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Construction of solutions of a semilinear parabolic equation with the aid of the linear Boltzmann equation

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1. Introduction

Consider the Cauchy problem for a semilinear parabolic equation of the following form:

(P) $$u_t + \sum_{i=1}^n A^i(u)_{x_i} = \nu \Delta u \quad (x \in R^n,\ t>0),$$
$$u(x, 0) = u_0(x)$$

where Δ denotes the Laplacian; ν is any fixed positive number; and A^i, $i=1,\dots, n$, are C^1 functions of a single real variable. As is well known (see [8]) the solution u of the problem (P) with bounded measurable initial value u_0 converges, as $\nu \to 0$, to a global weak solution satisfying the entropy condition of the following hyperbolic problem:

(H) $$u_t + \sum_{i=1}^n A^i(u)_{x_i} = 0 \quad (x \in R^n,\ t>0),$$
$$u(x, 0) = u_0(x).$$

On the other hand, Kobayashi [7] has recently proposed an approximation scheme to the problem (H), using the solutions of the linear Boltzmann equation:

(B) $$f_t + \sum_{i=1}^n \xi^i f_{x_i} = 0 \quad (x \in R^n,\ \xi=(\xi^1,\dots, \xi^n) \in R^n,\ t>0),$$
$$f(x, \xi, 0) = f_0(x, \xi).$$

He used the function $v(x, t)=\int f(x, \xi, t)d\xi$ under a suitable choice of the initial function f_0 in order to construct approximate solutions of (H), and this procedure is an analogy of getting macroscopic quantities in fluid mechanics by integrating the corresponding microscopic ones with respect to the velocity argument. In this paper we modify the method in [7] so as to obtain approximate solutions of the parabolic problem (P).

The relationship between the initial values of (P) (or (H)) and (B) is given in the following way (compare with [7]). Take any function $\chi(\xi)$ with the following properties;

$$\chi(\xi) \geqq 0 \quad \text{on} \quad R^n;\ \chi \in C_0^\infty(R^n) \quad \text{and} \quad \operatorname{supp}\chi \subset \{\xi \in R^n;\ |\xi| \leqq 1\}. \tag{1.1}$$

$$\chi(\xi) = \chi(|\xi|) \quad \text{and} \quad \int \chi(\xi)d\xi = 1. \tag{1.2}$$

Put $\chi_\varepsilon(\xi) = \varepsilon^n \chi(\varepsilon\xi)$ for any fixed $\varepsilon > 0$ and

$$F_\varepsilon(w, \xi) = \int_0^w \chi_\varepsilon(\xi - a(s))ds, \quad w \in R^1, \tag{1.3}$$

$$a(s) = (a^1(s), \dots, a^n(s)), \quad a^i(s) = dA^i(s)/ds.$$

The following are easily verified.

$$w = \int F_\varepsilon(w, \xi)d\xi \qquad \text{for} \quad w \in R^1 \tag{D}$$

$$A^i(w) - A^i(0) = \int \xi^i F_\varepsilon(w, \xi)d\xi \qquad \text{for} \quad w \in R^1. \tag{C}$$

Now let $\{U_\xi(t);\ t \geqq 0\}$ be the family of solution operators of the problem (B) and set, for any fixed $\varepsilon > 0$,

$$(S_t v)(x) = \int [U_\xi(t) f_0](x, \xi)d\xi \quad \text{with} \quad f_0(x, \xi) = F_\varepsilon(v(x), \xi). \tag{1.4}$$

Then conditions (C) and (D) together imply that the function $S_t u_0$ satisfies (at least formally) the problem (H) at $t = 0$. This suggests that the function $S_h^{[t/h]} u_0$, $h > 0$, approximates in some sense a solution of the problem (H), where $[a]$ denotes the greatest integer in $a \in R^1$. Also, note that if $v \in L^\infty(R^n)$ and if $\varepsilon \uparrow \infty$, then $S_t v$ tends to the function

$$\int_{-\infty}^{\infty} F(v(x - a(s)t), s)ds, \quad \text{where} \quad F(w, s) = \begin{cases} 1 & \text{if} \quad 0 < s \leqq w, \\ -1 & \text{if} \quad w \leqq s < 0, \\ 0 & \text{otherwise,} \end{cases}$$

in the sense of distributions on R^n. This function was used in the previous paper [4] to construct approximate solutions of the problem (H) by the method illustrated above. See also [5].

