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# A product formula approach to first order quasilinear equations

Dedicated to Professor Isao Miyadera on his 60th birthday

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## Introduction

This paper is concerned with the Cauchy problem (hereafter called $(CP)$) for the scalar quasilinear equation

$$(DE) \qquad u_t + \textstyle\sum_{i=1}^{d} (\phi_i(u))_{x_i} = 0 \qquad \text{for} \quad t > 0,\ x \in \boldsymbol{R}^d$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_d)$ is a smooth $\boldsymbol{R}^d$-valued function on $\boldsymbol{R}$ such that $\phi(0) = 0$.

We treat this problem from the point of view of the theory of nonlinear semigroups and establish a new operator theoretic algorithm for solving the problem in conjunction with product formulae. It is well-known that solutions of $(CP)$ can be constructed by both the method of vanishing viscosity and the finite difference method. Recently, Giga and Miyakawa proposed in [7] a new method for constructing solutions of $(CP)$ via the iterative scheme

$$(0.1) \qquad u_{k+1} = C_h u_k, \quad k = 0, 1, 2, \ldots,$$

where the operators $C_h$, $h > 0$, are defined by

$$(0.2) \qquad (C_h u)(x) = \int_{\boldsymbol{R}} 2^{-1}(\mathrm{sign}\,(u(x - h\phi'(\xi)) - \xi) + \mathrm{sign}\,(\xi))d\zeta$$

for $x \in \boldsymbol{R}^d$, where $h$ stands for a mesh size of time difference.

Let $u(t, x)$ be a function of $(t, x) \in (0, \infty) \times \boldsymbol{R}^d$ and $f(t, x, \xi)$ the function of $(t, x, \xi) \in (0, \infty) \times \boldsymbol{R}^d \times \boldsymbol{R}$ defined by

$$f(t, x, \xi) = 2^{-1}(\mathrm{sign}\,(u(t, x) - \xi) + \mathrm{sign}\,(\xi)),$$

where $\xi$ is understood to mean a parameter varying over $\boldsymbol{R}$. Then the function $u$ and $f$ satisfies the relation

$$u(t, x) = \int_{\boldsymbol{R}} f(t, x, \xi)d\xi$$

and

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$$\phi_i(u(t, x)) = \int_{\boldsymbol{R}} \phi_i'(\xi) f(t, x, \xi) d\xi,$$

for $i=1, 2,\dots, d$. (See Proposition 1.1 below.) Hence, if $f(t, x, \xi)$ satisfies the linear equation

$$f_t + \textstyle\sum_{i=1}^{d} \phi_i'(\xi) f_{x_i} = 0 \tag{0.3}$$

at a time $t$, then $u(t, x)$ satisfies (DE) at $t$. Since the solution $f(t, x, \xi)$ of (0.3) satisfies

$$f(t+h, x, \xi) = f(t, x-h\phi'(\xi), \xi)$$

for $t$, $h \geqq 0$, the above-mentioned suggests that a solution of $(CP)$ is approximated by the solution of the scheme (0.1). In fact, it is proved in [7] that approximate solution of $(CP)$ can be constructed through the scheme (0.1) and converge to a weak solution of $(CP)$. Although their idea is quite natural and interesting in the sense that their method is interpreted in terms of kinetic theory of gases, it is not explicitly discussed in [7] whether the limit of the approximate solutions is uniquely determined by initial data. It is well known that there can be an infinite number of weak solutions of $(CP)$ for the same initial value and that an additional condition, called the entropy condition, is needed to select "physically right" weak solutions which are uniquely determined by initial data.

The main objective of this work is to establish a convergence theorem for approximate solutions defined through the scheme (0.1) to the weak solutions of $(CP)$ satisfying the entropy condition.

It is already known that the problem $(CP)$ can be studied via nonlinear semigroup theory. For example, Crandall [4] and subsequently Oharu-Takahashi [14] constructed a semigroup $\{T(t)\}_{t\geqq 0}$ of nonlinear contractions on $L^1(\boldsymbol{R}^d)$ such that, for $u_0 \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$, $u(t, x)=(T(t)u_0)(x)$ gives a unique entropy solution of $(CP)$. In the first paper, the vanishing viscosity method is employed and the generation theorem due to Crandall-Liggett is directly applied, while in the second paper finite-difference approximation of $(CP)$ is discussed from the point of view of the approximation theory for nonlinear semigroups and a convergence theorem for nonlinear semigroups plays an essential role.

In this paper we discuss a new semigroup approach to the problem $(CP)$. Let $\{T(t)\}_{t\geqq 0}$ be the semigroup constructed in the works cited above. Then we obtain the following convergence theorem which is the main result of this paper.

THEOREM. Let $u \in L^1(\boldsymbol{R}^d)$. Then we have the convergence

$$T(t)u = \lim_{h\downarrow 0} C_h^{[t/h]} u \tag{0.4}$$

in $L^1(\boldsymbol{R}^d)$ for $t \geqq 0$ and the convergence is uniform in $t$ on compact subsets of $[0, \infty)$. (Here $[\xi]$ denotes the greatest integer in $\xi \in \boldsymbol{R}$.)

The above mentioned result not only shows that the method proposed by Giga and Miyakawa is a new method for constructing "entropy solutions" of $(CP)$ but also provides an operator theoretic algorithm for obtaining semigroup solutions of $(CP)$ in term of product formula (0.4).

The plan of the paper is as follows: In Section 1 the results of Crandall [4] and Oharu-Takahashi [14] are recalled and nonlinear dissipative operators are introduced in connection with the notion of entropy condition for weak solutions of $(CP)$. In Section 2 various stability properties of the scheme (0.1) are studied. Basic estimates concerning the consistency with $(CP)$ of the scheme are prepared in Section 3. Finally, in Section 4, the proof of our main theorem mentioned above is given and several consequences of the theorem are discussed.

