\mathrm{R}\mathfrak{N} \cap \dot{\mathrm{E}}(\lhd)\mathfrak{A}$ (Theorem 2.2 (1)) and that if H wasc L and $H/H_{sL} \in \mathfrak{G}$ then $H \leq^{\omega} L$ (Theorem 2.2 (2)). Furthermore, we shall show that if $H \leq^{\rho} L$, $K \leq^{\sigma} L$ and $[H, K] \subseteq H$ then $H+K \leq^{\sigma\rho} L$ (Theorem 2.5).

In Section 3 we shall show that various classes are subclasses of $\mathfrak{L}(\Delta)$ or $\mathfrak{L}^{\infty}(\Delta)$, where Δ is any one of the relations wasc and asc. For example, the class $\mathfrak{D}$(wasc)$\mathfrak{A}$, which contains all hypercentral-by-abelian Lie algebras, is a subclass of $\mathfrak{L}$(wasc), and the classes $\mathfrak{D}$(wasc)$(\mathfrak{F} \cap \mathfrak{A}_1)$ and $\mathfrak{D}$(wasc)$\mathfrak{S}$(wasc) are subclasses of $\mathfrak{L}^{\infty}$(wasc) (Theorem 3.9). The class $\mathfrak{D}$(asc)$\mathfrak{A} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$, which contains all hypercentral-by-abelian Lie algebras, is a subclass of $\mathfrak{L}$(asc), and the classes $\mathfrak{D}$(asc)$(\mathfrak{F} \cap \mathfrak{A}_1) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\mathfrak{D}$(asc)$(\mathfrak{G} \cap \mathfrak{S})$, the latter of which contains all hypercentral-by-finitely-generated-simple Lie algebras, are subclasses of $\mathfrak{L}^{\infty}(\mathrm{asc})$ (Theorem 3.10).

In Section 4 we shall first improve [6, Theorem 7] in Theorem 4.1. Secondly we shall show that various classes are subclasses of $\mathfrak{L}(\mathrm{asc})$ or $\mathfrak{L}^{\infty}(\mathrm{asc})$ over any field of characteristic zero. For example, over any field of characteristic zero the classes $\mathfrak{N}(\mathfrak{G}^{\mathrm{I}} \cap \text{Min-si})$ and $\mathfrak{N}(\mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_{*}{}^{\mathrm{I}})$ are subclasses of $\mathfrak{L}(\mathrm{asc})$, and the classes $\mathfrak{N}(\text{Min-si} \cap \text{Max-si})$ and $\mathfrak{N}(\text{Max-si} \cap \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_{*}{}^{\mathrm{I}})$ are subclasses of $\mathfrak{L}^{\infty}(\mathrm{asc})$ (Theorem 4.3).

In Section 5 we shall show that $\mathfrak{L}^{\infty}(\Delta) < \mathfrak{L}(\Delta) < \mathfrak{L}(\mathfrak{G}\text{-}\Delta)$, $\mathfrak{L}^{\infty}(\Delta) \not\leq \mathfrak{L}_{\infty}(\Delta)$ and $\mathfrak{L}_{\infty}(\Delta) \not\leq \mathfrak{L}(\mathfrak{G}\text{-}\Delta)$, where Δ is any one of the relations wasc and asc (Theorem 5.1).

The author wishes to express his thanks to Professor S. Tôgô for his valuable comments in preparing this paper.

1.

Throughout this paper we always consider not necessarily finite-dimensional Lie algebras over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified, and mostly follow [3] for the use of notations and terminology.

Let H be a subalgebra of a Lie algebra L. For an ordinal σ, H is a σ-step weakly ascendant subalgebra (resp. a σ-step ascendant subalgebra) of L, denoted by $H \leq^{\sigma} L$ (resp. $H \lhd^{\sigma} L$), if there exists an ascending chain $(H_{\alpha})_{\alpha \leq \sigma}$ of subspaces (resp. subalgebras) of L such that

(1) $H_0 = H$ and $H_{\sigma} = L$,

(2) $[H_{\alpha+1}, H] \subseteq H_{\alpha}$ (resp. $H_{\alpha} \lhd H_{\alpha+1}$) for any ordinal $\alpha < \sigma$,

(3) $H_{\lambda} = \bigcup_{\alpha<\lambda} H_{\alpha}$ for any limit ordinal $\lambda \leq \sigma$.

Then the chain $(H_{\alpha})_{\alpha \leq \sigma}$ is called a weakly ascending series (resp. an ascending series) from H to L. H is a weakly ascendant subalgebra (resp. an ascendant subalgebra) of L, denoted by H wasc L (resp. H asc L), if $H \leq^{\sigma} L$ (resp. $H \lhd^{\sigma} L$) for some ordinal σ. If H wasc L (resp. H asc L), then there exists the least ordinal μ such that $H \leq^{\mu} L$ (resp. $H \lhd^{\mu} L$). We call such an ordinal μ the *weakly ascendant index* (resp. the *ascendant index*) of H in L and denote it by $\mathrm{wasc}(L\colon H)$ (resp. $\mathrm{asc}(L\colon H)$). H is a weak subideal (resp. a subideal) of L, denoted by H wsi L (resp. H si L), if H wasc L (resp. H asc L) with $\mathrm{wasc}(L\colon H) < \omega$ (resp. $\mathrm{asc}(L\colon H) < \omega$).

Let Δ be one of the relations wasc and asc. We introduce the classes $\mathfrak{M}_1(\Delta)$, $\mathfrak{M}_2(\Delta)$ and $\mathfrak{M}(\Delta)$ of Lie algebras L as follows:

$L \in \mathfrak{M}_1(\Delta)$ if and only if $\Delta(L\colon H) \leq 1$ whenever $H \,\Delta\, L$;

$L \in \mathfrak{M}_2(\Delta)$ if and only if $\Delta(L\colon H) \leq 2$ whenever $H \,\Delta\, L$;

$L \in \mathfrak{M}(\Delta)$ if and only if $\Delta(L\colon H) < \omega$ whenever $H \,\Delta\, L$.

In [11] $\mathfrak{M}_1(\text{asc})$ and $\mathfrak{M}(\text{asc})$ are denoted by $\mathfrak{M}'$ and $\mathfrak{M}$ respectively.

We need the closure operation I defined as follows: For a class $\mathfrak{X}$ of Lie algebras, $\mathfrak{X}$ is I-closed if and only if $H \text{ si } L \in \mathfrak{X}$ implies $H \in \mathfrak{X}$. Analogously the closure operation $\text{I}(\Delta)$ is defined as follows: For a class $\mathfrak{X}$ of Lie algebras, $\mathfrak{X}$ is $\text{I}(\Delta)$-closed if and only if $H \,\Delta\, L \in \mathfrak{X}$ implies $H \in \mathfrak{X}$. In [2] $\text{I}(\text{asc})$ is denoted by $\acute{\text{I}}$. It is clear that the closure operations I and $\text{I}(\Delta)$ are unary operations in the sense of [8, p. 5]. Hence for any class $\mathfrak{X}$ of Lie algebras the largest I-closed (resp. $\text{I}(\Delta)$-closed) subclass of $\mathfrak{X}$ is well defined and denoted by $\mathfrak{X}^{\text{I}}$ (resp. $\mathfrak{X}^{\text{I}(\Delta)}$). Then we can easily see that for a Lie algebra L, $L \in \mathfrak{X}^{\text{I}}$ (resp. $\mathfrak{X}^{\text{I}(\Delta)}$) if and only if $H \text{ si } L$ (resp. $H \,\Delta\, L$) implies $H \in \mathfrak{X}$. In particular, $\mathfrak{G}^{\text{I}}$ is defined in [3, p. 66].

LEMMA 1.1. (1) $\mathfrak{G}^{\text{I}(\text{asc})} = \mathfrak{G}^{\text{I}} \leq \mathfrak{M}(\text{asc})$.

(2) $\mathfrak{G}^{\text{I}} \cap \acute{\text{E}}\mathfrak{A} = \mathfrak{F} \cap \text{E}\mathfrak{A}$.

PROOF. (1) It suffices to show that $\mathfrak{G}^{\text{I}} \leq \mathfrak{M}(\text{asc})$. Let $L \in \mathfrak{G}^{\text{I}}$ and $H \text{ asc } L$. There exists a strictly ascending series $(H_\alpha)_{\alpha \leq \sigma}$ from H to L. Assume that σ is an infinite ordinal. Then we can find a limit ordinal λ and a finite ordinal n such that $\sigma = \lambda + n$. Since $H_\lambda \lhd^n H_{\lambda+n} = L \in \mathfrak{G}^{\text{I}}$, we have $H_\lambda \in \mathfrak{G}$, so that $H_\lambda = H_\alpha$ for some $\alpha < \lambda$. This is a contradiction. Hence we have $\sigma < \omega$ and $H \text{ si } L$. Therefore we obtain $L \in \mathfrak{M}(\text{asc})$.

(2) Let $L \in \mathfrak{G}^{\text{I}} \cap \acute{\text{E}}\mathfrak{A}$. Then by (1) every ascendant $\mathfrak{A}$-subalgebra of L is finite-dimensional. Using [3, Corollary 9.3.6 (c)] we have $L \in \mathfrak{F} \cap \text{E}\mathfrak{A}$. Hence $\mathfrak{G}^{\text{I}} \cap \acute{\text{E}}\mathfrak{A} \leq \mathfrak{F} \cap \text{E}\mathfrak{A}$. The converse inclusion is trivial.

Let Δ be one of the relations wasc and asc. In [5] the family of subalgebras (resp. subideals) of a Lie algebra L is denoted by $\mathscr{S}_L(\leq)$ (resp. $\mathscr{S}_L(\text{si})$). We similarly denote by $\mathscr{S}_L(\Delta)$ the family of subalgebras H of L such that $H \,\Delta\, L$. Then we define the classes $\mathfrak{L}(\Delta)$, $\mathfrak{L}^\infty(\Delta)$ and $\mathfrak{L}_\infty(\Delta)$ of Lie algebras L as follows:

$L \in \mathfrak{L}(\Delta)$ if and only if $\langle H, K \rangle \in \mathscr{S}_L(\Delta)$ whenever $H, K \in \mathscr{S}_L(\Delta)$;

$L \in \mathfrak{L}^\infty(\Delta)$ if and only if $\langle H_\lambda \colon \lambda \in \Lambda \rangle \in \mathscr{S}_L(\Delta)$ whenever $\{H_\lambda \colon \lambda \in \Lambda\} \subseteq \mathscr{S}_L(\Delta)$;

$L \in \mathfrak{L}_\infty(\Delta)$ if and only if $\bigcap_{\lambda \in \Lambda} H_\lambda \in \mathscr{S}_L(\Delta)$ whenever $\{H_\lambda \colon \lambda \in \Lambda\} \subseteq \mathscr{S}_L(\Delta)$.

In [5] the classes, defined by replacing Δ with si in the above definitions, are denoted by $\mathfrak{L}$, $\mathfrak{L}^\infty$, $\mathfrak{L}_\infty$ respectively.

We need the following

LEMMA 1.2. *If $H \,\Delta\, L$ and $K \,\Delta\, L$, then $H \cap K \,\Delta\, L$ and*

$$\Delta(L \colon H \cap K) \leq \max\{\Delta(L \colon H), \Delta(L \colon K)\}.$$

PROOF. Here we only prove the lemma for the case that Δ is wasc, since for the other case it can be proved similarly. Let $\sigma = \max\{\text{wasc}(L \colon H),$

wasc$(L: K)\}$. Then there are weakly ascending series $(H_\alpha)_{\alpha \leq \sigma}$ from H to L and $(K_\alpha)_{\alpha \leq \sigma}$ from K to L. It is not hard to show that $(H_\alpha \cap K_\alpha)_{\alpha \leq \sigma}$ is a weakly ascending series from $H \cap K$ to L. Therefore we have $H \cap K$ wasc L with wasc$(L: H \cap K) \leq \sigma$.

For any Lie algebra L, we can regard $\mathscr{S}_L(\leq)$ as a lattice by introducing the usual lattice structure in it. Then by Lemma 1.2, $\mathscr{S}_L(\Delta)$ is a sublattice (resp. a complete sublattice) of $\mathscr{S}_L(\leq)$ if and only if $L \in \mathfrak{L}(\Delta)$ (resp. $\mathfrak{L}^\infty(\Delta) \cap \mathfrak{L}_\infty(\Delta)$). So it seems to be interesting to search subclasses of $\mathfrak{L}(\Delta)$ or $\mathfrak{L}^\infty(\Delta) \cap \mathfrak{L}_\infty(\Delta)$. On the other hand, we denote by $\mathscr{S}_L(\mathfrak{G}\text{-}\Delta)$ the family of $\mathfrak{G}$-subalgebras H of a Lie algebra L such that $H \Delta L$. Then we define the class $\mathfrak{L}(\mathfrak{G}\text{-}\Delta)$ of Lie algebras L as follows:

$L \in \mathfrak{L}(\mathfrak{G}\text{-}\Delta)$ if and only if $\langle H, K\rangle \in \mathscr{S}_L(\mathfrak{G}\text{-}\Delta)$ whenever $H, K \in \mathscr{S}_L(\mathfrak{G}\text{-}\Delta)$.