The same argument as in [4] shows that if $u_0 \in L^\infty(R^n) \cap L^1(R^n)$ and if $\varepsilon > 0$ is fixed, then $S_h^{[t/h]} u_0$ converges, as $h \downarrow 0$, to the solution of (H) satisfying the entropy condition. Kobayashi [7] proved this for $\varepsilon = 1$ by using nonlinear semigroup theory. In this paper we will show that the same function converges to the solution of the problem (P) if we let $h \downarrow 0$ and $\varepsilon \downarrow 0$ under the condition that h/ε^2 is some fixed constant. To state our result we recall a notion of weak solution of the Cauchy problem (P). Let u_0 be in $L^\infty(R^n) \cap L^1(R^n)$. Then a function $u(x, t)$

lying in $L^\infty(R^n\times(0,\infty))\cap C([0,\infty); L^1(R^n))$ is called a weak solution of the problem (P) if $u(\cdot, 0)=u_0$ and

$$\int_0^\infty dt\int[u(\phi_t+\nu\Delta\phi)+\sum_i A^i(u)\phi_{x_i}]dx = 0 \qquad \text{for all} \quad \phi\in C_0^\infty(R^n\times(0,\infty)).$$

In Section 3 we shall show the uniqueness of the weak solution in the sense stated above. We can now state our main result in this paper.

CONVERGENCE THEOREM. *Let χ be any function satisfying* (1.1), (1.2), *and let $h>0$, $\varepsilon>0$ satisfy the relation*

$$(h/2n\varepsilon^2)\int|\xi|^2\chi(\xi)d\xi = \nu, \tag{1.5}$$

where ν is the number specified in (P). *Then, if $u_0\in L^1(R^n)\cap L^\infty(R^n)$, the function $S_h^{[t/h]}u_0$ tends in $L^1(R^n)$ as $h\downarrow 0$ to the unique weak solution of the Cauchy problem* (P) *and the convergence is uniform for bounded $t\geqq 0$.*

In proving this result it seems impossible to apply the argument in [4] which is based on the compactness theorem for functions of bounded variation. Indeed, it would be difficult to obtain necessary estimates for time-derivatives of $S_h^{[t/h]}u_0$ which are uniform in $h>0$, because the propagation speed of their supports becomes arbitrarily large as $h\downarrow 0$ under the condition (1.5). So we shall prove our result by applying the approximation theorem for nonlinear semigroups which was first established by Brezis and Pazy [1] and then generalized by Oharu and Takahashi [10] to the form convenient for our use. We note that a similar (but more complicated) idea was employed by Douglis [3] to obtain solutions of (P) by using approximate solutions of (H).

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2. Estimates for $S_t v$

First we recall the approximation theorem for nonlinear semigroups due to Oharu and Takahashi [10]. Let X be a real Banach space with norm $|\cdot|$, and $\{X_m; m=1, 2,\dots\}$ an increasing sequence of closed convex subsets in X. We set $X_\infty=\bigcup_{m=1}^\infty X_m$. Suppose given a family $\{C_h; h>0\}$ of (nonlinear) operators C_h on X_∞ such that each C_h defines a contraction map: $X_m\to X_m$ for all m, and set $B_h=h^{-1}(C_h-1)$. Now let $\lambda>0$ and $v\in X_m$. Then applying the contraction mapping principle to the equation

$$w = h(h+\lambda)^{-1}v + \lambda(h+\lambda)^{-1}C_h w$$

we easily see that

(2.1) $$R(1-\lambda B_h) \supset X_m \quad \text{for all} \quad \lambda > 0 \quad \text{and} \quad m$$

and

(2.2) $$|v-w| \leqq |(1-\lambda B_h)v-(1-\lambda B_h)w| \quad \text{for all} \quad \lambda > 0 \quad \text{and} \quad v, w \in X_\infty.$$

Here $R(1-\lambda B_h)$ denotes the range of the operator $1-\lambda B_h$. The estimate (2.2) means that the operators B_h are dissipative in X; see [10]. From (2.1) and (2.2) we see that the equation $(1-\lambda B_h)w=v$ with $v \in X_\infty$ and $\lambda>0$ has a unique solution $w \in X_\infty$, which we denote by $(1-\lambda B_h)^{-1}v$.

THEOREM 2.1 ([10]). *Suppose that the limit*

$$J(\lambda)v = \lim_{h \downarrow 0} (1-\lambda B_h)^{-1}v$$

exists for all $v \in X_\infty$ and $\lambda>0$. Then we have:

(i) *There exists a dissipative operator B in X such that*

$$R(1-\lambda B) = X_\infty \supset D(B) \quad \text{and} \quad J(\lambda) = (1-\lambda B)^{-1} \quad \text{for all} \quad \lambda > 0$$

where $D(B)$ is the domain of the operator B.