## 1. Preliminaries

Let $\boldsymbol{R}^d$ denote the $d$-dimensional Euclidean space with norm $|\cdot|$. We denote by $x\cdot y$ the Euclidean inner product of $x$ and $y$.

Let $\phi=(\phi_1, \phi_2,\dots, \phi_d)$ be a fixed continuously differentiable function on $\boldsymbol{R}$ into $\boldsymbol{R}^d$. We assume that the function $\phi$ is normalised in the sense that $\phi(0)=0$. The derivative $(\phi_1', \phi_2',\dots, \phi_d')$ of $\phi$ is denoted by $\phi'$.

The spatial gradient $(f_{x_1}, f_{x_2},\dots, f_{x_d})$ of a function $f$ on $\boldsymbol{R}^d$ is written as $f_x$. We write $L^1(\boldsymbol{R}^d)$ and $L^\infty(\boldsymbol{R}^d)$ for the ordinary Lebesgue spaces with standard norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. Also $C_0^\infty(\boldsymbol{R}^d)$ is the usual space of smooth functions with compact supports. We use the function

$$\operatorname{sign}(\xi) = \begin{cases} -1, & \text{if } \xi < 0, \\ 0, & \text{if } \xi = 0, \\ 1, & \text{if } \xi > 0. \end{cases}$$

Given a $u_0 \in L^\infty(\boldsymbol{R}^d)$, a function $u(t, \cdot)$ on $[0, \infty)$ into $L^\infty(\boldsymbol{R}^d)$ is called an *entropy solution* of $(CP)$ *with initial value* $u_0$ if it satisfies the following conditions:

$(a_1)$ $\|u(t,\cdot)\|_\infty$ is uniformly bounded in $t \in [0, \infty)$.

$(a_2)$ For each $t \in [0, \infty)$ and each $r>0$,

$$\lim_{s\to t}\int_{|x|<r} |u(s, x)-u(t, x)|dx = 0$$

and

$$u(0, x) = u_0(x) \text{ a.e.}.$$

$(a_3)$ For each $k \in \boldsymbol{R}$ and each $f \in C_0^\infty((0, \infty)\times \boldsymbol{R}^d)$ with $f \geqq 0$,

$$(1.1)\qquad \int_0^\infty \int_{\boldsymbol{R}^d} \{|u(t, x)-k| f_t(t, x) + \operatorname{sign}(u(t, x)-k)(\phi(u(t, x))-\phi(k))\cdot f_x(t, x)\} dxdt \geqq 0.$$

Condition $(a_3)$ was proposed by Vol'pert [15] and is regarded as an *entropy condition* in the multi-dimensional case. Also $(a_3)$ implies that an entropy solution $u$ is a *weak solution* of $(DE)$, i.e., $u$ satisfies $(DE)$ in the sense of distributions. The existence and uniqueness of the entropy solution of $(CP)$ was established by Kružkov [9].

In order to treat $(CP)$ via nonlinear semigroup theory, it is required to define a *generator* $A$ such that

$$Au = -\sum_{i=1}^d (\phi_i(u))_{x_i}$$

in an appropriate sense. We here define two operators $A_0$ and $A$ in $L^1(\boldsymbol{R}^d)$ in the following way:

$(b_1)$ $u \in D(A_0)$ and $w \in A_0 u$ if and only if
$u, w \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and

$$(1.2)\qquad \int_{\boldsymbol{R}^d} \operatorname{sign}(u(x)-k)\{(\phi(u(x))-\phi(k))\cdot f_x(x) - w(x)f(x)\} dx \geqq 0,$$

for every $k \in \boldsymbol{R}$ an every $f \in C_0^\infty(\boldsymbol{R}^d)$ with $f \geqq 0$.

$(b_2)$ $A$ is the closure of $A_0$ in $L^1(\boldsymbol{R}^d)$, i.e., $u \in D(A)$ and $w \in Au$ if and only if there exist sequences $\{u_k\}$ in $D(A_0)$ and $\{w_k\}$ in $L^1(\boldsymbol{R}^d)$ such that $w_k \in A_0 u_k$ and $u_k \to u$, $w_k \to w$ in $L^1(\boldsymbol{R}^d)$ as $k \to \infty$.

The definition of the operator $A$ is due to Crandall [4]. The operator $A_0$ is in fact single-valued, $C_0^1(\boldsymbol{R}^d) \subset D(A_0)$ and it is represented as

$$A_0 u = -\sum_{i=1}^d (\phi_i(u))_{x_i} \qquad \text{for} \quad u \in D(A_0),$$

in the sense of distributions. (See [4], Lemma 1.1.) It follows from the results of [4] and [14] that $A$ is a densely defined, *m-dissipative operator* in $L^1(\boldsymbol{R}^d)$, i.e., $A$ satisfies conditions $(c_1)$ and $(c_2)$ below. (See the references [1], [11] and [12] for basic properties of dissipative operators.)

$(c_1)$ For $\lambda > 0$, $u_i \in D(A)$ and $w_i \in Au_i$, $i = 1, 2$, we have

$$\|u_1 - \lambda w_1 - (u_2 - \lambda w_2)\|_1 \geqq \|u_1 - u_2\|_1.$$

$(c_2)$ For $\lambda > 0$ and $v \in L^1(\boldsymbol{R}^d)$, there exists a $u \in D(A)$ such that $u - \lambda Au \ni v$.