Over a field of characteristic zero, it is not known whether the class $\mathfrak{G}$ is ascendantly coalescent, equivalently $\mathfrak{L}(\mathfrak{G}\text{-asc}) = \mathfrak{D}$. But over a field of characteristic $p > 0$, it is known that the class $\mathfrak{G}$ is not ascendantly coalescent and that $\mathfrak{F} \cap \mathfrak{A}\mathfrak{N}_2 \not\leq \mathfrak{L}(\mathfrak{G}\text{-asc})$ (cf. [3, Lemma 3.1.1]). It also seems to be interesting to search subclasses of $\mathfrak{L}(\mathfrak{G}\text{-}\Delta)$. For this purpose we define the class $\mathfrak{L}^*(\Delta)$ (resp. $\mathfrak{L}^*(\mathfrak{G}\text{-}\Delta)$) of Lie algebras as the largest $\mathrm{I}(\Delta)$-closed subclass of $\mathfrak{L}(\Delta)$ (resp. $\mathfrak{L}(\mathfrak{G}\text{-}\Delta)$) such that for any Lie algebra L, if $H, K \in \mathscr{S}_L(\Delta)$ (resp. $\mathscr{S}_L(\mathfrak{G}\text{-}\Delta)$) and $J = \langle H, K\rangle \in \mathfrak{L}^*(\Delta)$ (resp. $\mathfrak{L}^*(\mathfrak{G}\text{-}\Delta)$), then $J \in \mathscr{S}_L(\Delta)$. Then we have the following result, which supplements [5, Theorem 2.6].

LEMMA 1.3. (1) $\mathfrak{N}\mathfrak{L}^*(\Delta) \leq \mathfrak{L}(\Delta)$.
(2) $\mathfrak{N}\mathfrak{L}^*(\mathfrak{G}\text{-}\Delta) \leq \mathfrak{L}(\mathfrak{G}\text{-}\Delta)$.

PROOF. By replacing wsi with Δ in the proof of [5, Theorem 2.6], we can prove (1). (2) is proved similarly.

As a relationship between $\mathfrak{L}(\Delta)$ (resp. $\mathfrak{L}^*(\Delta)$) and $\mathfrak{L}(\mathfrak{G}\text{-}\Delta)$ (resp. $\mathfrak{L}^*(\mathfrak{G}\text{-}\Delta)$), we have

LEMMA 1.4. (1) $\mathfrak{G}^{\mathrm{I}(\Delta)} \cap \mathfrak{L}(\mathfrak{G}\text{-}\Delta) \leq \mathfrak{L}(\Delta) \leq \mathfrak{L}(\mathfrak{G}\text{-}\Delta)$.
(2) $\mathfrak{G}^{\mathrm{I}(\Delta)} \cap \mathfrak{L}^*(\mathfrak{G}\text{-}\Delta) \leq \mathfrak{L}^*(\Delta) \leq \mathfrak{L}^*(\mathfrak{G}\text{-}\Delta)$.

PROOF. (1) is trivial. $\mathfrak{G}^{\mathrm{I}(\Delta)} \cap \mathfrak{L}^*(\mathfrak{G}\text{-}\Delta)$ is an $\mathrm{I}(\Delta)$-closed subclass of $\mathfrak{L}(\Delta)$ by (1). If $H, K \in \mathscr{S}_L(\Delta)$ and $J = \langle H, K\rangle \in \mathfrak{G}^{\mathrm{I}(\Delta)} \cap \mathfrak{L}^*(\mathfrak{G}\text{-}\Delta)$, then $J \in \mathscr{S}_L(\Delta)$ as $H, K \in \mathfrak{G}$. It follows that $\mathfrak{G}^{\mathrm{I}(\Delta)} \cap \mathfrak{L}^*(\mathfrak{G}\text{-}\Delta) \leq \mathfrak{L}^*(\Delta)$. Clearly we have $\mathfrak{L}^*(\Delta) \leq \mathfrak{L}^*(\mathfrak{G}\text{-}\Delta)$.

We denote by Max-wasc (resp. Min-wasc) the class of Lie algebras satisfying the maximal (resp. minimal) condition for weakly ascendant subalgebras.

We characterize the class $\mathfrak{L}^\infty(\Delta)$ in the following proposition corresponding to [6, Theorem 1].

PROPOSITION 1.5. *Let Δ be one of the relations* wasc *and* asc. *Then we have*

$$\mathfrak{L}(\Delta) \cap (\mathfrak{L}^{\infty}(\Delta)\,\text{Max-}\Delta) = \mathfrak{L}^{\infty}(\Delta).$$

In particular,

$$\mathfrak{N}(\mathfrak{L}^{*}(\Delta) \cap \text{Max-}\Delta) \leq \mathfrak{L}^{\infty}(\Delta).$$

PROOF. As in the proof of [5, Lemma 3.3], we can prove that for an $\mathfrak{L}(\Delta)$-algebra L, $L \in \mathfrak{L}^{\infty}(\Delta)$ if and only if $\mathscr{S}_L(\Delta)$ is closed under the formation of unions of ascending (well-ordered) chains. By using this result and replacing si with Δ in the proof of [6, Theorem 1], we can prove the first half of the proposition. The latter half is immediately deduced from the first half and Lemma 1.3 (1).

A Lie algebra L lies in the class $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ (resp. $\grave{\mathrm{E}}(\lhd)\mathfrak{A}$) if L has an ascending ideal series (resp. a descending ideal series) with abelian factors. [10, Theorem 1] states that if $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ then $\mathscr{S}_L(\text{wasc}) = \mathscr{S}_L(\text{asc})$. Therefore we obtain

LEMMA 1.6. (1) *If $\mathfrak{X}$ is any one of the symbols $\mathfrak{M}_2$ and $\mathfrak{M}$, then*

$$\mathfrak{X}(\text{asc}) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \mathfrak{X}(\text{wasc}).$$

(2) *If $\mathfrak{X}$ is any one of the symbols $\mathfrak{M}_1$, $\mathfrak{L}$, $\mathfrak{L}^{\infty}$ and $\mathfrak{L}_{\infty}$, then*

$$\mathfrak{X}(\text{asc}) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} = \mathfrak{X}(\text{wasc}) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}.$$

2.

Concerning ascendant subalgebras of Lie algebras we know many properties enough to investigate their joins. However, very little are known concerning weakly ascendant subalgebras of Lie algebras. In this section we shall investigate general properties of weakly ascendant subalgebras of Lie algebras.

As in the proof of [2, Lemma 2.1], we can easily show the following

LEMMA 2.1. *Let H* wasc *L, let X be a finite subset of L and let $X_1, X_2, \ldots$ be finite subsets of H. Then there exists an integer $n = n(X, X_1, X_2, \ldots) > 0$ such that $[X, X_1, X_2, \ldots, X_n] \subseteq H$.*

We remark that the statements of [2, Corollaries 2.2 and 2.3] also hold for weakly ascendant subalgebras instead of ascendant subalgebras.

Let A and B be subspaces of a Lie algebra L. Recall that the permutiser $P_A(B)$ of B in A is defined as the largest subspace of A permuting with B (cf. [3, p. 34]). On the other hand, in [2, p. 28] the largest B-invariant subspace of A is denoted by A_B. Particularly if $A \leq L$, then A_L is called the core of A in

L and is usually denoted by $\mathrm{Core}_L(A)$. Furthermore, by [2, Lemma 5.2 (1)] we see that if $A \leq L$ then $A_B \lhd P_A(B) \leq L$. Next we consider the case that A is a subalgebra of L and B is an A-invariant subspace of L containing A. Then we define

$$A_{sB} = \{a \in A \colon [a, B] \subseteq A\}.$$

Especially, in [2, p. 7] A_{sL} is called the semi-core of A with respect to L. It is clear that $A_B \lhd A_{sB} \lhd A$.

[2, Theorem 2.5] states that if $H \operatorname{asc} L$ then $H/H_L \in \mathrm{L}\mathfrak{R}\mathfrak{N} \cap \grave{\mathrm{E}}\mathfrak{A}$, where $\grave{\mathrm{E}}\mathfrak{A}$ is the class of Lie algebras having a descending $\mathfrak{A}$-series. On the other hand, [10, Theorem 4] states that if $H \operatorname{wasc} L$ and $H \in \mathfrak{G}$ then $H \leq^{\omega} L$. We can generalize both statements in the following

THEOREM 2.2. (1) *If* $H \operatorname{wasc} L$, *then* $H/H_L \in \mathrm{L}\mathfrak{R}\mathfrak{N} \cap \grave{\mathrm{E}}(\lhd)\mathfrak{A}$.
(2) *If* $H \operatorname{wasc} L$ *and* $H/H_{sL} \in \mathfrak{G}$, *then* $H \leq^{\omega} L$ *and so* $H^{\omega} \lhd L$.

PROOF. (1) Without loss of generality we may assume that $H_L = 0$. Set $\mathscr{T} = \{F \colon F \leq H \text{ and } F \in \mathfrak{G}\}$ and $T = \bigcup_{F \in \mathscr{T}} F^{\omega}$. Clearly T is a subspace of H. Let $x \in L$ and $F \in \mathscr{T}$. There is a finite subset X of H such that $F = \langle X \rangle$. By Lemma 2.1 we can find an integer $n = n(x, X) > 0$ such that $[x,_n X] \subseteq H$. Set $F(x) = \langle X, [x,_n X] \rangle$. Then we have $F(x) \in \mathscr{T}$ and $[x,_n F] \subseteq F(x)$. It follows that

$$[x, F^{\omega}] \subseteq F(x)^{\omega} \subseteq T.$$

Hence we have $T \lhd L$, so that $T = 0$ as $H_L = 0$. Therefore we obtain $H \in \mathrm{L}\mathfrak{R}\mathfrak{N}$.

Next we shall show that $H \in \grave{\mathrm{E}}(\lhd)\mathfrak{A}$. Let $(M_{\alpha})_{\alpha \leq \sigma}$ be a weakly ascending series from H to L. For each ordinal $\alpha \leq \sigma$, set $H_{\alpha} = H_{sM_{\alpha}}$. Then it is not hard to see that

$$H_0 = H \quad \text{and} \quad H_{\sigma} = H_{sL},$$

$$H_{\alpha+1} \leq H_{\alpha} \lhd H \quad \text{for any ordinal} \quad \alpha < \sigma,$$

$$H_{\lambda} = \bigcap_{\alpha < \lambda} H_{\alpha} \quad \text{for any limit ordinal} \quad \lambda \leq \sigma.$$

For any ordinal $\alpha < \sigma$,

$$[H_{\alpha}^2, M_{\alpha+1}] \subseteq [M_{\alpha+1},_2 H_{\alpha}] \subseteq [M_{\alpha}, H_{\alpha}] \subseteq H$$

and so $H_{\alpha}^2 \leq H_{\alpha+1}$. Hence we have

$$H_{\alpha}/H_{\alpha+1} \in \mathfrak{A} \quad \text{for any ordinal} \quad \alpha < \sigma.$$

For each non-zero ordinal $\beta \leq \omega$, set $H_{\sigma+\beta} = (H_{sL})^{\beta} \lhd H$. Then

$$[L, H_{\sigma+\omega}] \subseteq \bigcap_{m\geq 0} [L, (H_{sL})^{m+2}] \subseteq \bigcap_{m\geq 0} [H,_{m+1} H_{sL}] \subseteq (H_{sL})^{\omega} = H_{\sigma+\omega}.$$

Hence we have $H_{\sigma+\omega} \leq H_L$, so that $H_{\sigma+\omega}=0$. Therefore $(H_\alpha)_{\alpha\leq\sigma+\omega}$ is a descending ideal series of H with abelian factors. Thus we obtain $H \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$.

(2) Since $H/H_{sL} \in \mathfrak{G}$, we can find a $\mathfrak{G}$-subalgebra F of H such that $H=F+H_{sL}$. Let $x \in L$. Then by the proof of (1), there exists an integer $n=n(x, F)>0$ such that $[x,_n F] \subseteq H$. It follows that

$$[x,_n H] \subseteq [x,_n F] + H = H.$$

Hence we have $H \leq^{\omega} L$. Using [4, Lemma 2.10] we obtain $H^{\omega} \lhd L$.

Let H wasc L and assume that $H/H_{sL} \in \mathfrak{G}$. Then we can easily see that $H/H_{sL} \in \mathrm{R}\mathfrak{N}$ and $H/H_L \in \mathrm{R}\mathfrak{N}$.

Recall that the set of left Engel elements of a Lie algebra L is denoted by $\mathfrak{e}(L)$.