(ii) *B generates a C_0 semigroup $\{T(t);\ t \geqq 0\}$ of nonlinear contractions on the closure $\overline{D(B)}$ of $D(B)$ such that $T(t)[X_m \cap \overline{D(B)}] \subset X_m \cap \overline{D(B)}$ for all m and $t \geqq 0$.*

(iii) $\lim_{h \downarrow 0} C_h^{[t/h]}v = T(t)v$ *for $v \in X_\infty \cap \overline{D(B)}$ uniformly for bounded $t \geqq 0$.*

For the proof we refer to [10, §2]. We wish to apply this theorem to the case where $C_h = S_h$, $X = L^1(R^n)$,

(2.3) $$X_m = \{v \in L^1(R^n) \cap L^\infty(R^n);\ |v|_\infty \leqq m\},$$

and B is an appropriate operator associated to the problem (P). (Here and hereafter $|\cdot|_p$ denotes the norm of the Banach space $L^p(R^n)$, $1 \leqq p \leqq \infty$.) To this end we prepare some basic estimates for the operators S_h, $h>0$. First we note that, by definition,

(2.4) $$(S_h v)(x) = \int F_\varepsilon(v(x-\xi h), \xi)d\xi$$

whenever the right-hand side makes sense.

LEMMA 2.2. *The following are valid:*

(i) $\tau_y S_h = S_h \tau_y$ *for* $y \in R^n$, *where* $(\tau_y v)(x) = v(x+y)$.

(ii) $|S_h v|_p \leqq |v|_p$ *for* $v \in L^p(R^n)$ $(p=1, \infty)$ *and* $h \geqq 0$.

(iii) $|S_h v - S_h w|_1 \leqq |v-w|_1$ *for* $v, w \in L^1(R^n)$ *and* $h \geqq 0$.

PROOF. Assertion (i) is obvious from (2.4). By (1.1)–(1.3), the function

$F_\varepsilon(w, \xi)$ is nondecreasing in w; hence

$$F_\varepsilon(-r, \xi) \leqq F_\varepsilon(v(x-\xi h), \xi) \leqq F_\varepsilon(r, \xi)$$

if $v \in L^\infty(R^n)$ and $|v|_\infty = r$. Integrating this with respect to ξ and then using condition (D), we obtain assertion (ii) with $p=\infty$. We next consider the case: $p=1$. By (2.4) and Fubini's theorem we have

$$\begin{aligned} |S_h v|_1 &\leqq \int dx \int |F_\varepsilon(v(x-\xi h), \xi)| d\xi = \int d\xi \int |F_\varepsilon(v(x-\xi h), \xi)| dx \\ &= \int d\xi \int |F_\varepsilon(v(x), \xi)| dx = \int dx \int |F_\varepsilon(v(x), \xi)| d\xi. \end{aligned}$$

Since $|w| = \int |F_\varepsilon(w, \xi)| d\xi$ for $w \in R^1$, the last term equals $|v|_1$. This shows (ii) with $p=1$. Assertion (iii) is similarly proved by using the identity:

$$|v-w| = \int |F_\varepsilon(v, \xi) - F_\varepsilon(w, \xi)| d\xi \qquad \text{for} \quad v, w \in R^1.$$

This completes the proof.

Lemma 2.2 above shows that the operators S_h, $h>0$, satisfy all the conditions imposed on C_h in Theorem 2.1. Thus the operators

$$B_h = h^{-1}(S_h - 1), \quad h > 0 \tag{2.5}$$

satisfy (2.1) and (2.2) with $|\cdot| = |\cdot|_1$. Moreover, Lemma 2.2 (ii) implies

(2.6) $\quad |v|_p \leqq |(1-\lambda B_h)v|_p \; (p=1, \infty) \qquad$ for $\lambda > 0$ and $v \in L^1(R^n) \cap L^\infty(R^n)$.

In the next section we discuss the behavior of the functions $(1-\lambda B_h)^{-1}v$, with $v \in X_\infty = L^1(R^n) \cap L^\infty(R^n)$, as h tends to 0 and prove our result (Convergence Theorem stated in the Introduction) by applying Theorem 2.1.