By virtue of the above-mentioned properties of $A$, the generation theorem of nonlinear semigroups due to Crandall-Liggett [5] can be applied to conclude that there exists a *semigroup* $\{T(t)\}_{t\geqq 0}$ of nonlinear contractions on $L^1(\boldsymbol{R}^d)$ into itself such that

$$(1.3) \qquad T(u)u = \lim_{\lambda\downarrow 0}(I-\lambda A)^{-[t/\lambda]}u \quad \text{in} \quad L^1(\boldsymbol{R}^d)$$

for $u\in L^1(\boldsymbol{R}^d)$ and $t\geqq 0$, where $I$ stands for the identity operator on $L^1(\boldsymbol{R}^d)$. For each $u_0\in L^1(\boldsymbol{R}^d)$, the function $u(t)=T(t)u_0$ gives a solution in a generalized sense of the abstract Cauchy problem

$$(ACP) \qquad du/dt\in Au, \quad u(0) = u_0$$

in the Banach space $L^1(\boldsymbol{R}^d)$. Moreover it is shown in [4] and [14] that $u(t, x)=(T(t)u_0)(x)$ is an entropy solution of $(CP)$ with initial value $u_0$ in $L^1(\boldsymbol{R}^d)\cap L^\infty(\boldsymbol{R}^d)$.

Let $\{C_h\}_{h\geqq 0}$ be the family of operators defined by (0.2) and set

$$A_h = h^{-1}(C_h-I) \qquad \text{for} \quad h>0.$$

Then the iterative scheme (0.1) can be rewirtten in the following form:

$$(1.4) \qquad h^{-1}(u^{k+1}-u^k) = A_h u^k, \quad k = 0, 1, 2,\dots.$$

In order to prove the main theorem via the approximation theory for nonlinear semigroups, we employ Theorem 3.2 in Brezis-Pazy [3]. Hence it suffices to show that the family $\{C_h\}_{h>0}$ satisfies the following two conditions.

$(d_1)$ Each $C_h$ is a contraction operator on $L^1(\boldsymbol{R}^d)$ into itself in the sense that

$$\|C_h u - C_h v\|_1 \leqq \|u-v\|_1 \qquad \text{for} \quad u, v\in L^1(\boldsymbol{R}^d).$$

$(d_2)$ For each $\lambda>0$ and each $v\in L^1(\boldsymbol{R}^d)$,

$$(I-\lambda A)^{-1}v = \lim_{h\downarrow 0}(I-\lambda A_h)^{-1}v \quad \text{in} \quad L^1(\boldsymbol{R}^d).$$

Condition $(d_1)$ implies that $C_h^k$ is a contraction operator on $L^1(\boldsymbol{R}^d)$ into itself for every $h>0$ and $k=0, 1, 2,\dots$. In this sense, $(d_1)$ ensures the *stability* of the scheme (0.1) or (1.4). Although we cannot expect that $A_h$ converges directly to $A_0u$ as $h\downarrow 0$ even if $u\in D(A_0)$, we may understand that the family of operators $A_h$, $h>0$, approximates the operator $A$ on $(ACP)$, because condition $(d_2)$ implies that $u\in D(A)$ and $w\in Au$ if and only if there exist $u_h\in L^1(\boldsymbol{R}^d)$ such that $u_h\to u$ and $A_h u_h\to w$ in $L^1(\boldsymbol{R}^d)$ as $h\downarrow 0$. (See [13] and [12].) Noting that, under condition $(d_1)$, $(d_2)$ yields the convergence (0.4), we call hereafter $(d_2)$ the *consistency condition.*

Following Giga and Miyakawa [7], we employ the function $F$ on $\boldsymbol{R}\times\boldsymbol{R}$ defined by

$$F(a, \xi) = 2^{-1}(\operatorname{sign}(a-\xi)+\operatorname{sign}(\xi)), \qquad \text{for} \quad a, \xi\in\boldsymbol{R}.$$

Using this function, we can rewrite the operator $C_h$ as

$$(1.5)\qquad (C_h u)(x) = \int_{\boldsymbol{R}} F(u(x-h\phi'(\xi)),\ \xi)d\xi \qquad \text{for} \quad x \in \boldsymbol{R}^d.$$

We here list some basic properties of the function $F$ in the next proposition:

PROPOSITION 1.1. *(i) If $a \geqq b$, then $F(a, \xi) \geqq F(b, \xi)$ for $\xi \in \boldsymbol{R}$.*
*(ii) If $f$ is a locally integrable function on $\boldsymbol{R}$ and $a, b \in \boldsymbol{R}$, then*

$$(1.6)\qquad \int_{\boldsymbol{R}} f(\xi)(F(a, \xi)-F(b, \xi))d\xi = \int_b^a f(\xi)d\xi$$

*and*

$$(1.7)\qquad \int_{\boldsymbol{R}} f(\xi)|F(a, \xi)-F(b, \xi)|d\xi = \operatorname{sign}(a-b)\int_b^a f(\xi)d\xi.$$

(iii) *For each $a \in \boldsymbol{R}$, $F(a, \xi)=0$ for $|\xi|>|a|$ and $F(0, \xi)=0$ for all $\xi \in \boldsymbol{R}$.*
(iv) *For each $a, b \in \boldsymbol{R}$,*

$$\int_{\boldsymbol{R}} F(a, \xi)d\xi = a, \quad \int_{\boldsymbol{R}} |F(a, \xi)|d\xi = |a|$$