COROLLARY 2.3. *Let L be a locally finite, non-abelian simple Lie algebra. Then we have*

$$\langle H \colon H \text{ wasc } L \text{ and } H \neq L \rangle = \langle \mathfrak{e}(L) \rangle.$$

PROOF. Let H be a proper weakly ascendant subalgebra of L. Then by Theorem 2.2 (1) we have $H \in \mathrm{LR}\mathfrak{N}$. Let $x \in L$ and $y \in H$. By Lemma 2.1 there exists an integer $n=n(x, y)>0$ such that $[x,_n y] \in H$. Hence we have $\langle [x,_n y], y \rangle \in \mathfrak{F} \cap \mathrm{R}\mathfrak{N} = \mathfrak{F} \cap \mathfrak{N}$. Therefore $[x,_{n+m} y]=0$ for some integer $m>0$. It follows that $y \in \mathfrak{e}(L)$. Hence $H \subseteq \mathfrak{e}(L)$ and therefore

$$\langle H \colon H \text{ wasc } L \text{ and } H \neq L \rangle \leq \langle \mathfrak{e}(L) \rangle.$$

The converse inclusion is clear from [10, Lemma 5].

Next we consider under what conditions joins of pairs of weakly ascendant subalgebras are weakly ascendant. A key lemma for this purpose is the following

LEMMA 2.4. *Let $H \leq^{\rho} L$, $K \leq^{\sigma} L$ and $J=\langle H, K \rangle$. Assume that there exists a weakly ascending series $(H_\alpha)_{\alpha\leq\rho}$ from H to L such that H_α is K-invariant for any $\alpha \leq \rho$. Then we have $J \leq^{\sigma\rho} L$.*

PROOF. Let $(K_\beta)_{\beta\leq\sigma}$ be a weakly ascending series from K to L. For each pair (β, α) of ordinals $\beta \leq \sigma$ and $\alpha \leq \rho$, we define the subspace $J_{(\beta,\alpha)}$ of L by

$$J_{(\beta,\alpha)} = H_\alpha + (H_{\alpha+1} \cap K_\beta) + K,$$

where we put $H_{\rho+1}=L$. Then it is easy to see that

$$J_{(0,0)} = J \quad \text{and} \quad J_{(0,\rho)} = L,$$

$$J_{(0,\alpha)} \subseteq J_{(0,\alpha+1)} = J_{(\sigma,\alpha)} \quad \text{for any ordinal} \quad \alpha < \rho,$$

$$J_{(0,\lambda)} = \cup_{\alpha<\lambda} J_{(0,\alpha)} \quad \text{for any limit ordinal} \quad \lambda \leq \rho.$$

Let $\alpha<\rho$. Then for any $\beta<\sigma$, $J_{(\beta,\alpha)} \subseteq J_{(\beta+1,\alpha)}$ and

$$\begin{aligned}[J_{(\beta+1,\alpha)}, J] &= [H_\alpha+(H_{\alpha+1} \cap K_{\beta+1})+K, H+K] \\ &\subseteq H_\alpha + [H_{\alpha+1} \cap K_{\beta+1}, K] + K \\ &\subseteq H_\alpha + (H_{\alpha+1} \cap K_\beta) + K = J_{(\beta,\alpha)}.\end{aligned}$$

Moreover, for any limit ordinal $\mu \leq \sigma$, $J_{(\mu,\alpha)} = \cup_{\beta<\mu} J_{(\beta,\alpha)}$. Hence $(J_{(\beta,\alpha)})$ is a weakly ascending series from J to L. Therefore we obtain $J \leq^{\sigma\rho} L$.

Now we can prove the second main theorem of this section, which is useful to search subclasses of $\mathfrak{L}$(wasc).

THEOREM 2.5. *Let $H \leq^\rho L$, $K \leq^\sigma L$ and $J=\langle H, K\rangle$. If $[H, K] \subseteq H$, then $J \leq^{\sigma\rho} L$.*

PROOF. Let $(H_\alpha)_{\alpha\leq\rho}$ be a weakly ascending series from H to L. Then $(H_\alpha^K)_{\alpha\leq\rho}$ is an ascending chain of subspaces of L. Evidently we have

$$H_0^K = H \quad \text{and} \quad H_\rho^K = L.$$

Let $\alpha<\rho$. By induction on n we show that

$$[[H_{\alpha+1},{}_n K], H] \subseteq H_\alpha^K \quad (n \geq 0).$$

It is trivial for $n=0$. Let $n>0$ and suppose that the result is true for $n-1$. By Jacobi identity, since $[H, K] \subseteq H$, we have

$$\begin{aligned}[[H_{\alpha+1},{}_n K], H] &\subseteq [[H_{\alpha+1},{}_{n-1} K], H, K] + [[H_{\alpha+1},{}_{n-1} K], H] \\ &\subseteq [H_\alpha^K, K] + H_\alpha^K = H_\alpha^K.\end{aligned}$$

Therefore we have

$$[H_{\alpha+1}^K, H] = \sum_{n\geq 0} [[H_{\alpha+1},{}_n K], H] \subseteq H_\alpha^K.$$

On the other hand, we can easily see that for any limit ordinal $\lambda \leq \rho$

$$H_\lambda^K = \cup_{\alpha<\lambda} H_\alpha^K.$$

Hence $(H_\alpha^K)_{\alpha\leq\rho}$ is a weakly ascending series from H to L all of whose terms are K-invariant. By Lemma 2.4 we obtain $J \leq^{\sigma\rho} L$.

It is not known whether the statement of Theorem 2.5 holds for ascendant subalgebras instead of weakly ascendant subalgebras. However, Amayo [2] shows that if $H \operatorname{asc} L$, $K \operatorname{asc} L$ and $K/H_K \cap K \in \mathfrak{G}$ then $H_K+K \operatorname{asc} L$. We shall show that the analogous result holds for weakly ascendant subalgebras.

Let H and K be weakly ascendant subalgebras of a Lie algebra L, and let $(H_\alpha)_{\alpha\leq\rho}$ be a weakly ascending series from H to L. For each ordinal $\alpha\leq\rho$, set $N_\alpha=(H_\alpha)_K$ and $P_\alpha=P_{H_\alpha}(K)$. Then we have

LEMMA 2.6. (1) *For any ordinal* $\alpha<\rho$,

$$[P_{\alpha+1}, H_K] \subseteq N_\alpha \subseteq P_\alpha.$$

(2) *If* $K/H_K \cap K \in \mathfrak{G}$, *then for any limit ordinal* $\lambda\leq\rho$

$$P_\lambda = \cup_{\alpha<\lambda} P_\alpha.$$

PROOF. (1) Let $\alpha<\rho$. Then we have

$$[P_{\alpha+1}, H_K] \subseteq [H_{\alpha+1}, H] \subseteq H_\alpha.$$

We use Jacobi identity to see that

$$[[P_{\alpha+1}, H_K], K] \subseteq [P_{\alpha+1}+K, H_K] + [P_{\alpha+1}, H_K] \subseteq [P_{\alpha+1}, H_K] + H_K.$$

It follows that $[P_{\alpha+1}, H_K]+H_K$ is a K-invariant subspace of H_α. Therefore we have $[P_{\alpha+1}, H_K]\subseteq N_\alpha$. It is clear that $N_\alpha\subseteq P_\alpha$.

(2) Suppose that $K/H_K \cap K\in\mathfrak{G}$. Then there exists a finite subset Y of K such that $K=\langle Y\rangle+M$ where $M=H_K\cap K$. Let λ be any limit ordinal $\leq\rho$, and let $x\in P_\lambda$. By Lemma 2.1 we can find an integer $n=n(x, Y)>0$ such that $[x,_n Y]\subseteq K$. By simple induction on i we have $[x,_i Y]\subseteq P_\lambda+K$ $(i\geq 0)$. Since $[x,_i Y]$ is finite-dimensional, there are finitely many elements $x_{ij}+y_{ij}$ $(1\leq j\leq m_i)$ spanning $[x,_i Y]$, where $x_{ij}\in P_\lambda$ and $y_{ij}\in K$ $(0\leq i<n)$. Here we may assume that $m_0=1$, $x_{01}=x$ and $y_{01}=0$. Let F be the subspace of L spanned by $\{x_{ij}$: $0\leq i<n$, $1\leq j\leq m_i\}$ and G be the one spanned by $[x,_n Y]$ and $\{y_{ij}$: $0\leq i<n$, $1\leq j\leq m_i\}$. Then F and G are finite-dimensional subspaces of P_λ and K respectively. Moreover, we have

$$x\in F \quad\text{and}\quad x^Y \subseteq F + G^Y.$$

Since $x_{ij}\in[x,_i Y]+G$, we have

$$[x_{ij}, Y] \subseteq x^Y + G^Y \subseteq F + G^Y.$$

It follows that $[F, Y]\subseteq F+G^Y$. Hence we have

$$F^Y \subseteq F + G^Y \subseteq F + K.$$

By using [3, Lemma 2.2.3], we have

$$[F^M, K] \subseteq F^K = (F^Y)^M \subseteq (F+K)^M \subseteq F^M + K,$$

that is, F^M permutes with K. Since F is a finite-dimensional subspace of H_λ, there exists an ordinal $\alpha<\lambda$ such that $F\subseteq H_\alpha$. Then we have $F^M\subseteq H_\alpha$, so that $x\in F^M\subseteq P_\alpha$. Hence $P_\lambda\subseteq\bigcup_{\alpha<\lambda}P_\alpha$. The converse inclusion is clear, and $P_\lambda=\bigcup_{\alpha<\lambda}P_\alpha$.

The following result corresponds to [2, Lemma 6.5].

PROPOSITION 2.7. *Let $H\leq^\rho L$ and $K\leq^\sigma L$, and assume that $K/H_K\cap K\in\mathfrak{G}$. Then we have $H_K\leq^{\rho+1}L$ and $H_K+K\leq^{\sigma(\rho+1)}L$.*

PROOF. Let $(H_\alpha)_{\alpha\leq\rho}$ be a weakly ascending series from H to L, and let $N_\alpha=(H_\alpha)_K$ and $P_\alpha=P_{H_\alpha}(K)$ $(\alpha\leq\rho)$. First we shall construct a weakly ascending series $(M_\alpha)_{\alpha\leq\rho+1}$ from H_K to L such that $[M_\alpha, K]\subseteq M_\alpha$ for all $\alpha\leq\rho+1$. We define the terms M_α as follows: For each non-limit ordinal $\alpha\leq\rho+1$,

$$M_\alpha=\begin{cases} N_\alpha & \text{if } \rho+1\neq\alpha<\omega \\ N_\rho & \text{if } \alpha=\rho+1<\omega \\ N_{\alpha-1} & \text{if } \alpha>\omega; \end{cases}$$

and for each limit ordinal $\alpha\leq\rho+1$,

$$M_\alpha=\bigcup_{\beta<\alpha}N_\beta.$$

Then we have $[M_\alpha, K]\subseteq M_\alpha$ for all $\alpha\leq\rho+1$. It is clear that $M_0=H_K$ and $M_{\rho+1}=L$. Let $\alpha<\rho+1$. Obviously $M_\alpha\subseteq M_{\alpha+1}$. If α is not a limit ordinal, then $[M_{\alpha+1}, H_K]\subseteq M_\alpha$ by Lemma 2.6 (1). Suppose that α is a limit ordinal. Then by using Lemma 2.6 we have

$$[M_{\alpha+1}, H_K]\subseteq[P_\alpha, H_K]=\bigcup_{\beta<\alpha}[P_\beta, H_K]\subseteq\bigcup_{\beta<\alpha}N_\beta=M_\alpha.$$

Furthermore, we can easily see that $M_\alpha=\bigcup_{\beta<\alpha}N_\beta=\bigcup_{\beta<\alpha}M_\beta$. Therefore $(M_\alpha)_{\alpha\leq\rho+1}$ is a weakly ascending series from H_K to L all of whose terms are K-invariant. Thus we have $H_K\leq^{\rho+1}L$, so that $H_K+K\leq^{\sigma(\rho+1)}L$ by Lemma 2.4.

3.

Throughout this section we always denote by Δ any one of the relations wasc and asc. The purpose of this section is to search subclasses of $\mathfrak{L}(\mathfrak{G}\text{-}\Delta)$, $\mathfrak{L}(\Delta)$, $\mathfrak{L}^\infty(\Delta)$ or $\mathfrak{L}^\infty(\Delta)\cap\mathfrak{L}_\infty(\Delta)$. We begin by showing two propositions which present subclasses of $\mathfrak{L}^*(\mathfrak{G}\text{-}\Delta)$ or $\mathfrak{L}^*(\Delta)$.

PROPOSITION 3.1. (1) $\mathfrak{M}_1(\text{wasc})\leq\mathfrak{L}^*(\text{wasc})$.
(2) $\mathfrak{M}_1(\text{asc})\leq\mathfrak{L}^*(\mathfrak{G}\text{-asc})$ *and* $\mathfrak{G}^I\cap\mathfrak{M}_1(\text{asc})\leq\mathfrak{L}^*(\text{asc})$.