3. Proof of Convergence Theorem

We begin by proving the following lemma, which is important in the subsequent argument. Let B_h be defined by (2.5).

LEMMA 3.1. *Let $v \in L^\infty(R^n)$, $k \in R^1$ and $\phi \in C_0^\infty(R^n)$ with $\phi \geqq 0$. Then*

$$\begin{aligned} &\int \operatorname{sgn}(v-k)\phi B_h v dx \\ &\quad \leqq h^{-1} \iint \operatorname{sgn}(v-k)[F_\varepsilon(v, \xi) - F_\varepsilon(k, \xi)](\phi(x+\xi h) - \phi(x)) dx d\xi \end{aligned} \tag{3.1}$$

where $\operatorname{sgn}(y) = y/|y|$ *if* $y \in R^1$, $y \neq 0$, *and* $\operatorname{sgn}(0) = 0$.

PROOF. We note that $B_h k=0$ by (2.4). Thus direct calculation gives

$$\int \operatorname{sgn}(v-k)\phi B_h v dx = \int \operatorname{sgn}(v-k)\phi(B_h v - B_h k)dx$$

$$= h^{-1}\iint \operatorname{sgn}(v(x+\xi h)-k)\phi(x+\xi h)[F_\varepsilon(v(x),\xi)-F_\varepsilon(k,\xi)]dxd\xi$$

$$- h^{-1}\iint \operatorname{sgn}(v(x)-k)\phi(x)[F_\varepsilon(v(x),\xi)-F_\varepsilon(k,\xi)]dxd\xi$$

$$= h^{-1}\iint \operatorname{sgn}(v(x)-k)[F_\varepsilon(v(x),\xi)-F_\varepsilon(k,\xi)](\phi(x+\xi h)-\phi(x))dxd\xi$$

$$+ h^{-1}\iint [F_\varepsilon(v(x),\xi)-F_\varepsilon(k,\xi)]\cdot$$

$$[\operatorname{sgn}(v(x+\xi h)-k)-\operatorname{sgn}(v(x)-k)]\phi(x+\xi h)dxd\xi.$$

Since $[F_\varepsilon(v,\xi)-F_\varepsilon(k,\xi)]\operatorname{sgn}(v-k)\geqq 0$ and $\phi\geqq 0$, the last term is nonpositive; so we obtain the inequality (3.1). This completes the proof.

We now define an operator B in $L^1(R^n)$ by

$$(3.2)\qquad Bv = v\Delta v - \textstyle\sum_{i=1}^n A^i(v)_{x_i} \qquad \text{for}\quad v\in D(B);$$

$$D(B) = \{v\in X_\infty \cap H^2(R^n);\ Bv\in X_\infty\}$$

where $H^2(R^n)$ is the usual Sobolev space. (Recall that $X_\infty = L^1(R^n)\cap L^\infty(R^n)$.) The following can be shown in the same way as in [2, Proposition 2.3].

PROPOSITION 3.2. *The operator B defined by* (3.2) *is dissipative in* $L^1(R^n)$.

In view of Theorem 2.1 and Proposition 3.2, the following result ensures the convergence of $S_h^{[t/h]}u_0$, $u_0\in X_\infty$, as $h\downarrow 0$.

PROPOSITION 3.3. *Let $v\in X_\infty$ and $\lambda>0$. Then $R(1-\lambda B)=X_\infty$ and*

$$(1-\lambda B_h)^{-1}v \longrightarrow (1-\lambda B)^{-1}v \quad in \quad L^1(R^n) \quad as \quad h\downarrow 0$$

provided that h and ε satisfy the relation (1.5).

We prove this result in two steps. Set $v_h^\lambda=(1-\lambda B_h)^{-1}v$ for $v\in X_\infty$.

LEMMA 3.4. *If h and ε satisfy* (1.5), *then the set $\{v_h^\lambda;\ h\in(0,\delta)\}$ is precompact in $L^1(R^n)$ for any fixed $\lambda>0$, $v\in X_\infty$ and $\delta>0$.*