*and*

$$\int_{\boldsymbol{R}} |F(a, \xi)-F(b, \xi)|d\xi = |a-b|.$$

PROOF. Since the function $\operatorname{sign}(\cdot)$ is nondecreasing on $\boldsymbol{R}$,

$$F(a, \xi) - F(b, \xi) = 2^{-1}(\operatorname{sign}(a-\xi)-\operatorname{sign}(b-\xi)) \geqq 0$$

for $a \geqq b$ and $\xi \in \boldsymbol{R}$. Let $f$ be locally integrable on $\boldsymbol{R}$ and let $a>b$. Then

$$(1.8)\qquad F(a, \xi) - F(b, \xi) = \begin{cases} 1/2 & \text{if} \quad \xi = a, \\ 1 & \text{if} \quad a > \xi > b, \\ -1/2 & \text{if} \quad \xi = b, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have (1.6). Similarly, (1.6) holds in the case $a \leqq b$. By (i),

$$|F(a, \xi)-F(b, \xi)| = \operatorname{sign}(a-b)(F(a, \xi)-F(b, \xi))$$

for $a, b, \xi \in \boldsymbol{R}$. Hence (1.6) implies (1.7). Furthermore, the function $\operatorname{sign}(\cdot)$ is odd, and so

$$F(0, \xi) = 2^{-1}(\operatorname{sign}(\xi)+\operatorname{sign}(-\xi)) = 0 \qquad \text{for} \quad \xi \in \boldsymbol{R}.$$

Therefore, $F(a, \xi)=0$ for $|\xi|>|a|$ by (1.8). The properties of $F$ stated in (iv) are easily deduced from those of $F$ listed in (ii) and (iii). Q.E.D.

REMARK. In the following argument, any other properties of $F$ as mentioned above will not be necessary. So, the function

$$F(a, \xi) = \begin{cases} 1 & \text{if } 0 \leqq \xi \leqq a, \\ -1 & \text{if } a \leqq \xi < 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is employed in [7], can be employed for the definition of the operator $C_h$.

## 2. Stability of the scheme

First we prepare basic estimates concerning the stability of the operators $C_h$ defined by (1.5). Although those estimates are essentially proved in [7], we here give a proof of them for the sake of completeness. For each $y \in \boldsymbol{R}^d$, we define $\tau^y$: $L^1(\boldsymbol{R}^d) \to L^1(\boldsymbol{R}^d)$ by

$$(\tau^y u)(x) = u(x+y) \qquad \text{for} \quad x \in \boldsymbol{R}^d \text{ and } u \in L^1(\boldsymbol{R}^d).$$

PROPOSITION 2.1. *Let* $h > 0$. *Then*:

(i) $C_h$ *is a contraction operator on* $L^1(\boldsymbol{R}^d)$ *into itself and* $\|C_h u\|_1 \leqq \|u\|_1$ *for* $u \in L^1(\boldsymbol{R}^d)$.

(ii) $C_h \tau^y = \tau^y C_h$ *for* $y \in \boldsymbol{R}^d$.

(iii) *If* $u \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$, *then* $C_h u \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ *and* $\|C_h u\|_\infty \leqq \|u\|_\infty$.

PROOF. Let $u \in L^1(\boldsymbol{R}^d)$. Then, by Fubini's theorem,

$$\begin{aligned} &\int_{\boldsymbol{R}^d} |(C_h u)(x)| dx \\ &\leqq \iint_{\boldsymbol{R}^d \times \boldsymbol{R}} |F(u(x - h\phi'(\xi)), \xi)| dx d\xi \\ &= \iint_{\boldsymbol{R}^d \times \boldsymbol{R}} |F(u(x), \xi)| dx d\xi. \end{aligned}$$

Since

$$\int_{\boldsymbol{R}} |F(u(x), \xi)| d\xi = |u(x)|$$

by (iv) of Proposition 1.1, we have $C_h u \in L^1(\boldsymbol{R}^d)$ and $\|C_h u\|_1 \leqq \|u\|_1$. Similarly, it follows from Fubini's theorem and (iv) of Proposition 1.1 that

$$\begin{aligned}
&\int_{\boldsymbol{R}^d} |(C_h u)(x)-(C_h v)(x)|dx \\
&\leqq \iint_{\boldsymbol{R}^d \times \boldsymbol{R}} |F(u(x-h\phi'(\xi)), \xi)-F(v(x-h\phi'(\xi)), \xi)|dxd\xi \\
&= \iint_{\boldsymbol{R}^d \times \boldsymbol{R}} |F(u(x), \xi)-F(v(x), \xi)|dxd\xi \\
&= \int_{\boldsymbol{R}^d} |u(x)-v(x)|dx
\end{aligned}$$

for $u, v \in L^1(\boldsymbol{R}^d)$. Assertion (ii) is evident from the definition of $C_h$. It now remains to prove (iii). Let $u \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$. Then, by (i) of Proposition 1.1,

$$F(-\|u\|_\infty, \xi) \leqq F(u(x-h\phi'(\xi), \xi) \leqq F(\|u\|_\infty, \xi) \text{ a.e.}.$$

Integrating the above terms with respect to $\xi$ and using (iv) of Proposition 1.1, we have

$$-\|u\|_\infty \leqq (C_h u)(x) \leqq \|u\|_\infty \text{ a.e.}.$$

Therefore, $C_h u \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and $\|C_h u\|_\infty \leqq \|u\|_\infty$. Q.E.D.