PROOF. Clearly we have $\mathrm{I}(\Delta)\mathfrak{M}_1(\Delta)=\mathfrak{M}_1(\Delta)\leq\mathfrak{L}(\Delta)$. Let H, $K\in\mathscr{S}_L(\Delta)$

and assume that $J=\langle H, K\rangle \in \mathfrak{M}_1(\Delta)$. Then $H \lhd J$ as $H \,\Delta\, J$. Hence H is K-invariant. In case that Δ is wasc, by Theorem 2.5 we have J wasc L. In case that Δ is asc, if $H, K \in \mathfrak{G}$ then by [2, Lemma 6.5] we have J asc L. It follows that

$$\mathfrak{M}_1(\text{wasc}) \leq \mathfrak{L}^*(\text{wasc}) \quad \text{and} \quad \mathfrak{M}_1(\text{asc}) \leq \mathfrak{L}^*(\mathfrak{G}\text{-asc}).$$

By Lemmas 1.1 (1) and 1.4 (2) we have

$$\mathfrak{G}^{\mathrm{I}} \cap \mathfrak{M}_1(\text{asc}) \leq \mathfrak{L}^*(\text{asc}).$$

In [5] we considered the classes $\mathfrak{A}_0$ and $\mathfrak{A}_1$ of metabelian Lie algebras, which are defined as follows: For a Lie algebra L, $L \in \mathfrak{A}_1$ if and only if either $L \in \mathfrak{A}$, or $L \in \mathfrak{A}^2$ with $\dim(L/L^2)=1$; $L \in \mathfrak{A}_0$ if and only if either $L \in \mathfrak{A}$, or $L \in \mathfrak{A}_1 \backslash \mathfrak{A}$ with L/L^2 acting on L^2 as scalar multiplications. By using [5, Proposition 2.8], it is easy to show that $\mathfrak{M}_1(\Delta) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} = \mathfrak{A}_0$.

PROPOSITION 3.2. (1) $\mathfrak{M}(\text{wasc}) \cap \mathfrak{A}_1 \leq \mathfrak{M}_2(\text{wasc}) \cap \mathfrak{L}^*(\text{wasc})$.
(2) $\mathfrak{M}(\text{asc}) \cap \mathfrak{A}_1 \leq \mathfrak{M}_2(\text{asc}) \cap \mathfrak{L}^*(\mathfrak{G}\text{-asc})$ *and* $\mathfrak{F} \cap \mathfrak{A}_1 \leq \mathfrak{L}^*(\text{asc})$.

PROOF. Using [5, Lemma 2.9] we see that for any $J \in (\mathfrak{M}(\Delta) \cap \mathfrak{A}_1) \backslash \mathfrak{A}$

$$\mathscr{S}_J(\Delta) = \{H\colon H \leq J^2 \text{ or } H=J\} = \{H\colon H \lhd^2 J\}. \tag{$*$}$$

Hence it is not hard to see that

$$\mathrm{I}(\Delta)(\mathfrak{M}(\Delta) \cap \mathfrak{A}_1) = \mathfrak{M}(\Delta) \cap \mathfrak{A}_1 \leq \mathfrak{M}_2(\Delta) \cap \mathfrak{L}(\Delta).$$

Let $H, K \in \mathscr{S}_L(\Delta)$ and assume that $J=\langle H, K\rangle \in \mathfrak{M}(\Delta) \cap \mathfrak{A}_1$, $H<J$ and $K<J$. If $J \notin \mathfrak{A}$ then by $(*)$ $J=\langle H, K\rangle \leq J^2 \in \mathfrak{A}$, a contradiction. Therefore we have $J \in \mathfrak{A}$, so that $[H, K]=0$. In case that Δ is wasc, by Theorem 2.5 we have J wasc L. In case that Δ is asc, if $H, K \in \mathfrak{G}$ then by [2, Lemma 6.5] we have J asc L. Thus we obtain

$$\mathfrak{M}(\text{wasc}) \cap \mathfrak{A}_1 \leq \mathfrak{L}^*(\text{wasc}) \quad \text{and} \quad \mathfrak{M}(\text{asc}) \cap \mathfrak{A}_1 \leq \mathfrak{L}^*(\mathfrak{G}\text{-asc}).$$

It then follows from Lemma 1.4 (2) that

$$\mathfrak{G}^{\mathrm{I(asc)}} \cap \mathfrak{M}(\text{asc}) \cap \mathfrak{A}_1 \leq \mathfrak{L}^*(\text{asc}).$$

By using Lemma 1.1, we can immediately show that

$$\mathfrak{F} \cap \mathfrak{A}_1 = \mathfrak{G}^{\mathrm{I(asc)}} \cap \mathfrak{M}(\text{asc}) \cap \mathfrak{A}_1.$$

We should note that $\mathfrak{F} \cap \mathfrak{A}_1 < \mathfrak{M}(\Delta) \cap \mathfrak{A}_1$.

For subalgebras H and K of a Lie algebra L, the circle product $H \circ K$ of H and K is defined as $H \circ K = [H, K]^{H \cup K}$. We have the following two lemmas corresponding to [5, Lemma 2.4].

LEMMA 3.3. *Let* H wasc L, K wasc L *and* $J=\langle H, K\rangle$. *Then the following conditions are equivalent*:

(1) J wasc L, (2) $\langle H^K\rangle$ wasc L, (3) $H\circ K$ wasc L.

PROOF. Since $H\circ K \lhd \langle H^K\rangle \lhd J$, it suffices to show that (3) implies (1). Suppose that $H\circ K$ wasc L. By using Theorem 2.5 we have $\langle H^K\rangle = H\circ K + H$ wasc L, so that $J=\langle H^K\rangle + K$ wasc L.

LEMMA 3.4. *Let* H asc L, K asc L *and* $J=\langle H, K\rangle$.

(1) *Assume that* $K\in\mathfrak{G}$. *Then* J asc L *if and only if* $\langle H^K\rangle$ asc L.

(2) *Assume that* $H, K\in\mathfrak{G}$. *Then* J asc L *if and only if* $H\circ K$ asc L.

PROOF. (1) The 'only if' part is trivial as $\langle H^K\rangle \lhd J$. If $\langle H^K\rangle$ asc L, then by [2, Lemma 6.5] we have

$$J = \langle H^K\rangle + K = \langle H^K\rangle_K + K \text{ asc } L.$$

(2) The 'only if' part is trivial as $H\circ K \lhd J$. If $H\circ K$ asc L, then by [2, Lemma 6.5] we have

$$\langle H^K\rangle = H\circ K + H = (H\circ K)_H + H \text{ asc } L.$$

It follows from (1) that J asc L.

Now we denote by $\mathfrak{D}(\Delta)$ (resp. $\mathfrak{D}(\mathfrak{G}\text{-}\Delta)$) the class of Lie algebras L such that $H\,\Delta\,L$ whenever $H\leq L$ (resp. $H\in\mathfrak{G}$ and $H\leq L$). We should remark that the class $\mathfrak{D}(\Delta)$ coincides with the class of Lie algebras satisfying the idealiser condition, that is, that $L\in\mathfrak{D}(\Delta)$ if and only if $H<L$ implies $H<I_L(H)$. However, for convenience' sake we use the notations $\mathfrak{D}$(wasc) and $\mathfrak{D}$(asc) separately. On the other hand, owing to [2, Theorem 4.6 and Corollary 4.7] we have $\mathfrak{D}(\mathfrak{G}\text{-asc}) = \mathfrak{Gr} \leq \mathrm{L}\mathfrak{N}$, where $\mathfrak{Gr}$ is the class of Gruenberg Lie algebras, that is, Lie algebras L such that $\langle x\rangle$ asc L for any $x\in L$. Therefore we see that

$$\begin{aligned}\mathfrak{Z} &\leq \mathfrak{D}(\text{asc}) = \mathfrak{D}(\text{wasc})\\ &\leq \mathfrak{D}(\mathfrak{G}\text{-asc}) = \mathfrak{Gr}\\ &\leq \mathrm{L}\mathfrak{N} \leq \mathfrak{D}(\mathfrak{G}\text{-wasc}) \leq \mathfrak{E},\end{aligned}$$

where $\mathfrak{Z}$ is the class of hypercentral Lie algebras and $\mathfrak{E}$ is the class of Engel Lie algebras.

PROPOSITION 3.5. (1) $\mathfrak{D}(\text{wasc})\mathfrak{A} \leq \mathfrak{L}(\text{wasc})$.

(2) $\mathfrak{D}(\text{asc})\mathfrak{A} \leq \mathfrak{L}(\mathfrak{G}\text{-asc})$ *and* $\mathfrak{D}(\text{asc})\mathfrak{A} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \mathfrak{L}(\text{asc})$.

PROOF. Let $L\in\mathfrak{D}(\Delta)\mathfrak{A}$ and $H, K\in\mathscr{S}_L(\Delta)$. Then we have $H\circ K\,\Delta\,L^2$ since

$H \circ K \leq L^2 \in \mathfrak{D}(\Delta)$. It follows that $H \circ K \, \Delta \, L$. In case that Δ is wasc, by Lemma 3·3 we have $\langle H, K \rangle$ wasc L. In case that Δ is asc, if $H, K \in \mathfrak{G}$ then by Lemma 3.4 we have $\langle H, K \rangle$ asc L. Therefore we obtain

$$\mathfrak{D}(\Delta)\mathfrak{A} \leq \mathfrak{L}(\text{wasc}) \cap \mathfrak{L}(\mathfrak{G}\text{-asc}).$$

It then follows from Lemma 1.6 (2) that $\mathfrak{D}(\Delta)\mathfrak{A} \cap \acute{E}(\lhd)\mathfrak{A} \leq \mathfrak{L}(\text{asc})$.

We here define the class $\mathfrak{S}(\Delta)$ of Lie algebras L as follows:

$L \in \mathfrak{S}(\Delta)$ if and only if $H \, \Delta \, L$ implies that $H=0$ or $H=L$.

Then the class $\mathfrak{S}(\Delta)$ is a proper subclass of $\mathfrak{M}_1(\Delta)$. Owing to [7] (or [2, Theorem 3.8]), we see that the class $\mathfrak{S}(\text{asc})$ coincides with the class of simple Lie algebras, denoted by $\mathfrak{S}$ in [5]. By Corollary 2.3 a locally finite, non-abelian simple Lie algebra L lies in $\mathfrak{S}(\text{wasc})$ if and only if $\mathfrak{e}(L)=0$. Hence we have

$$\mathfrak{S}(\text{wasc}) < \mathfrak{S}(\text{asc}) = \mathfrak{S}.$$

The class $\mathfrak{S}(\text{wasc})$ is not so large, but is non-trivial. In fact, let $\mathfrak{k}$ be a formal real field and let S be a 3-dimensional simple Lie algebra over $\mathfrak{k}$ with basis $\{x, y, z\}$ such that

$$[x, y] = z, \quad [y, z] = x, \quad [z, x] = y.$$

Then we can prove that $S \in \mathfrak{S}(\text{wasc})$ and so $\mathfrak{e}(S)=0$ (cf. [5, Example 4.3]).

THEOREM 3.6. (1) $\mathfrak{D}(\text{wasc})\mathfrak{S}(\text{wasc}) \leq \mathfrak{L}^{\infty}(\text{wasc})$.

(2) $\mathfrak{D}(\text{asc})\mathfrak{S} \leq \mathfrak{L}(\mathfrak{G}\text{-asc})$ *and* $\mathfrak{D}(\text{asc})(\mathfrak{G} \cap \mathfrak{S}) \leq \mathfrak{L}^{\infty}(\text{asc})$.