PROOF. First we note that (2.2), (2.6) and Lemma 2.2 (i) together imply

$$(3.3)\qquad |v_h^\lambda|_p \leqq |v|_p \ (p=1,\infty),$$

$$(3.4)\qquad \int |v_h^\lambda(x+y)-v_h^\lambda(x)|dx \leqq \int |v(x+y)-v(x)|dx$$

for all $h>0$ and $y\in R^n$. We next show that

$$\text{(3.5)}\qquad \lim_{\rho\uparrow\infty}\int_{|x|>\rho}|v_h^\lambda(x)|dx = 0$$

uniformly in $h\in(0,\delta)$ if h and ε satisfy (1.5). Lemma 3.4 then follows from the Fréchet-Kolmogorov theorem ([12, p. 275]). To show (3.5) we first note that if $v\in X_\infty$, the estimate (3.1) with $k=0$ holds for any bounded continuous function $\phi\geqq 0$ with bounded and continuous derivatives up to and including order 2. Fixing any such ϕ, we use (3.1) with $v=v_h^\lambda$ and $k=0$. Since $B_h v_h^\lambda=\lambda^{-1}(v_h^\lambda-v)$, we have

$$\begin{aligned}\text{(3.6)}\quad \lambda^{-1}\Big[\int|v_h^\lambda|\phi dx-\int|v|\phi dx\Big] &\leqq \int \operatorname{sgn}(v_h^\lambda)\phi B_h v_h^\lambda dx\\
&\leqq h^{-1}\iint \operatorname{sgn}(v_h^\lambda)F_\varepsilon(v_h,\xi)[\phi(x+\xi h)-\phi(x)]dxd\xi\\
&= \textstyle\sum_i\displaystyle\iint \operatorname{sgn}(v_h^\lambda)\xi^i F_\varepsilon(v_h^\lambda,\xi)\phi_{x_i}(x)dxd\xi\\
&\quad + h\textstyle\sum_{i,j}\displaystyle\iint \operatorname{sgn}(v_h^\lambda)\xi^i\xi^j F_\varepsilon(v_h^\lambda,\xi)\Big[\int_0^1(1-\theta)\phi_{x_ix_j}(x+\theta\xi h)d\theta\Big]dxd\xi\\
&= I_1+I_2.\end{aligned}$$

By condition (C) we obtain

$$I_1=\textstyle\sum_i\displaystyle\int \operatorname{sgn}(v_h^\lambda)[A^i(v_h^\lambda)-A^i(0)]\phi_{x_i}dx=\textstyle\sum_i\displaystyle\int|v_h^\lambda|b^i\phi_{x_i}dx,$$

where $b^i(x)=\int_0^1 a^i(\theta v_h^\lambda(x))d\theta$. Thus (3.3) implies

$$\text{(3.7)}\qquad |I_1|\leqq(\sup_{|s|\leqq m}|a(s)|)|v_h^\lambda|_1\sup|D\phi|\leqq(\sup_{|s|\leqq m}|a(s)|)|v|_1\sup|D\phi|$$

where $D\phi=(\phi_{x_1},\dots,\phi_{x_n})$ and $m=|v|_\infty$. On the other hand, by the change of variables: $\varepsilon\xi=\eta$,

$$\begin{aligned}I_2 &= h\textstyle\sum_{i,j}\displaystyle\iint \operatorname{sgn}(v_h^\lambda)\Big[\int_0^1 d\theta\int_0^{v_h^\lambda}(\xi^i+a^i(s))(\xi^j+a^j(s))\chi_\varepsilon(\xi)\times\\
&\qquad\qquad\times(1-\theta)\phi_{x_ix_j}(x+\theta(\xi+a(s))h)ds\Big]dxd\xi\\
&= h\varepsilon^{-2}\textstyle\sum_{i,j}\displaystyle\iint \operatorname{sgn}(v_h^\lambda)\Big[\int_0^1 d\theta\int_0^{v_h^\lambda}(\eta^i+\varepsilon a^i(s))(\eta^j+\varepsilon a^j(s))\chi(\eta)\times\\
&\qquad\qquad\times(1-\theta)\phi_{x_ix_j}(x+\theta(\eta+\varepsilon a(s))h\varepsilon^{-1})ds\Big]dxd\eta.\end{aligned}$$

In what follows, we assume that the number $\delta>0$ is so chosen that $0<\varepsilon<1$

whenever $h \in (0, \delta)$. Since $|\eta| \leqq 1$ for $\eta \in \operatorname{supp} \chi$, we obtain

$$(3.8) \qquad |I_2| \leqq n^2 h\varepsilon^{-2}(1+\sup_{|s|\leqq m}|a(s)|)^2 |v|_1 \sup |D^2\phi|$$

where $D^2\phi = (\phi_{x_i x_j})_{i,=1}^n$. From (3.6)–(3.8) we have

$$(3.9) \qquad \lambda^{-1}\int |v_h^\lambda|\phi dx \leqq \lambda^{-1}\int |v|\phi dx + (\sup_{|s|\leqq m}|a(s)|)|v|_1 \sup|D\phi| + n^2 h\varepsilon^{-2}(1+\sup_{|s|\leqq m}|a(s)|)^2|v|_1 \sup|D^2\phi|.$$