Asserion (i) of Proposition 2.1 implies that each $A_h = h^{-1}(C_h - I)$ is $m$-dissipative in $L^1(\boldsymbol{R}^d)$. Hence the resolvent

$$J_{\lambda,h} = (I-\lambda A_h)^{-1}$$

exists for each $\lambda > 0$ and each $h > 0$. Then, as easily seen, we have the relations

$$J_{\lambda,h}v = h(\lambda+h)^{-1}v + \lambda(\lambda+h)^{-1}C_h J_{\lambda,h}v \tag{2.1}$$

and

$$A_h J_{\lambda,h}v = \lambda^{-1}(J_{\lambda,h}v - v) \tag{2.2}$$

for $v \in L^1(\boldsymbol{R}^d)$ and $\lambda, h > 0$. Basic properties of the resolvents $J_{\lambda,h}$ of $A_h$ may be stated in the following form:

PROPOSITION 2.2. *Let $h, \lambda > 0$. Then:*

(i) *$J_{\lambda,h}$ is a contraction operator in $L^1(\boldsymbol{R}^d)$ into itself and $\|J_{\lambda,h}v\|_1 \leqq \|v\|_1$ for $v \in L^1(\boldsymbol{R}^d)$.*

(ii) *$J_{\lambda,h}\tau^y = \tau^y J_{\lambda,h}$ for $y \in \boldsymbol{R}^d$.*

(iii) *If $v \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$, then $J_{\lambda,h}v \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and $\|J_{\lambda,h}v\|_\infty \leqq \|v\|_\infty$.*

PROOF. Since $A_h$ is an $m$-dissipative operator in $L^1(\boldsymbol{R}^d)$, each of its resolvents

$J_{\lambda,h}$ is a contraction operator on $L^1(\boldsymbol{R}^d)$ into itself. Let $v \in L^1(\boldsymbol{R}^d)$. Then, by (i) of Proposition 2.1,

$$\|C_h J_{\lambda,h} v\|_1 \leqq \|J_{\lambda,h} v\|_1.$$

Therefore, (2.1) implies that

$$\|J_{\lambda,h} v\|_1 \leqq h(\lambda+h)^{-1}\|v\|_1 + \lambda(\lambda+h)^{-1}\|J_{\lambda,h} v\|_1;$$

and hence

$$\|J_{\lambda,h} v\|_1 \leqq \|v\|_1.$$

It now remains to prove (ii) and (iii). For each $v \in L^1(\boldsymbol{R}^d)$ we define

$$K^v u = h(\lambda+h)^{-1} v + \lambda(\lambda+h)^{-1} C_h u \qquad \text{for} \quad u \in L^1(\boldsymbol{R}^d).$$

Then Proposition 2.1 (i) implies that each $K^v$ is a strict contraction operator (with Lipschitz constant less than or equal to $\lambda(\lambda+h)^{-1}$) on $L^1(\boldsymbol{R}^d)$ into itself. Therefore, each $K^v$ has a unique fixed point in $L^1(\boldsymbol{R}^d)$. But the relation (2.1) states that, for each $v \in L^1(\boldsymbol{R}^d)$, $J_{\lambda,h} v$ itself is the unique fixed point of $K^v$. To prove (ii), let $v \in L^1(\boldsymbol{R}^d)$ and $y \in \boldsymbol{R}^d$. Then the application of Proposition 2.1 (ii) and the relation (2.1) yields

$$\tau^y J_{\lambda,h} v = h(\lambda+h)^{-1} \tau^y v + \lambda(\lambda+h)^{-1} C_h \tau^y J_{\lambda,h} v.$$

This means that $\tau^y J_{\lambda,h} v$ is a fixed point of $K^{\tau^y v}$, and we have $\tau^y J_{\lambda,h} v = J_{\lambda,h} \tau^y v$ by the unicity of the fixed point. Finally, let $v \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and set $X^v = \{u \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d);\ \|u\|_\infty \leqq \|v\|_\infty\}$. Then $X^v$ is a nonempty closed convex subset of $L^1(\boldsymbol{R}^d)$. Furthermore, Proposition 2.1 (iii) implies that $K^v$ maps $X^v$ into itself. Consequently, the fixed point $J_{\lambda,h} v$ of $K^v$ belongs to $X^v$. Hence, $J_{\lambda,h} v \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and $\|J_{\lambda,h} v\|_\infty \leqq \|v\|_\infty$. Q.E.D.

## 3. Consistency of the scheme

We begin by establishing the following result, which is the core of our argument below.

PROPOSITION 3.1. *Let $u \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and $h > 0$. Then*

$$\begin{aligned} &\int_{\boldsymbol{R}^d} (|(C_h u)(x) - k| - |u(x) - k|) f(x)\,dx \\ &\leqq \int_{\boldsymbol{R}^d} \operatorname{sign}(u(x) - k) \int_k^{u(x)} (f(x + h\phi'(\xi)) - f(x))\,d\xi\,dx \end{aligned} \tag{3.1}$$

*for every $k \in \boldsymbol{R}$ and every $f \in C_0^\infty(\boldsymbol{R}^d)$ with $f \geqq 0$.*

PROOF. Let $k \in \boldsymbol{R}$, $f \in C_0^\infty(\boldsymbol{R}^d)$, and assume that $f \geqq 0$. By (iv) of Proposition 1.1 we have

$$(C_h u)(x) - k = \int_{\boldsymbol{R}} F(u(x-h\phi'(\xi)), \xi) - F(k, \xi)d\xi, \ x \in \boldsymbol{R}^d.$$

Hence

$$|(C_h u)(x) - k| \leqq \int_{\boldsymbol{R}} |F(u(x-h\phi'(\xi)), \xi) - F(k, \xi)|d\xi, \quad x \in \boldsymbol{R}^d.$$

On the other hand, Proposition 1.1 (iv) yields

$$|u(x) - k| = \int_{\boldsymbol{R}} |F(u(x), \xi) - F(k, \xi)|d\xi, \quad x \in \boldsymbol{R}^d.$$

Therefore, the application of Fubini's theorem yields

$$\begin{aligned}
&\int_{\boldsymbol{R}^d} (|(C_h u) - k| - |u(x) - k|)f(x)dx \\
&\leqq \int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}} \{|F(u(x-h\phi'(\xi)), \xi) - F(k, \xi)|f(x) - |F(u(x), \xi) - F(k, \xi)|f(x)d\xi dx \\
&= \int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}} |F(u(x), \xi) - F(k, \xi)|(f(x+h\phi'(\xi)) - f(x))d\xi dx,
\end{aligned}$$

We now apply Proposition 1.1 (ii) to the above estimate to obtain the desired inequality (3.1). Q.E.D.