PROOF. Let $L \in \mathfrak{D}(\Delta)\mathfrak{S}(\Delta)$. Then there exists a $\mathfrak{D}(\Delta)$-ideal I of L such that $L/I \in \mathfrak{S}(\Delta)$. Let $H, K \in \mathscr{S}_L(\Delta)$ and $J=\langle H, K \rangle$. If Δ is asc, then we assume that $H, K \in \mathfrak{G}$. First we shall show that $J \, \Delta \, L$. Since $(H+I)/I \, \Delta \, L/I \in \mathfrak{S}(\Delta)$, we have $H+I=L$ or $H \leq I$. Similarly $K+I=L$ or $K \leq I$. If $H+I=L$, then by modular law

$$J = J \cap (H+I) = H + (J \cap I).$$

Since $I \in \mathfrak{D}(\Delta)$, we have $\langle (J \cap I)^H \rangle = J \cap I \, \Delta \, L$. It then follows from Lemma 3.3 or 3.4 that $J \, \Delta \, L$. The case $K+I=L$ is similar. Finally if $H \leq I$ and $K \leq I$, then $J \leq I \in \mathfrak{D}(\Delta)$ and so $J \, \Delta \, L$. Therefore we have

$$\mathfrak{D}(\text{wasc})\mathfrak{S}(\text{wasc}) \leq \mathfrak{L}(\text{wasc})$$

and

$$\mathfrak{D}(\text{asc})\mathfrak{S} = \mathfrak{D}(\text{asc})\mathfrak{S}(\text{asc}) \leq \mathfrak{L}(\mathfrak{G}\text{-asc}).$$

Since $\mathfrak{D}(\text{wasc}) \leq \mathfrak{L}^{\infty}(\text{wasc})$ and $\mathfrak{S}(\text{wasc}) \leq \text{Max-wasc}$, by Proposition 1.5 we have

$$\mathfrak{D}(\text{wasc})\mathfrak{S}(\text{wasc}) \leq \mathfrak{L}^{\infty}(\text{wasc}).$$

Next we prove that $\mathfrak{D}(\mathrm{asc})(\mathfrak{G} \cap \mathfrak{S}) \leq \mathfrak{L}^{\infty}(\mathrm{asc})$. Let $L \in \mathfrak{D}(\mathrm{asc})(\mathfrak{G} \cap \mathfrak{S})$. Then there exists a $\mathfrak{D}(\mathrm{asc})$-ideal I of L such that $L/I \in \mathfrak{G} \cap \mathfrak{S}$. Let $H, K \in \mathscr{S}_L(\mathrm{asc})$ and $J = \langle H, K \rangle$. We shall show that $J \operatorname{asc} L$. Clearly $H+I=L$ or $H \leq I$, and $K+I=L$ or $K \leq I$. If $H+I=L$, then by the above argument we have $J=H+(J \cap I)$ and $J \cap I \operatorname{asc} L$. Moreover,

$$H/(J \cap I)_H \cap H = H/H \cap I \cong L/I \in \mathfrak{G}.$$

Using [2, Lemma 6.5] we have $J \operatorname{asc} L$. The case $K+I=L$ is similar. Finally if $H \leq I$ and $K \leq I$, then $J \leq I \in \mathfrak{D}(\mathrm{asc})$ and so $J \operatorname{asc} L$. Therefore we have $L \in \mathfrak{L}(\mathrm{asc})$. Since $\mathfrak{S} = \mathfrak{S}(\mathrm{asc}) \leq \text{Max-asc}$, by Proposition 1.5 we have $L \in \mathfrak{L}^{\infty}(\mathrm{asc})$. Thus we obtain

$$\mathfrak{D}(\mathrm{asc})(\mathfrak{G} \cap \mathfrak{S}) \leq \mathfrak{L}^{\infty}(\mathrm{asc}).$$

As a consequence of Theorem 3.6 we see that the Lie algebra constructed in [5, Example 4.3] is an $\mathfrak{L}^{\infty}(\Delta)$-algebra. We can present the other type of $\mathfrak{L}^{\infty}(\Delta)$-algebras in the following

THEOREM 3.7. (1) $\mathfrak{D}(\mathrm{wasc})(\mathfrak{F} \cap \mathfrak{A}_1) \leq \mathfrak{L}^{\infty}(\mathrm{wasc})$.
(2) $\mathfrak{D}(\mathrm{asc})(\mathfrak{F} \cap \mathfrak{A}_1) \leq \mathfrak{L}(\mathfrak{G}\text{-asc})$ *and* $\mathfrak{D}(\mathrm{asc})(\mathfrak{F} \cap \mathfrak{A}_1) \cap \acute{E}(\lhd)\mathfrak{A} \leq \mathfrak{L}^{\infty}(\mathrm{asc})$.

PROOF. Let $L \in \mathfrak{D}(\Delta)(\mathfrak{F} \cap \mathfrak{A}_1)$. Then there exists a $\mathfrak{D}(\Delta)$-ideal I of L such that $L/I \in \mathfrak{F} \cap \mathfrak{A}_1$. Let $H, K \in \mathscr{S}_L(\Delta)$ and $J = \langle H, K \rangle$. If Δ is asc, then we assume that $H, K \in \mathfrak{G}$. We shall show that $J \Delta L$. If $L/I \in \mathfrak{A}$, then it is trivial from Proposition 3.5. Hence we suppose that $L/I \in \mathfrak{A}_1 \backslash \mathfrak{A}$. It is clear that

$$(H+I)/I \,\Delta\, L/I \quad \text{with} \quad \Delta(L/I : (H+I)/I) < \omega.$$

By using [5, Lemma 2.9], we have

$$(H+I)/I \leq (L^2+I)/I \quad \text{or} \quad H+I=L.$$

If $H+I=L$, then it is easy to see that

$$J = H + (J \cap I) \quad \text{and} \quad \langle (J \cap I)^H \rangle = J \cap I \,\Delta\, L.$$

Therefore by Lemma 3.3 or 3.4 we have $J \Delta L$. If $(H+I)/I \leq (L^2+I)/I$ and $(K+I)/I \leq (L^2+I)/I$, then we have

$$(J+I)/I \leq (L^2+I)/I \in \mathfrak{A}.$$

This implies that $H \circ K \leq J^2 \leq I$. Hence we have $H \circ K \,\Delta\, L$, so that $J \Delta L$ by Lemma 3.3 or 3.4. Therefore we obtain

$$\mathfrak{D}(\Delta)(\mathfrak{F} \cap \mathfrak{A}_1) \leq \mathfrak{L}(\mathrm{wasc}) \cap \mathfrak{L}(\mathfrak{G}\text{-asc}).$$

It now follows from Proposition 1.5 that

$$\mathfrak{D}(\mathrm{wasc})(\mathfrak{F} \cap \mathfrak{A}_1) \leq \mathfrak{L}^{\infty}(\mathrm{wasc}).$$

By Lemma 1.6 (2) we have

$$\mathfrak{D}(\mathrm{asc})(\mathfrak{F} \cap \mathfrak{A}_1) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \mathfrak{L}^{\infty}(\mathrm{asc}).$$

By the proof of Theorem 5.1 (3) below, we have $\mathfrak{A}_1 \not\leq \mathfrak{L}_{\infty}(\Delta)$, so that $\mathfrak{A}\mathfrak{F}_1 \not\leq \mathfrak{L}_{\infty}(\Delta)$. Therefore it seems that the class $\mathfrak{L}_{\infty}(\Delta)$ is not large enough to present many interesting subclasses of $\mathfrak{L}^{\infty}(\Delta) \cap \mathfrak{L}_{\infty}(\Delta)$. However, we can prove

LEMMA 3.8. Min-$\Delta \leq \mathfrak{L}_{\infty}(\Delta)$.

PROOF. Let $L \in$ Min-Δ and $\{H_\lambda : \lambda \in \Lambda\} \subseteq \mathscr{S}_L(\Delta)$. Set $J = \bigcap_{\lambda \in \Lambda} H_\lambda$. We must show that $J \in \mathscr{S}_L(\Delta)$. Assume, to the contrary, that $J \notin \mathscr{S}_L(\Delta)$. Let the elements of Λ be well ordered, that is, $\Lambda = \{\alpha : \alpha < \rho\}$ for some ordinal ρ. Then we construct the descending chain $(J_\alpha)_{\alpha \leq \rho}$ of subalgebras of L as follows:

$$J_0 = L, \quad J_\alpha = \bigcap_{\beta < \alpha} H_\beta \quad (0 < \alpha \leq \rho).$$

Since $J_\rho = J \notin \mathscr{S}_L(\Delta)$, there exists the least ordinal $\mu \leq \rho$ such that $J_\mu \notin \mathscr{S}_L(\Delta)$. Evidently $\mu > 0$. If μ is not a limit ordinal, then by Lemma 1.2

$$J_\mu = J_{\mu-1} \cap H_{\mu-1} \in \mathscr{S}_L(\Delta),$$

a contradiction. Hence μ is a limit ordinal. It follows that $J_\mu = \bigcap_{\alpha < \mu} J_\alpha$. Since $L \in$ Min-Δ and $\{J_\alpha : \alpha < \mu\} \subseteq \mathscr{S}_L(\Delta)$, we can find an ordinal $\lambda < \mu$ such that $J_\lambda \leq J_\alpha$ for all $\alpha < \mu$. Then we have $J_\mu = J_\lambda \in \mathscr{S}_L(\Delta)$. This is also a contradiction. Therefore we have $J \in \mathscr{S}_L(\Delta)$, so that $L \in \mathfrak{L}_{\infty}(\Delta)$.

We now set about showing the main theorems of this section.

THEOREM 3.9. (1) *The following are subclasses of* $\mathfrak{L}(\mathfrak{G}\text{-wasc})$:

$$\mathrm{L}\mathfrak{N}, \quad \mathfrak{D}(\mathfrak{G}\text{-wasc}).$$

(2) *The following are subclasses of* $\mathfrak{L}(\mathrm{wasc})$:

$$\mathfrak{Z}\mathfrak{A}, \quad \mathfrak{D}(\mathrm{wasc})\mathfrak{A}, \quad \mathfrak{N}\mathfrak{A}_0, \quad \mathfrak{N}(\mathfrak{M}(\mathrm{wasc}) \cap \mathfrak{A}_1), \quad \mathfrak{N}\mathfrak{M}_1(\mathrm{wasc}).$$

(3) *The following are subclasses of* $\mathfrak{L}^{\infty}(\mathrm{wasc})$:

$$\mathfrak{A}_1, \quad \mathfrak{Z}(\mathfrak{F} \cap \mathfrak{A}_1), \quad \mathfrak{D}(\mathrm{wasc})(\mathfrak{F} \cap \mathfrak{A}_1),$$

$$\mathfrak{Z}\mathfrak{S}(\mathrm{wasc}), \quad \mathfrak{D}(\mathrm{wasc})\mathfrak{S}(\mathrm{wasc}), \quad \mathfrak{N}(\mathrm{Max}\text{-}\lhd \cap \mathfrak{M}_1(\mathrm{wasc})).$$

(4) *The following are subclasses of* $\mathfrak{L}^{\infty}(\mathrm{wasc}) \cap \mathfrak{L}_{\infty}(\mathrm{wasc})$:

$$\mathfrak{M}_2(\mathrm{asc}) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}, \quad \mathfrak{F} \cap (\mathfrak{N}\mathfrak{A}_1),$$

$$(\mathfrak{F} \cap \mathfrak{N})\mathfrak{S}(\mathrm{wasc}), \quad (\mathfrak{F} \cap \mathfrak{N})(\mathrm{Min}\text{-}\lhd \cap \mathrm{Max}\text{-}\lhd \cap \mathfrak{M}_1(\mathrm{wasc})).$$

PROOF. (1) is trivial.

(2) By Proposition 3.5 (1) we have

$$\mathfrak{Z}\mathfrak{A} \leq \mathfrak{D}(\mathrm{wasc})\mathfrak{A} \leq \mathfrak{L}(\mathrm{wasc}).$$

By Propositions 3.1 (1) and 3.2 (1) we have

$$\mathfrak{M}_1(\mathrm{wasc}) \leq \mathfrak{L}^*(\mathrm{wasc}) \quad \text{and} \quad \mathfrak{M}(\mathrm{wasc}) \cap \mathfrak{A}_1 \leq \mathfrak{L}^*(\mathrm{wasc}).$$

It then follows from Lemma 1.3 (1) that

$$\mathfrak{N}\mathfrak{A}_0 \leq \mathfrak{N}\mathfrak{M}_1(\mathrm{wasc}) \leq \mathfrak{L}(\mathrm{wasc})$$

and

$$\mathfrak{N}(\mathfrak{M}(\mathrm{wasc}) \cap \mathfrak{A}_1) \leq \mathfrak{L}(\mathrm{wasc}).$$

(3) By Theorem 3.7 (1) we have

$$\mathfrak{A}_1 \leq \mathfrak{Z}(\mathfrak{F} \cap \mathfrak{A}_1) \leq \mathfrak{D}(\mathrm{wasc})(\mathfrak{F} \cap \mathfrak{A}_1) \leq \mathfrak{L}^{\infty}(\mathrm{wasc}).$$

By Theorem 3.6 (1) we have

$$\mathfrak{Z}\mathfrak{S}(\mathrm{wasc}) \leq \mathfrak{D}(\mathrm{wasc})\mathfrak{S}(\mathrm{wasc}) \leq \mathfrak{L}^{\infty}(\mathrm{wasc}).$$

It is immediately deduced from Propositions 1.5 and 3.1 (1) that

$$\mathfrak{N}(\mathrm{Max}\text{-}\lhd \cap \mathfrak{M}_1(\mathrm{wasc})) \leq \mathfrak{L}^{\infty}(\mathrm{wasc}).$$

(4) By Theorem 3.10 (4) below, we have $\mathfrak{M}_2(\mathrm{asc}) \leq \mathfrak{L}^{\infty}(\mathrm{asc}) \cap \mathfrak{L}_{\infty}(\mathrm{asc})$. It follows from Lemma 1.6 (2) that

$$\mathfrak{M}_2(\mathrm{asc}) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \mathfrak{L}^{\infty}(\mathrm{wasc}) \cap \mathfrak{L}_{\infty}(\mathrm{wasc}).$$