Now choose a function $g \in C^\infty(R^1)$ such that

$$g(s) = 1 \quad \text{if} \quad s \geqq 1;\ g(s) = 0 \quad \text{if} \quad s \leqq 0; \quad \text{and} \quad 0 \leqq g(s) \leqq 1 \qquad \text{for} \quad s \in R^1$$

and define for $\rho > \tau > 0$ the function $g_{\rho,\tau}(s)$ as the even function so that

$$g_{\rho,\tau}(s) = g[(s-\tau)(\rho-\tau)^{-1}] \qquad \text{for} \quad s \geqq 0.$$

By definition we easily see that $0 \leqq g_{\rho,\tau}(s) \leqq 1$ for $s \in R^1$ and

$$g_{\rho,\tau}(s) = 1 \quad \text{if} \quad |s| \geqq \rho;\ g_{\rho,\tau}(s) = 0 \quad \text{if} \quad |s| \leqq \tau;$$
$$\sup|g'_{\rho,\tau}| \longrightarrow 0 \quad \text{and} \quad \sup|g''_{\rho,\tau}| \longrightarrow 0 \quad \text{as} \quad \rho \uparrow \infty.$$

So if we set $\phi_{\rho,\tau}(x) = \sum_i g_{\rho,\tau}(x_i)$, then,

$$(3.10) \quad 0 \leqq \phi_{\rho,\tau} \leqq n;\ \phi_{\rho,\tau}(x) \geqq 1 \text{ if } |x| \geqq \rho n^{1/2}; \text{ and } \phi_{\rho,\tau}(x) = 0 \text{ if } |x| \leqq \tau,$$

$$(3.11) \quad \sup|D\phi_{\rho,\tau}| \longrightarrow 0 \text{ and } \sup|D^2\phi_{\rho,\tau}| \longrightarrow 0 \text{ as } \rho \uparrow \infty.$$

Substituting $\phi = \phi_{\rho,\tau}$ into (3.9) and then using (3.10)–(3.11), we obtain

$$\limsup_{\rho\uparrow\infty}\int_{|x|>\rho n^{1/2}} |v_h^\lambda(x)|dx \leqq n\int_{|x|>\tau}|v(x)|dx,$$

since $h\varepsilon^{-2} = \text{const.}$. Since $\tau > 0$ is arbitrary, this proves (3.5).

The proof of Proposition 3.3 will be complete if we show the following

LEMMA 3.5. *Suppose that h and ε satisfy* (1.5) *and let v^λ be any cluster point of the set $\{v_h^\lambda\}$ as $h \downarrow 0$. Then*

$$v^\lambda \in D(B) \quad \textit{and} \quad v^\lambda = (1-\lambda B)^{-1}v$$

where B is the operator defined by (3.2). *Consequently, $v_h^\lambda \to v^\lambda$ in $L^1(R^n)$ as $h \downarrow 0$.*

PROOF. We may assume, without loss of generality, that $v_h^\lambda \to v^\lambda$ in $L^1(R^n)$ and $v_h^\lambda \to v^\lambda$ a.e. in R^n as $h \downarrow 0$. First we show that the function v^λ satisfies the equation

$$\lambda^{-1}(v^\lambda - v) = \nu \Delta v^\lambda - \sum_i A^i(v^\lambda)_{x_i} \tag{3.12}$$

in the sense of distributions. Since $\lambda^{-1}(v_h^\lambda - v) = B_h v_h^\lambda$, we have, for $\phi \in C_0^\infty$,

$$\begin{aligned}\lambda^{-1}\int (v_h^\lambda - v)\phi dx &= h^{-1}\iint F_\varepsilon(v_h^\lambda, \xi)(\phi(x+\xi h)-\phi(x))dxd\xi \\ &= \sum_i \iint \xi^i F_\varepsilon(v_h^\lambda, \xi)\phi_{x_i}dxd\xi + (h/2)\sum_{i,j}\iint \xi^i\xi^j F_\varepsilon(v_h^\lambda, \xi)\phi_{x_ix_j}dxd\xi \\ &\quad + h\sum_{i,j}\iint \xi^i\xi^j F_\varepsilon(v_h^\lambda(x), \xi)\times \\ &\qquad\times\left[\int_0^1 (1-\theta)(\phi_{x_ix_j}(x+\theta\xi h)-\phi_{x_ix_j}(x))d\theta\right]dxd\xi \\ &= J_1 + J_2 + J_3.\end{aligned} \tag{3.13}$$