To show the consistency of our scheme with the problem $(CP)$, we need a few more estimates which are derived from (3.1).

Let $P$ be the set of all functions $p$: $\boldsymbol{R} \to \boldsymbol{R}$ satisfying

(i) $p$ is nondecreasing and Lipcshitz continuous;
(ii) the derivative $p'$ has compact support:

and

(iii) $p(+\infty) + p(-\infty) = 0$.

The next inequality (3.2) involving the operator $A_h$ corresponds to the inequality (1.2) which specifies the operator $A_0$.

PROPOSITION 3.2. *Let $u \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and $h > 0$. Let $p \in P$. Then,*

$$\begin{aligned}
(3.2) \qquad &\int_{\boldsymbol{R}^d} p(u(x))(A_h u)(x) f(x)dx \\
&\leqq \int_{\boldsymbol{R}^d}\int_k^{u(x)} p(s)h^{-1}(f(x+h\phi'(s)) - f(x))ds dx
\end{aligned}$$

*for every $k \in \boldsymbol{R}$ and every $f \in C_0^\infty(\boldsymbol{R}^d)$ with $f \geqq 0$.*

PROOF. We follow the argument of the proof of Lemma A in [4]. (See also the proof of Theorem 5.3 in [14].) Choose a positive number $m$ so that $\|u\|_\infty \leqq m$ and the support of $p'$ is contained in the open interval $(-m, m)$. Then we have

$$p(m) + p(-m) = 0, \tag{3.3}$$

since $p(+\infty)+p(-\infty)=0$.

Let $k \in \boldsymbol{R}, f \in C_0^\infty(\boldsymbol{R}^d)$ and assume that $f \geqq 0$. Set

$$g(s) = \int_{\boldsymbol{R}^d} \operatorname{sign}(u(x)-s)(A_h u)(x) f(x) dx$$

and

$$h(s) = \int_{\boldsymbol{R}^d} \operatorname{sign}(u(x)-s) \int_s^{u(x)} h^{-1}(f(x+h\phi'(\xi)) - f(x)) d\xi dx$$

for $s \in \boldsymbol{R}$. Since

$$\begin{aligned} &\operatorname{sign}(u(x)-s)(A_h u)(x) \\ &\quad = h^{-1}[((C_h u)(x)-s)\operatorname{sign}(u(x)-s) - (u(x)-s)\operatorname{sign}(u(x)-s)] \\ &\quad \leqq h^{-1}\{|(C_h u)(x)-s| - |u(x)-s|\}, \end{aligned}$$

it follows from Proposition 3.1 that

$$g(s) \leqq h(s) \qquad \text{for all} \quad s \in \boldsymbol{R}.$$

Consequently, we have

$$\int_{-m}^{m} p'(s) g(s) ds \leqq \int_{-m}^{m} p'(s) h(s) ds. \tag{3.4}$$

On the other hand, we have

$$\int_{-m}^{m} p'(s) g(s) ds = \int_{\boldsymbol{R}^d} \left\{ \int_{-m}^{m} p'(s) \operatorname{sign}(u(x)-s) ds \right\} (A_h u)(x) f(x) dx,$$

by Fubini's theorem and

$$\int_{-m}^{m} p'(s) \operatorname{sign}(u(x)-s) ds = \int_{-m}^{u(x)} p'(s) - \int_{u(x)}^{m} p'(s) ds = 2p(u(x))$$

by (3.3). Hence

$$\int_{-m}^{m} p'(s) g(s) ds = 2\int_{\boldsymbol{R}^d} p(u(x))(A_h u)(x) f(x) dx. \tag{3.5}$$

In the same way as above we have

$$(3.6)\qquad \int_{-m}^{m} p'(s)h(s)ds$$

$$= \int_{\boldsymbol{R}^d}\left[\left(\int_{m}^{u(x)} - \int_{u(x)}^{m}\right)p'(s)\left\{\int_{s}^{u(x)} h^{-1}(f(x+h\phi'(\xi))-f(x))d\xi\right\}ds\right]dx.$$

Therefore integration by part yields

$$(3.7)\qquad \int_{-m}^{u(x)} p'(s)\left\{\int_{s}^{u(x)} h^{-1}(f(x+h\phi'(\xi))-f(x))d\xi\right\}ds$$

$$= -p(-m)\int_{-m}^{u(x)} h^{-1}(f(x+h\phi'(\xi))-f(x))d\xi$$

$$+\int_{-m}^{u(x)} p(s)h^{-1}(f(x+h\phi'(s))-f(x))ds$$

and

$$(3.8)\qquad \int_{u(x)}^{m} p'(s)\left\{\int_{s}^{u(x)} h^{-1}(f(x+h\phi'(\xi))-f(x))d\xi\right\}ds$$

$$= p(m)\int_{m}^{u(x)} h^{-1}(f(x+h\phi'(\xi))-f(x))d\xi$$

$$+\int_{u(x)}^{m} p(s)h^{-1}(f(x+h\phi'(s))-f(x))ds.$$

Moreover, observe that

$$\int_{\boldsymbol{R}^d}\left\{\int_{a}^{b} h^{-1}(f(x+h\phi'(\xi))-f(x))d\xi\right\}dx = 0$$