Using [3, Proposition 8.5.1] we have $\mathrm{Min}\text{-}\mathrm{wasc} \cap \mathfrak{D}(\mathrm{wasc}) = \mathfrak{F} \cap \mathfrak{N}$. Therefore by (3) and Lemma 3.8 we obtain

$$\begin{aligned}\mathfrak{F} \cap (\mathfrak{N}\mathfrak{A}_1) &= \mathrm{Min}\text{-}\mathrm{wasc} \cap \mathfrak{D}(\mathrm{wasc})(\mathfrak{F} \cap \mathfrak{A}_1) \\ &\leq \mathfrak{L}^{\infty}(\mathrm{wasc}) \cap \mathfrak{L}_{\infty}(\mathrm{wasc})\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{F} \cap \mathfrak{N})\mathfrak{S}(\mathrm{wasc}) &\leq (\mathfrak{F} \cap \mathfrak{N})(\mathrm{Min}\text{-}\lhd \cap \mathrm{Max}\text{-}\lhd \cap \mathfrak{M}_1(\mathrm{wasc})) \\ &= \mathrm{Min}\text{-}\mathrm{wasc} \cap \mathfrak{N}(\mathrm{Max}\text{-}\lhd \cap \mathfrak{M}_1(\mathrm{wasc}))\end{aligned}$$

$$\leq \mathfrak{L}^{\infty}(\text{wasc}) \cap \mathfrak{L}_{\infty}(\text{wasc}).$$

THEOREM 3.10. (1) *The following are subclasses of* $\mathfrak{L}(\mathfrak{G}\text{-asc})$:

$$\mathfrak{G}\mathfrak{r}, \quad \mathfrak{D}(\text{asc})\mathfrak{A}, \quad \mathfrak{D}(\text{asc})(\mathfrak{F} \cap \mathfrak{A}_1), \quad \mathfrak{Z}\mathfrak{S}, \quad \mathfrak{D}(\text{asc})\mathfrak{S},$$
$$\mathfrak{N}(\mathfrak{M}(\text{asc}) \cap \mathfrak{A}_1), \quad \mathfrak{N}\mathfrak{M}_1(\text{asc}).$$

(2) *The following are subclasses of* $\mathfrak{L}(\text{asc})$:

$$\mathfrak{Z}\mathfrak{A}, \quad \mathfrak{D}(\text{asc})\mathfrak{A} \cap \acute{E}(\lhd)\mathfrak{A},$$
$$\mathfrak{N}\mathfrak{A}_0, \quad \mathfrak{N}(\mathfrak{G}^{\mathrm{I}} \cap \mathfrak{M}_1(\text{asc})), \quad (\mathfrak{F} \cap \mathfrak{N})\mathfrak{M}_1(\text{asc}).$$

(3) *The following are subclasses of* $\mathfrak{L}^{\infty}(\text{asc})$:

$$\mathfrak{A}_1, \quad \mathfrak{Z}(\mathfrak{F} \cap \mathfrak{A}_1), \quad \mathfrak{D}(\text{asc})(\mathfrak{F} \cap \mathfrak{A}_1) \cap \acute{E}(\lhd)\mathfrak{A},$$
$$\mathfrak{Z}(\mathfrak{G} \cap \mathfrak{S}), \quad \mathfrak{D}(\text{asc})(\mathfrak{G} \cap \mathfrak{S}),$$
$$\mathfrak{N}(\text{Max-}\lhd \cap \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{M}_1(\text{asc})), \quad (\mathfrak{F} \cap \mathfrak{N})(\text{Max-}\lhd \cap \mathfrak{M}_1(\text{asc})).$$

(4) *The following are subclasses of* $\mathfrak{L}^{\infty}(\text{asc}) \cap \mathfrak{L}_{\infty}(\text{asc})$:

$$\mathfrak{M}_2(\text{asc}), \quad \mathfrak{F} \cap (\mathfrak{N}\mathfrak{A}_1),$$
$$(\mathfrak{F} \cap \mathfrak{N})\mathfrak{S}, \quad (\mathfrak{F} \cap \mathfrak{N})(\text{Min-}\lhd \cap \text{Max-}\lhd \cap \mathfrak{M}_1(\text{asc})).$$

PROOF. (1) Since $\mathfrak{G}\mathfrak{r} = \mathfrak{D}(\mathfrak{G}\text{-asc})$, we have $\mathfrak{G}\mathfrak{r} \leq \mathfrak{L}(\mathfrak{G}\text{-asc})$. By Proposition 3.5 (2) and Theorem 3.7 (2), we have

$$\mathfrak{D}(\text{asc})\mathfrak{A} \leq \mathfrak{L}(\mathfrak{G}\text{-asc}) \quad \text{and} \quad \mathfrak{D}(\text{asc})(\mathfrak{F} \cap \mathfrak{A}_1) \leq \mathfrak{L}(\mathfrak{G}\text{-asc}).$$

Moreover, by Theorem 3.6 (2) we have

$$\mathfrak{Z}\mathfrak{S} \leq \mathfrak{D}(\text{asc})\mathfrak{S} \leq \mathfrak{L}(\mathfrak{G}\text{-asc}).$$

By Propositions 3.1 (2) and 3.2 (2), we have

$$\mathfrak{M}_1(\text{asc}) \leq \mathfrak{L}^*(\mathfrak{G}\text{-asc}) \quad \text{and} \quad \mathfrak{M}(\text{asc}) \cap \mathfrak{A}_1 \leq \mathfrak{L}^*(\mathfrak{G}\text{-asc}).$$

It then follows from Lemma 1.3 (2) that

$$\mathfrak{N}\mathfrak{M}_1(\text{asc}) \leq \mathfrak{L}(\mathfrak{G}\text{-asc}) \quad \text{and} \quad \mathfrak{N}(\mathfrak{M}(\text{asc}) \cap \mathfrak{A}_1) \leq \mathfrak{L}(\mathfrak{G}\text{-asc}).$$

(2) Since every hypercentral Lie algebra has an ascending $\mathfrak{A}$-series all of whose terms are characteristic ideals, it is easy to see that $\mathfrak{Z}\mathfrak{A} \leq \acute{E}(\lhd)\mathfrak{A}$. Hence by Proposition 3.5 (2) we have

$$\mathfrak{Z}\mathfrak{A} \leq \mathfrak{D}(\text{asc})\mathfrak{A} \cap \acute{E}(\lhd)\mathfrak{A} \leq \mathfrak{L}(\text{asc}).$$

By Theorem 3.9 (2) and Lemma 1.6 (2), we have

$$\mathfrak{N}\mathfrak{A}_0 \leq \mathfrak{L}(\text{wasc}) \cap \mathrm{E}\mathfrak{A} \leq \mathfrak{L}(\text{asc}).$$

It is directly deduced from Proposition 3.1 (2) and Lemma 1.3 (1) that

$$\mathfrak{N}(\mathfrak{G}^{\mathrm{I}} \cap \mathfrak{M}_1(\text{asc})) \leq \mathfrak{L}(\text{asc}).$$

Using [5, Theorem 2.7 (2)], we can easily show that

$$(\mathfrak{F} \cap \mathfrak{N})\mathfrak{M}_1(\text{asc}) \leq \mathfrak{M}(\text{asc}) \cap (\mathfrak{N}\mathfrak{M}_1(\text{asc})) \leq \mathfrak{L}(\text{asc}).$$

(3) By Theorem 3.7 (2) we have

$$\mathfrak{A}_1 \leq \mathfrak{Z}(\mathfrak{F} \cap \mathfrak{A}_1) \leq \mathfrak{D}(\text{asc})(\mathfrak{F} \cap \mathfrak{A}_1) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} \leq \mathfrak{L}^{\infty}(\text{asc}).$$

By Theorem 3.6 (2) we have

$$\mathfrak{Z}(\mathfrak{G} \cap \mathfrak{S}) \leq \mathfrak{D}(\text{asc})(\mathfrak{G} \cap \mathfrak{S}) \leq \mathfrak{L}^{\infty}(\text{asc}).$$

Using Proposition 1.5 we can deduce from (2) that

$$\mathfrak{N}(\text{Max-}\lhd \cap \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{M}_1(\text{asc})) \leq \mathfrak{L}^{\infty}(\text{asc})$$

and

$$(\mathfrak{F} \cap \mathfrak{N})(\text{Max-}\lhd \cap \mathfrak{M}_1(\text{asc})) \leq \mathfrak{L}^{\infty}(\text{asc}).$$

(4) By using [6, Proposition 11], we have

$$\mathfrak{M}_2(\text{asc}) \leq \mathfrak{M}(\text{asc}) \cap \mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty} \leq \mathfrak{L}^{\infty}(\text{asc}) \cap \mathfrak{L}_{\infty}(\text{asc}).$$

It is easily deduced from (3) and Lemma 3.8 that

$$\begin{aligned}\mathfrak{F} \cap (\mathfrak{N}\mathfrak{A}_1) &= \text{Min-asc} \cap \mathfrak{D}(\text{asc})(\mathfrak{F} \cap \mathfrak{A}_1) \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A} \\ &\leq \mathfrak{L}^{\infty}(\text{asc}) \cap \mathfrak{L}_{\infty}(\text{asc})\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{F} \cap \mathfrak{N})\mathfrak{S} &\leq (\mathfrak{F} \cap \mathfrak{N})(\text{Min-}\lhd \cap \text{Max-}\lhd \cap \mathfrak{M}_1(\text{asc})) \\ &= \text{Min-asc} \cap (\mathfrak{F} \cap \mathfrak{N})(\text{Max-}\lhd \cap \mathfrak{M}_1(\text{asc})) \\ &\leq \mathfrak{L}^{\infty}(\text{asc}) \cap \mathfrak{L}_{\infty}(\text{asc}).\end{aligned}$$

4.

In this section we shall mainly consider Lie algebras over a field of characteristic zero to search more interesting subclasses of $\mathfrak{L}(\mathfrak{G}\text{-asc})$, $\mathfrak{L}(\text{asc})$, $\mathfrak{L}^{\infty}(\text{asc})$ or $\mathfrak{L}^{\infty}(\text{asc}) \cap \mathfrak{L}_{\infty}(\text{asc})$.

In [6, Theorem 7] we proved that over any field of characteristic zero, $\mathfrak{E} \cap (\mathfrak{L}^{\infty}\text{Max-si}) \leq \mathfrak{L}^{\infty}$. We shall first improve this result in the following theorem, which corresponds to [8, Theorem 3.25] and characterizes the class $\mathfrak{L}^{\infty}$ over a field of characteristic zero.

THEOREM 4.1. *Over any field of characteristic zero,*

$$\mathfrak{L}^{\infty}\text{Max-si} = \mathfrak{L}^{\infty}.$$

PROOF. Let $L \in \mathfrak{L}^{\infty}\text{Max-si}$. Then there exists an $\mathfrak{L}^{\infty}$-ideal I of L such that $L/I \in \text{Max-si}$. Let $H_{\lambda} \text{ si } L\ (\lambda \in \Lambda)$ and $J = \langle H_{\lambda} \colon \lambda \in \Lambda \rangle$. Set $\mathscr{T} = \{T \colon T \leq J$ and $T \text{ si } L\}$. First we show that $\mathscr{T}$ has a maximal element. To do this we use Zorn's lemma. Let $\{T_{\alpha} \colon \alpha \in A\}$ be a totally ordered subset of $\mathscr{T}$ and set $T = \bigcup_{\alpha \in A} T_{\alpha}$. Since $L/I \in \text{Max-si}$, we can find a $\beta \in A$ such that for all $\alpha \in A$

$$(T_{\alpha} + I)/I \leq (T_{\beta} + I)/I.$$

Then we have $T + I = T_{\beta} + I$, so that

$$T = T \cap (T_{\beta} + I) = T_{\beta} + (T \cap I).$$

Since $T_{\alpha} \cap I \text{ si } I \in \mathfrak{L}^{\infty}$ for all $\alpha \in A$, we have $T \cap I \text{ si } L$. By using [3, Lemma 2.1.4] we have $T \text{ si } L$, so that $T \in \mathscr{T}$. Therefore by Zorn's lemma $\mathscr{T}$ has a maximal element. Owing to [1, Theorem 1.2], we have $J \text{ si } L$. Thus we obtain $L \in \mathfrak{L}^{\infty}$. This completes the proof.