By condition (C) and (3.3) we have

$$J_1 = \sum_i \int A^i(v_h^\lambda)\phi_{x_i}dx \longrightarrow \sum_i \int A^i(v^\lambda)\phi_{x_i}dx \quad \text{as} \quad h \downarrow 0. \tag{3.14}$$

J_2 is rewritten as

$$\begin{aligned}J_2 &= (h/2)\sum_{i,j}\int \phi_{x_ix_j}\left[\int_0^{v_h^\lambda} ds \int (\xi^i + a^i(s))(\xi^j + a^j(s))\chi_\varepsilon(\xi)d\xi\right]dx \\ &= (h/2\varepsilon^2)\sum_{i,j}\int \phi_{x_ix_j}\left[\int_0^{v_h^\lambda} ds\int (\eta^i + \varepsilon a^i(s))(\eta^j + \varepsilon a^j(s))\chi(\eta)d\eta\right]dx.\end{aligned}$$

Since χ is assumed to be a radial function (see (1.2)), we have

$$\int \eta^i\chi(\eta)d\eta = 0; \int \eta^i\eta^j\chi(\eta)d\eta = 0 \quad \text{if} \quad i \neq j; \int (\eta^i)^2\chi(\eta)d\eta = n^{-1}\int |\eta|^2\chi(\eta)d\eta.$$

Hence,

$$\begin{aligned}J_2 &= (h/2n\varepsilon^2)\left[\int |\eta|^2\chi(\eta)d\eta\right]\int v_h^\lambda \Delta\phi dx \\ &\qquad + (h/2)\sum_{i,j}\int \phi_{x_ix_j}\left[\int_0^{v_h^\lambda} a^i(s)a^j(s)ds\right]dx \\ &= J_{21} + J_{22}\end{aligned} \tag{3.15}$$

and, by (1.5),

$$J_{21} = \nu\int v_h^\lambda(x)\Delta\phi(x)dx \longrightarrow \nu\int v^\lambda(x)\Delta\phi(x)dx \quad \text{as} \quad h \downarrow 0; \tag{3.16}$$

$$|J_{22}| \leqq \text{const.}\, h|v|_\infty \sup_{|s|\leqq m}|a(s)|^2 \longrightarrow 0 \quad \text{as} \quad h \downarrow 0 \tag{3.17}$$

where $m = |v|_\infty$. On the other hand, since $|\eta| \leqq 1$ for $\eta \in \operatorname{supp}\chi$, we obtain after a change of variables,

$$|J_3| \leqq c(n)mh\varepsilon^{-2}[\sup\{|\eta+\varepsilon a(s)|;\ |\eta|\leqq 1,\ |s|\leqq m\}]^2 \times (\sup\chi) \times$$
$$\times \sup\left\{\iint |D^2\phi(x+\theta\eta\varepsilon^{-1}h+\theta a(s)h)-D^2\phi(x)|dx;\ |s|\leqq m,\ |\eta|\leqq 1,\ 0\leqq\theta\leqq 1\right\},$$

where $c(n)$ is a constant depending only on n. From this and (1.5) it follows that

$$J_3 \longrightarrow 0 \quad \text{as} \quad h\downarrow 0. \tag{3.18}$$

Combining (3.13)–(3.17) and (3.18) we conclude that (3.12) is valid. In view of the definition (3.2) of the operator B, it remains to show that $v^\lambda \in D(B)$. Since $v^\lambda \in X_\infty$ by (3.3), it suffices to show that $v^\lambda \in H^2(R^n)$. We write the equation (3.12) as

$$\nu\Delta v^\lambda = \lambda^{-1}(v^\lambda - v) + \textstyle\sum_i A^i(v^\lambda)_{x_i}. \tag{3.19}$$

Since v^λ and v are in X_∞, $\lambda^{-1}(v^\lambda - v)$ is in $L^2(R^n)$. Also, the functions

$$A^i(v^\lambda) - A^i(0) = b^i v^\lambda, \quad i = 1,\dots, n,$$

belong to $L^2(R^n)$ because $b^i = \int_0^1 a^i(\theta v^\lambda)d\theta$ are bounded functions. Hence,

$$\textstyle\sum_i A^i(v^\lambda)_{x_i} \in H^{-1}(R^n).$$

This, together with the equation (3.19), implies that v^λ is in $H^1(R^n)$; so as in the proof of the chain rule ([6, Lemma 7.5]), we obtain

$$\textstyle\sum_i A^i(v^\lambda)_{x_i} = \sum_i a^i(v^\lambda)v^\lambda_{x_i} \in L^2(R^n).$$

Consequently, the right-hand side of (3.19) is in $L^2(R^n)$. Hence $v^\lambda \in H^2(R^n)$, which completes the proof of Lemma 3.5; and so Proposition 3.3 is proved.