for every $a$, $b\in\boldsymbol{R}$. Hence, the substitution of (3.7) and (3.8) into (3.6) gives

$$(3.9)\qquad \int_{-m}^{m} p'(s)h(s)ds$$

$$= \int_{\boldsymbol{R}^d}\left[-(p(-m)+p(m))\int_{k}^{u(x)} h^{-1}(f(x+h\phi'(\xi))-f(x))d\xi\right.$$

$$\left.+\left(\int_{k}^{u(x)} - \int_{u(x)}^{k}\right)p(s)h^{-1}(f(x+h\phi'(s))-f(x))ds\right]dx$$

$$= 2\int_{\boldsymbol{R}^d}\left[\int_{k}^{u(x)} p(s)h^{-1}(f(x+h\phi'(s))-f(x))ds\right]dx,$$

where we have used (3.3) again. Combining (3.4), (3.5) and (3.9), we obtain the desired inequality (3.2). Q.E.D.

The following result states that

$$\lim_{\rho\to\infty}\sup_{h>0}\int_{|x|>\rho}|(J_{\lambda,h}v)(x)|dx = 0$$

for $v \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and $\lambda > 0$.

PROPOSITION 3.3. *Let $v \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and $\lambda > 0$. Then,*

$$\begin{aligned}(3.10)\qquad & \int_{|x|>\rho} (J_{\lambda,h}v)(x)|dx \\ & \leqq \int_{|x|>r} |v(x)|dx + \lambda(\rho - r)^{-1} M\|v\|_1\end{aligned}$$

for $\rho > r > 0$ and $h > 0$, where $M = \sup\{|\phi'(\xi)|;\ |\xi| \leqq \|v\|_\infty\}$.

PROOF. We follow the argument as in the proof of Lemma 4.3 in [14]. Let $v \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$, $\lambda > 0$, and set

$$u_h = J_{\lambda,h}v \qquad \text{for} \quad h > 0.$$

Then, it follows from Proposition 2.2 that $u_h \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and

$$(3.11)\qquad \|u_h\|_p \leqq \|v\|_p, \quad p = 1, \infty.$$

Let $f$ be a uniformly bounded, nonnegative and Lipschitz continuous function on $\boldsymbol{R}$. Then we see in the same way as in the proof of Proposition 3.1 that the following inequality holds:

$$\begin{aligned}& \int_{\boldsymbol{R}^d} (|(C_h u_h)(x)| - |u_h(x)|) f(x)dx \\ & \leqq \int_{\boldsymbol{R}^d} \operatorname{sign}(u_h(x)) \int_0^{u_h(x)} (f(x + h\phi'(\xi)) - f(x))d\xi dx.\end{aligned}$$

Let Lip $(f)$ denote the smallest Lipschitz constant of $f$. Then,

$$\begin{aligned}& \int_{\boldsymbol{R}^d} \operatorname{sign}(u_h(x)) \int_0^{u_h(x)} (f(x + h\phi'(\xi)) - f(x))d\xi dx \\ & \leqq h \operatorname{Lip}(f) \int_{\boldsymbol{R}^d} \left| \int_0^{u_h(x)} |\phi'(\xi)|d\xi \right| dx \\ & \leqq hM \operatorname{Lip}(f) \int_{\boldsymbol{R}^d} |u_h(x)|dx \\ & \leqq hM \operatorname{Lip}(f)\|v\|_1,\end{aligned}$$

where we have used (3.11). Therefore, we have

$$(3.12)\qquad \int_{\boldsymbol{R}^d} (|(C_h u_h)(x)| - |u_h(x)|) f(x)dx \leqq hM \operatorname{Lip}(f)\|v\|_1.$$

On the other hand, the relation (2.1) implies that

$$|u_h(x)| \leqq h(\lambda + h)^{-1}|v(x)| + \lambda(\lambda + h)^{-1}|(C_h u_h)(x)|$$

or

$$h(|u_h(x)|-|v(x)|) \leqq |(C_h u_h)(x)| - |u_h(x)|.$$

Combining this with (3.12) yields

$$\int_{\boldsymbol{R}^d} |u_h(x)|f(x)dx \leqq \int_{\boldsymbol{R}^d} |v(x)|f(x)dx + M\,\mathrm{Lip}\,(f)\|v\|_1. \tag{3.13}$$

Let $\rho > r > 0$ and let $\delta^{r,\rho}$ be a function on $[0, \infty)$ such that

$$\delta^{r,\rho}(s) = \begin{cases} 0 & \text{if } \ 0 \leqq s < r \\ (\rho - r)^{-1}(s-r) & \text{if } \ r \leqq s < \rho, \\ 1 & \text{otherwise.} \end{cases}$$

Set

$$f^{r,\rho}(x) = \delta^{r,\rho}(|x|) \qquad \text{for} \quad x \in \boldsymbol{R}^d.$$

Since

$$0 \leqq f^{r,\rho}(x) \leqq 1 \quad \text{and} \quad |f^{r,\rho}(x) - f^{r,\rho}(y)| \leqq (\rho - r)^{-1}|x-y|$$

for $x, y \in \boldsymbol{R}^d$, the substitution $f = f^{r,\rho}$ into (3.13) now yields the desired estimate (3.10). Q.E.D.

## 4. Proof of theorem

In this section, we give the proof of our main theorem. Assertion (i) of Proposition 2.1 states that the stability condition $(\mathrm{d}_1)$ holds. Hence it remains to prove the consistency condition $(d_2)$. For this purpose, we prepare the following lemma.