Let $\mathfrak{X}$ be a class of Lie algebras. In [3] $\mathfrak{X}$ is said to be subjunctive if in any Lie algebra the join of any pair of $\mathfrak{X}$-subideals is always a subideal. We analogously say that $\mathfrak{X}$ is *ascendantly subjunctive* if in any Lie algebra the join of any pair of ascendant $\mathfrak{X}$-subalgebras is always an ascendant subalgebra. Then we have the following

LEMMA 4.2. *If $\mathfrak{X}$ is an* I(asc)*-closed class of Lie algebras such that $\mathfrak{G} \cap \mathfrak{X}$ is ascendantly subjunctive, then*

$$\mathrm{L}\mathfrak{X} \leq \mathfrak{L}^{*}(\mathfrak{G}\text{-asc}) \quad \textit{and} \quad \mathfrak{G}^{1} \cap \mathfrak{X} \leq \mathfrak{L}^{*}(\text{asc}).$$

PROOF. Let $H, K \in \mathscr{S}_{L}(\mathfrak{G}\text{-asc})$ and $J = \langle H, K \rangle$. If $L \in \mathrm{L}\mathfrak{X}$, then there exists an $\mathfrak{X}$-subalgebra of L containing J. Since $\mathfrak{X}$ is I(asc)-closed, we have $H, K \in \mathfrak{G} \cap \mathfrak{X}$. Therefore we obtain $J \text{ asc } L$ as $\mathfrak{G} \cap \mathfrak{X}$ is ascendantly subjunctive. Hence $\mathrm{L}\mathfrak{X} \leq \mathfrak{L}(\mathfrak{G}\text{-asc})$. Similarly if $J \in \mathrm{L}\mathfrak{X}$ then $J \text{ asc } L$. Since the class $\mathrm{L}\mathfrak{X}$ is I(asc)-closed, we have $\mathrm{L}\mathfrak{X} \in \mathfrak{L}^{*}(\mathfrak{G}\text{-asc})$. It now follows from Lemmas 1.1 (1) and 1.4 (2) that $\mathfrak{G}^{1} \cap \mathfrak{X} = \mathfrak{G}^{1} \cap \mathrm{L}\mathfrak{X} \leq \mathfrak{L}^{*}(\text{asc})$.

In view of Lemma 4.2 it is effective for the purpose of giving subclasses of

$\mathfrak{L}(\mathfrak{G}\text{-asc})$ or $\mathfrak{L}(\text{asc})$ to search ascendantly subjunctive subclasses of $\mathfrak{G}$. However, it does not seem to be easy to do this. Over fields of characteristic $p>0$ there is no hope of success (cf. [3, Lemma 3.1.1]). Hence we must restrict ourselves to fields of characteristic zero. Then by using results of [2] and [3], we have the following classes as ascendantly subjunctive subclasses of $\mathfrak{G}$:

$$\mathfrak{G} \cap \text{Min-}\lhd, \quad \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_{*}, \quad \mathfrak{G} \cap \text{Max-asc} \cap \mathfrak{E}_{*},$$

where $\mathfrak{E}_{*}$ is the class of Lie algebras L such that $L=\langle \mathrm{e}(L)\rangle$.

For convenience' sake, we define the left-normed products of classes $\mathfrak{X}$ and $\mathfrak{Y}$ recursively by

$$\mathfrak{X}\cdot_0\mathfrak{Y} = \mathfrak{X}, \quad \mathfrak{X}\cdot_{n+1}\mathfrak{Y} = (\mathfrak{X}\cdot_n\mathfrak{Y})\mathfrak{Y} \quad (n\geq 0).$$

Then we have the following result as the main theorem of this section.

THEOREM 4.3. *Over any field of characteristic zero, we have*

(1) *The following are subclasses of* $\mathfrak{L}^{*}(\mathfrak{G}\text{-asc})$:

$$\text{LMin-si}, \quad \text{L}(\mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_{*}^{\mathrm{I}}), \quad \text{L}(\text{Max-asc} \cap \mathfrak{E}_{*}^{\mathrm{I}}).$$

(2) *The following are subclasses of* $\mathfrak{L}^{*}(\text{asc})$:

$$\text{Min-si} \cap \text{Max-si}, \quad \mathfrak{G}^{\mathrm{I}} \cap \text{Min-si}, \quad \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_{*}^{\mathrm{I}}.$$

(3) *The following are subclasses of* $\mathfrak{L}(\mathfrak{G}\text{-asc})$:

$$\mathfrak{N}(\text{LMin-si}), \quad \mathfrak{N}(\text{L}(\mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_{*}^{\mathrm{I}})), \quad \mathfrak{N}(\text{L}(\text{Max-asc} \cap \mathfrak{E}_{*}^{\mathrm{I}})).$$

(4) *The following are subclasses of* $\mathfrak{L}(\text{asc})$:

$$\text{Min-si}, \quad \mathfrak{G}^{\mathrm{I}}, \quad \mathfrak{N}(\mathfrak{G}^{\mathrm{I}} \cap \text{Min-si}), \quad \mathfrak{N}(\mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_{*}^{\mathrm{I}}).$$

(5) *The following are subclasses of* $\mathfrak{L}^{\infty}(\text{asc})$:

$$\text{Max-asc}, \quad \mathfrak{N}(\text{Min-si} \cap \text{Max-si}), \quad \mathfrak{N}(\text{Max-si} \cap \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_{*}^{\mathrm{I}}).$$

(6) *The following are subclasses of* $\mathfrak{L}^{\infty}(\text{asc}) \cap \mathfrak{L}_{\infty}(\text{asc})$:

$$(\text{Min-si} \cap \mathfrak{M}_2(\text{asc}))\cdot_n(\text{Min-si} \cap \text{Max-si}),$$

$$(\mathfrak{F} \cap \mathfrak{N})\cdot_n(\text{Min-si} \cap \text{Max-si}) \quad (n\geq 0).$$

PROOF. (1) Let $\mathfrak{X}$ be one of the following classes:

$$\text{Min-si}, \quad \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_{*}^{\mathrm{I}}, \quad \text{Max-asc} \cap \mathfrak{E}_{*}^{\mathrm{I}}.$$

By [9] we have Min-si $=$ Min-asc $\leq \mathfrak{M}(\text{asc})$. On the other hand, by Lemma 1.1 (1) and [11, Theorem 2.1 (2)] we have $\mathfrak{G}^{\mathrm{I}} \cup \text{Max-asc} \leq \mathfrak{M}(\text{asc})$. Hence $\mathfrak{X}$ is I(asc)-closed. Using [3, Theorem 3.2.5] or [2, Corollary 6.3 or 6.4], we can easily see that the class $\mathfrak{G} \cap \mathfrak{X}$ is ascendantly subjunctive. It then follows from Lemma 4.2 that

$$\mathrm{L}\mathfrak{X} \leq \mathfrak{L}^*(\mathfrak{G}\text{-asc}) \quad \text{and} \quad \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{X} \leq \mathfrak{L}^*(\text{asc}). \tag{$*$}$$

By the first one of (∗), (1) is proved.

(2) By the second one of (∗), we have

$$\mathfrak{G}^{\mathrm{I}} \cap \text{Min-si} \leq \mathfrak{L}^*(\text{asc}) \quad \text{and} \quad \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_*{}^{\mathrm{I}} \leq \mathfrak{L}^*(\text{asc}).$$

Since $\text{Min-si} \leq \mathfrak{M}(\text{asc})$, the class $\text{Min-si} \cap \text{Max-si}$ is I(asc)-closed. Let $H, K \in \mathscr{S}_L(\text{asc})$ and $J = \langle H, K \rangle$. If $L \in \text{Min-si} \cap \text{Max-si}$, then $H, K \in \text{Min-}\lhd \cap \text{Max-}\lhd$. By [3, Theorem 3.2.5] the class $\text{Min-}\lhd \cap \text{Max-}\lhd$ is ascendantly coalescent. Hence we have $J \operatorname{asc} L$. It follows that $\text{Min-si} \cap \text{Max-si} \leq \mathfrak{L}(\text{asc})$. Similarly if $J \in \text{Min-si} \cap \text{Max-si}$ then $J \operatorname{asc} L$. Thus we obtain

$$\text{Min-si} \cap \text{Max-si} \leq \mathfrak{L}^*(\text{asc}).$$

(3) is directly deduced from (1) and Lemma 1.3 (2).

(4) By [6, Corollary 5] we have $\text{Min-si} \cap \mathfrak{G}^{\mathrm{I}} \leq \mathfrak{L}$. It follows that

$$\text{Min-si} \cup \mathfrak{G}^{\mathrm{I}} \leq \mathfrak{M}(\text{asc}) \cap \mathfrak{L} \leq \mathfrak{L}(\text{asc}).$$

By (2) and Lemma 1.3 (1) we have

$$\mathfrak{N}(\mathfrak{G}^{\mathrm{I}} \cap \text{Min-si}) \leq \mathfrak{L}(\text{asc}) \quad \text{and} \quad \mathfrak{N}(\mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_*{}^{\mathrm{I}}) \leq \mathfrak{L}(\text{asc}).$$

(5) Using [6, Corollary 5] we have

$$\text{Max-asc} \leq \mathfrak{M}(\text{asc}) \cap \mathfrak{L} \leq \mathfrak{L}(\text{asc}).$$

It then follows from Proposition 1.5 that

$$\text{Max-asc} \leq \mathfrak{L}^\infty(\text{asc}).$$

By (2) and Proposition 1.5 we have

$$\mathfrak{N}(\text{Min-si} \cap \text{Max-si}) \leq \mathfrak{L}^\infty(\text{asc}).$$

and

$$\mathfrak{N}(\text{Max-si} \cap \mathfrak{G}^{\mathrm{I}} \cap \mathfrak{E}_*{}^{\mathrm{I}}) \leq \mathfrak{L}^\infty(\text{asc}).$$

(6) By [6, Proposition 11] we have $\mathfrak{M}_2(\text{asc}) \leq \mathfrak{L}^\infty$. Since the class Min-si is $\{\mathrm{E}, \mathrm{I}, \mathrm{Q}\}$-closed, by Theorem 4.1 we can easily show that for any integer $n \geq 0$

$$
\begin{aligned}
&(\text{Min-si} \cap \mathfrak{M}_2(\text{asc})) \cdot_n (\text{Min-si} \cap \text{Max-si}) \\
&\quad = \text{Min-si} \cap (\mathfrak{M}_2(\text{asc}) \cdot_n \text{Max-si}) \\
&\quad \leq \text{Min-si} \cap \mathfrak{L}^\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathfrak{F} \cap \mathfrak{N}) \cdot_n (\text{Min-si} \cap \text{Max-si}) \\
&\quad = \text{Min-si} \cap (\mathfrak{N} \cdot_n \text{Max-si}) \\
&\quad \leq \text{Min-si} \cap \mathfrak{L}^\infty.
\end{aligned}
$$

Since Min-si$\leq \mathfrak{M}$(asc), by Lemma 3.8 we obtain

$$\text{Min-si} \cap \mathfrak{L}^\infty \leq \mathfrak{L}^\infty(\text{asc}) \cap \mathfrak{L}_\infty(\text{asc}).$$

It is not hard to see that over any field of characteristic zero

$$\mathfrak{G}^{\text{I}} \cap \mathfrak{L}^\infty(\text{asc}) = \mathfrak{G}^{\text{I}} \cap \text{Max-asc},$$

that is, with respect to $\mathfrak{G}^{\text{I}}$ the class $\mathfrak{L}^\infty$(asc) coincides with the class of Lie algebras satisfying the maximal condition for ascendant subalgebras.

Finally we should note that

$$\mathfrak{E} < \mathfrak{E}_*{}^{\text{I}} < \mathfrak{E}_*.$$

(See the examples described in [3, p. 340].)

5.

In this section we shall first distinguish between the classes $\mathfrak{L}(\mathfrak{G}\text{-}\Delta)$, $\mathfrak{L}(\Delta)$, $\mathfrak{L}^\infty(\Delta)$ and $\mathfrak{L}_\infty(\Delta)$, where Δ is any one of the relations wasc and asc, and secondly present an example of Lie algebras L such that $L \in \mathfrak{L}^*(\mathfrak{G}\text{-asc}) \cap \mathfrak{L}^\infty(\text{asc}) \cap \mathfrak{L}_\infty(\text{asc})$ and $L \notin$ Min-si.

THEOREM 5.1. *Let Δ be one of the relations* wasc *and* asc.
(1) $\mathfrak{L}^\infty(\Delta) < \mathfrak{L}(\Delta)$ *over any field of characteristic $p>0$.*
(2) $\mathfrak{L}(\Delta) < \mathfrak{L}(\mathfrak{G}\text{-}\Delta)$ *over any field.*
(3) $\mathfrak{L}^\infty(\Delta) \nleq \mathfrak{L}_\infty(\Delta)$ *over any field.*
(4) $\mathfrak{L}_\infty(\Delta) \nleq \mathfrak{L}(\mathfrak{G}\text{-}\Delta)$ *over any field of characteristic $p>0$.*

PROOF. (1) Let $\mathfrak{k}$ be a field of characteristic $p>0$ and A the (additive) abelian group of type p^∞. Let X be an abelian Lie algebra over $\mathfrak{k}$ with basis $\{x_a: a \in A\}$. For each $b \in A$, define $\delta(b) \in \text{Der}(X)$ by

$$x_a\delta(b) = \sum_{i=0}^{p-1} x_{a+ib} \quad (a \in A).$$

Set $Y=\langle\delta(b)\colon b\in A\rangle\leq \mathrm{Der}(X)$ and construct the split extension $L=X\dot{+}Y$ of X by Y, which is the Lie algebra constructed in [5, Example 4.2]. Then we have $L\in\mathfrak{A}^2\cap\mathfrak{F}\mathfrak{t}$ as was proved there. Moreover, [5, Remark to Example 4.2] states that L does not satisfy the idealiser condition, that is, $L\notin\mathfrak{D}(\varDelta)$. By Proposition 3.5 we have $L\in\mathfrak{L}(\varDelta)$. However, since $L\in\mathfrak{F}\mathfrak{t}\leq\mathfrak{G}\mathfrak{r}\leq\mathfrak{D}(\mathfrak{G}\text{-}\varDelta)$, we have $L\in\mathfrak{D}(\mathfrak{G}\text{-}\varDelta)\backslash\mathfrak{D}(\varDelta)$, so that $L\notin\mathfrak{L}^\infty(\varDelta)$. Therefore (1) is proved.