We are now in a position to prove our main result. By Proposition 3.3 and Theorem 2.1 there exists a function u in $C([0, \infty); L^1(R^n)) \cap L^\infty(R^n \times (0, \infty))$ such that $u(\cdot, 0) = u_0$ and, as $h\downarrow 0$,

$$S_h^{[t/h]}u_0 \longrightarrow u(\cdot, t) \quad \text{in} \quad L^1(R^n) \quad \text{uniformly for bounded} \quad t \geqq 0.$$

Hence we have only to show that the limit function u is the desired weak solution of the problem (P). Since $(S_h u_h)(x, t) = u_h(x, t+h)$ for $u_h = S_h^{[t/h]}u_0$, we have

$$h^{-1}[u_h(x, t+h) - u_h(x, t)] = (B_h u_h)(x, t),$$

so that, for $\phi \in C_0^\infty(R^n \times (0, \infty))$ and small $h > 0$,

$$h^{-1}\int_0^\infty dt \int u_h(x, t)(\phi(x, t-h) - \phi(x, t))dx$$
$$= h^{-1}\int_0^\infty dt \iint F_\varepsilon(u_h(x, t), \xi)(\phi(x+\xi h, t) - \phi(x, t))dxd\xi.$$

Hence the same argument as in the proof of (3.12) yields

$$\int_0^\infty dt \int [u(\phi_t + \nu\Delta\phi) + \sum_i A^i(u)\phi_{x_i}]dx = 0$$

which shows that u is a weak solution of (P). Finally, we prove the uniqueness of weak solutions. Suppose that there is another weak solution v with $v(\cdot, 0) = u_0$ lying in $C([0, \infty); L^1(R^n)) \cap L^\infty(R^n \times (0, \infty))$ and set $w = u - v$. After the substitution: $w \to e^{\nu t}w$, we see that $w(\cdot, 0) = 0$ and

$$w_t + \nu(1-\Delta)w + e^{-\nu t}\sum_i [A^i(e^{\nu t}u) - A^i(e^{\nu t}v)]_{x_i} = 0 \tag{3.20}$$

in the sense of distributions. Since

$$A^j(e^{\nu t}u) - A^i(e^{\nu t}v) = e^{\nu t}b^i w, \quad b^i = \int_0^1 a^i[e^{\nu t}((1-\theta)v + \theta u)]d\theta \tag{3.21}$$

and since $w(\cdot, t) \in X_\infty \subset L^2(R^n)$ for a.e. $t \geqq 0$, (3.20) implies that w_t is in $L^\infty(0, T; H^{-2}(R^n))$ for every $T > 0$. So, as in [9, p. 71], we obtain

$$(d/dt)|w(t)|^2_{-1,2} = 2((1-\Delta)^{-1}w_t(t), w(t)). \tag{3.22}$$

Here and in the following $|\cdot|_{s,2}$ denotes the norm of the Sobolev space $H^s(R^n)$ and $(\cdot, \cdot)$ the inner product of $L^2(R^n)$; the operator $(1-\Delta)^{-1}$ is defined via the Fourier transform (see [11]). From (3.20)–(3.22) we have

$$\begin{aligned}(d/dt)|w(t)|^2_{-1,2} + 2\nu|w(t)|^2_{0,2} &= -2\sum_i((1-\Delta)^{-1}(b^i w)_{x_i}(t), w(t))\\ &= 2\sum_i((b^i w)(t), (1-\Delta)^{-1}w_{x_i}(t)) \leqq \text{const.}\ |w(t)|_{0,2}|w(t)|_{-1,2}\\ &\leqq \nu|w(t)|^2_{0,2} + C_\nu|w(t)|^2_{-1,2}\end{aligned}$$

with a constant $C_\nu > 0$ independent of w. Hence $w = 0$ by Gronwall's lemma. This proves Convergence Theorem.

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