(2) Let L be the Lie algebra constructed in the proof of [3, Lemma 2.1.11], that is, let L be the split extension of the abelian ideal $C=A\oplus B$ by the $\mathfrak{N}_2$-subalgebra J with A and B spanned respectively by $\{a(P)\colon P\in\mathscr{S}\}$ and $\{b(P)\colon P\in\mathscr{S}\}$, where $\mathscr{S}$ is the set of all infinite sequences $P=(p_0, p_1,\ldots)$ of integers, and with J spanned by $\{x_m, y_n, z_{m,n}\colon m, n\in\boldsymbol{Z}\}$. Then the following conditions are satisfied:

(a) $a(P)=(-1)^r a(P')$ and $b(P)=(-1)^r b(P')$ for any $P, P'\in\mathscr{S}$ such that by a finite number, r, of transpositions P can be transformed into P';

(b) $a(P)=b(P)=0$ if by a finite number of transpositions P cannot be transformed into any element of the subset $\mathscr{T}$ of $\mathscr{S}$ consisting of the strictly increasing sequences;

(c) For any $P\in\mathscr{S}$ and any $m, n\in\boldsymbol{Z}$

$$[x_m, x_n]=[y_m, y_n]=0,\quad [x_m, y_n]=z_{m,n},$$

$$[a(P), x_m]=b(m, P),\quad [b(P), x_m]=0,$$

$$[a(P), y_m]=0,\quad [b(P), y_m]=a(m, P),$$

where $(m, P)=(m, p_0, p_1,\ldots)$ if $P=(p_0, p_1,\ldots)$.
By the proof of [3, Lemma 2.1.11] we see that

$$H\lhd^3 L,\quad K\lhd^3 L,\quad J=\langle H, K\rangle,\quad I_L(J)=J\quad\text{and}\quad J^L=L,$$

where $H=\langle x_m\colon m\in\boldsymbol{Z}\rangle$ and $K=\langle y_m\colon m\in\boldsymbol{Z}\rangle$. Hence J is neither an ascendant nor a descendant subalgebra of L. It follows that $L\notin\mathfrak{L}(\mathrm{asc})$. Since $L\in\mathfrak{A}\mathfrak{N}_2\leq\mathfrak{A}^3$, by Lemma 1.6 (2) we have $L\notin\mathfrak{L}(\varDelta)$.

Next we shall show that $L\in\mathfrak{L}(\mathfrak{G}\text{-}\varDelta)$. Let $m\in\boldsymbol{Z}$. We denote by $\mathscr{T}_m$ the subset of $\mathscr{T}$ consisting of the sequences which involve m as terms. Set

$$C_m=\langle a(P), b(P)\colon P\in\mathscr{T}_m\rangle.$$

Then we have $C_m\lhd L$ and

$$[x_m, L]=[x_m, A]+[x_m, K]\subseteq C_m+\textstyle\sum_{n\in\boldsymbol{Z}}\langle z_{m,n}\rangle.$$

Hence we can easily show that

$$[x_m,{}_{i+1}L]\subseteq C_m\quad(i\geq 1).$$

It follows that

$$\langle x_m^L \rangle \subseteq C_m + \langle x_m \rangle + \sum_{n \in \mathbf{Z}} \langle z_{m,n} \rangle .$$

By (b) we see that for any $P \in \mathscr{T}_m$ and any $n \in \boldsymbol{Z}$

$$[a(P), z_{m,n}] = [b(P), z_{m,n}] = [a(P), x_m] = 0.$$

Therefore we have $\langle x_m^L \rangle \in \mathfrak{A}$. Similarly we have $\langle y_m^L \rangle \in \mathfrak{A}$. It is clear that

$$L = C + \sum_{m \in \mathbf{Z}} \langle x_m^L \rangle + \sum_{m \in \mathbf{Z}} \langle y_m^L \rangle .$$

Hence L is the sum of abelian ideals. This implies that $L \in \mathfrak{F}\mathfrak{t} \leq \mathfrak{G}\mathfrak{r}$. Therefore we have $L \in \mathfrak{L}(\mathfrak{G}\text{-}\Delta)$, so that $\mathfrak{L}(\Delta) < \mathfrak{L}(\mathfrak{G}\text{-}\Delta)$.

(3) Let A be an abelian Lie algebra with basis $\{a_i: i \in \boldsymbol{Z}\}$. Define $x \in \mathrm{Der}(A)$ by $a_i x = a_{i-1}$ $(i \in \boldsymbol{Z})$, and construct the split extension $L = A \dotplus \langle x \rangle$ of A by $\langle x \rangle$. Then it is clear that $L \in \mathfrak{A}_1$. Hence by Theorems 3.9 (3) and 3.10 (3) we have $L \in \mathfrak{L}^{\infty}(\Delta)$. For each $n \in \boldsymbol{Z}$, set $A_n = \langle a_i: i \leq n \rangle$. Then we have

$$\langle A_n, x \rangle \lhd \langle A_{n+1}, x \rangle \lhd \cdots \bigcup_{m \geq n} \langle A_m, x \rangle = L,$$

so that $\langle A_n, x \rangle \lhd^{\omega} L$ for any $n \in \boldsymbol{Z}$. But it is not hard to show that

$$\bigcap_{n \in \mathbf{Z}} \langle A_n, x \rangle = \langle x \rangle = I_L(\langle x \rangle) .$$

Hence $L \notin \mathfrak{L}_{\infty}(\mathrm{asc})$ and therefore $L \notin \mathfrak{L}_{\infty}(\Delta)$ by Lemma 1.6 (2). Thus we obtain $\mathfrak{L}^{\infty}(\Delta) \not\leq \mathfrak{L}_{\infty}(\Delta)$.

(4) Owing to [3, Lemma 3.1.1], we see that over any field of characteristic $p > 0$

$$\mathfrak{F} \cap (\mathfrak{A}\mathfrak{N}_2) \not\leq \mathfrak{L}(\mathfrak{G}\text{-}\Delta) .$$

By Lemma 3.8 we have $\mathfrak{F} \leq \mathfrak{L}_{\infty}(\Delta)$. Therefore (4) is proved.

REMARK. By using the result shown in the proof of Theorem 5.1 (2), we can easily see that there exists a Lie algebra having a serial (resp. weakly serial [4]) subalgebra which is neither ascendant (resp. weakly ascendant) nor descendant (resp. weakly descendant [4]). In fact, let L be the Lie algebra constructed in the proof of [3, Lemma 2.1.11]. Then L is an abelian-by-nilpotent Lie algebra having a subalgebra J which is neither ascendant nor descendant in L. On the other hand, by the proof of Theorem 5.1 (2) L is a Fitting algebra. It follows that L is locally nilpotent. Using [3, Proposition 13.2.4], we see that every subalgebra of L is a serial subalgebra of L. In particular, J is a serial subalgebra of L. Since J is not ascendant in L and L is solvable, J is not also a weakly ascendant subalgebra of L by [10, Theorem 1]. Since J is not descendant in L and L is abelian-by-nilpotent, J is not also a weakly descendant subalgebra of L by [4, Theorem 4.2].

Finally we shall construct a Lie algebra which lies in the class $\mathfrak{L}^*(\mathfrak{G}\text{-asc}) \cap \mathfrak{M}_2(\text{asc})$ and which does not satisfy the minimal condition for subideals.

EXAMPLE 5.2. Let $\mathfrak{k}$ be a field of characteristic zero. We regard the group $\boldsymbol{Z}$ of integers as a subgroup of the additive group of $\mathfrak{k}$. Let $W = \mathscr{W}_{\boldsymbol{Z}}$ be a generalized Witt algebra (cf. [3, p. 206]), that is, W be a Lie algebra over $\mathfrak{k}$ with basis $\{w_i: i \in \boldsymbol{Z}\}$ such that

$$[w_i, w_j] = (i-j)w_{i+j} \quad (i, j \in \boldsymbol{Z}).$$

Then by [3, Theorem 10.3.1] we have $W \in \mathfrak{S}$. Furthermore, it is clear that

$$W = \langle w_{-2}, w_{-1}, w_1, w_2 \rangle.$$

Hence we have $W \in \mathfrak{G} \cap \mathfrak{S}$.

Let A be a vector space over $\mathfrak{k}$ with basis $\{a_i: i \in \boldsymbol{Z}\}$. Under the linear map $w_i \mapsto a_i$ $(i \in \boldsymbol{Z})$, we regard A as the underlying vector space W. Then A is a W-module with respect to the adjoint action of W. Consider A as an abelian Lie algebra and construct the split extension $L = A \dotplus W$ of A by W. Since every subspace of A is a subideal of L, L does not satisfy the minimal condition for subideals. By Theorem 3.6 (2) we have

$$L \in \mathfrak{A}(\mathfrak{G} \cap \mathfrak{S}) \leq \mathfrak{L}^\infty(\text{asc}).$$

Moreover, we can prove that $L \in \mathfrak{M}_2(\text{asc})$, so that $L \in \mathfrak{L}^\infty(\text{asc}) \cap \mathfrak{L}_\infty(\text{asc})$ by Theorem 3.10 (4). To do this it is sufficient to show that

$$\mathscr{S}_L(\text{asc}) = \{H: H \leq A \text{ or } H = L\} = \{H: H \lhd^2 L\}.$$

Clearly we have $\{H: H \leq A \text{ or } H = L\} \subseteq \{H: H \lhd^2 L\} \subseteq \mathscr{S}_L(\text{asc})$. Let $H \in \mathscr{S}_L(\text{asc})$ and assume that $H \nleq A$. Then

$$0 \neq (H+A)/A \text{ asc } L/A \cong W \in \mathfrak{S} = \mathfrak{S}(\text{asc})$$

and so $H + A = L$. Hence we can find an $h \in H$ such that $w_0 \equiv h \bmod A$. For any integer $i \neq 0$, by Lemma 2.1 there exists an integer $n = n(i) > 0$ such that

$$i^n a_i = [a_{i},{}_n w_0] = [a_{i},{}_n h] \in H.$$

It follows that $a_i \in H$. Therefore we have

$$\{a_i: 0 \neq i \in \boldsymbol{Z}\} \subseteq H \quad \text{and} \quad L = H + \langle a_0 \rangle.$$

Hence we can find a $k \in H$ and an $\alpha \in \mathfrak{k}$ such that $k = w_1 + \alpha a_0$. Then we have

$$[w_2, k] = [w_2, w_1] - \alpha[a_0, w_2] = w_3 + 2\alpha a_2.$$

By induction on j it is easy to show that

$$[w_{2,j}\,k] \equiv (j!)w_{j+2} \bmod \textstyle\sum_{i\geq 1}\langle a_i\rangle \quad (j\geq 1).$$

By Lemma 2.1 there exists an integer $m>0$ such that $[w_{2,m}\,k]\in H$. Since $\sum_{i\geq 1}\langle a_i\rangle\subseteq H$, we have $(m!)w_{m+2}\in H$, so that $w_{m+2}\in H$. It follows that

$$2(m+2)a_0 = [w_{m+2},\, a_{-m-2}]\in H.$$

Hence $a_0\in H$ and therefore $L=H+\langle a_0\rangle=H$. Thus we obtain

$$\mathscr{S}_L(\mathrm{asc}) = \{H\colon H\leq A \ \text{ or } \ H=L\} = \{H\colon H \lhd^2 L\}.$$

Finally we shall show that $L\in\mathfrak{L}^*(\mathfrak{G}\text{-asc})$. Let $\mathfrak{X}$ denote the class $\mathfrak{A}\cup(L)$ of Lie algebras over $\mathfrak{k}$, where (L) is the smallest class containing L. Obviously $\mathfrak{X}\leq\mathfrak{L}(\mathfrak{G}\text{-asc})$. Since $\mathscr{S}_L(\mathrm{asc})=\{H\colon H\leq A$ or $H=L\}$, the class $\mathfrak{X}$ is $\mathrm{I}(\mathrm{asc})$-closed. Let M be any Lie algebra over $\mathfrak{k}$ and let $H, K\in\mathscr{S}_M(\mathfrak{G}\text{-asc})$. Assume that $J=\langle H, K\rangle\in\mathfrak{X}$, $H<J$ and $K<J$. If $J\notin\mathfrak{A}$, then J is embedded in A, a contradiction. Hence we have $J\in\mathfrak{A}$. By using [2, Lemma 6.5] we have $J \operatorname{asc} M$. It follows that $\mathfrak{X}\leq\mathfrak{L}^*(\mathfrak{G}\text{-asc})$. Therefore we obtain $L\in\mathfrak{L}^*(\mathfrak{G}\text{-asc})$.

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