LEMMA 4.1. *Let $v \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and $\lambda > 0$. Let $u_h = J_{\lambda,h}v$ for $h > 0$. Then we have:*

(i) *The set $\{u_h;\ h > 0\}$ is precompact in $L^1(\boldsymbol{R}^d)$.*

(ii) *If $\{h(n)\}$ is a null sequence such that $u_{h(n)}$ converges a.e. to a limit $u \in L^1(\boldsymbol{R}^d)$ as $n \to \infty$, then $u \in D(A_0)$ and $\lambda^{-1}(u-v) = A_0 u$.*

PROOF. Firstly Proposition 2.2 (i) states that

$$\sup_{h>0} \|u_h\|_1 \leqq \|v\|_1. \tag{4.1}$$

Secondly Proposition 2.2 (i) and (ii) together imply

$$\begin{aligned} \|\tau^y u_h - u_h\|_1 &= \|J_{\lambda,h}\tau^y v - J_{\lambda,h} v\|_1 \\ &\leqq \|\tau^y v - v\|_1 \end{aligned}$$

for $h>0$ and $y \in \boldsymbol{R}^d$. Hence

$$\sup_{h>0} \|\tau^y u_h - u_h\|_1 \longrightarrow 0 \quad \text{as} \quad y \longrightarrow 0. \tag{4.2}$$

Furthermore, Proposition 3.3 implies that

$$\sup_{h>0} \int_{|x|>\rho} |u_h(x)|dx \longrightarrow 0 \quad \text{as} \quad \rho \longrightarrow \infty. \tag{4.3}$$

In view of (4.1), (4.2) and (4.3), the Fréchet-Kolmogorov theorem can be applied to imply the first assertion (i).

It now remains to prove (ii). Let $\{h(n)\}$ be a null sequence such that $u_{h(n)}$ converges a.e. to some limit $u \in L^1(\boldsymbol{R}^d)$. By Proposition (iii),

$$\|u_h\|_\infty \leqq \|v\|_\infty \qquad \text{for} \quad h > 0. \tag{4.4}$$

Hence, $u \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ and $\|u\|_\infty \leqq \|v\|_\infty$. Let $k \in \boldsymbol{R}$ and take $f \in C_0^\infty(\boldsymbol{R}^d)$ with $f \geqq 0$. Let $p \in P$. Inserting $u_h$ into $u$ on (3.2) yields

$$\begin{aligned} &\int_{\boldsymbol{R}^d} p(u_h(x))(A_h u_h)(x) f(x) dx \\ &\leqq \int_{\boldsymbol{R}^d} \left\{ \int_k^{u_h(x)} p(s) h^{-1}(f(x+h\phi'(s)) - f(x)) ds \right\} dx. \end{aligned} \tag{4.5}$$

Notice that $\{\|u_h\|_\infty\}$ is uniformly bounded in $h$ by (4.4), and that

$$A_h u_h = \lambda^{-1}(u_h - v)$$

by (2.2). Hence, putting $h = h(n)$ in (4.5) and letting $n$ tend to the infinity in the resultant inequality, we have

$$\begin{aligned} &\int_{\boldsymbol{R}^d} p(u(x)) \lambda^{-1}(u(x) - v(x)) f(x) dx \\ &\leqq \int_{\boldsymbol{R}^d} \left\{ \int_k^{u(x)} p(s) \phi'(s) \cdot f_x(x) ds \right\} dx \end{aligned} \tag{4.6}$$

by use of the Lebesgue convergence theorem. We then set

$$p_\ell(s) = \begin{cases} -1 & \text{if} \quad s \leqq -1/\ell \\ \ell s & \text{if} \quad |s| < 1/\ell, \\ 1 & \text{if} \quad s \geqq 1/\ell, \end{cases}$$

for $\ell = 1, 2, \ldots$. Choose $p(s) \equiv p_\ell(s-k)$ as the function $p$ on (4.6) and let $\ell$ tend to the infinity. Then we have

$$\begin{aligned}&\int_{\boldsymbol{R}^d} \operatorname{sign}(u(x)-k)\lambda^{-1}(u(x)-v(x))f(x)dx\\&\leqq \int_{\boldsymbol{R}^d}\int_k^{u(x)} \operatorname{sign}(s-k)\phi'(s)\cdot f_x(x)dsdx\\&= \int_{\boldsymbol{R}^d} \operatorname{sign}(u(x)-k)(\phi(u(x))-\phi(k))\cdot f_x(x)dx.\end{aligned}$$

This shows that $u\in D(A_0)$ and $\lambda^{-1}(u-v)=A_0u$. Q.E.D.

We can now prove the consistency condition $(d_2)$. Let $v\in L^1(\boldsymbol{R}^d)$ and $\lambda>0$. Choose a sequence $v_k\in L^1(\boldsymbol{R}^d)\cap L^\infty(\boldsymbol{R}^d)$ so that $v_k\to v$ in $L^1(\boldsymbol{R}^d)$ as $k\to\infty$. Set

$$u_h = J_{\lambda,h}v \quad \text{and} \quad u_{h,k} = J_{\lambda,h}v_k$$

for $h>0$ and $k=1, 2,\dots$. By (i) of Lemma 4.1, there exists a null sequence $\{h(n)\}$ such that, for each $k$, $u_{h(n),k}$ converges a.e. and in $L^1(\boldsymbol{R}^d)$ to some limit $u_k\in L^1(\boldsymbol{R}^d)$ as $n\to\infty$. Then, it follows from Lemma 4.1 (ii) that

$$u_k\in D(A_0) \quad \text{and} \quad \lambda^{-1}(u_k-v_k) = A_0u_k \tag{4.7}$$

for $k=1, 2,\dots$. By (i) of Proposition 2.2, we have

$$\begin{aligned}\|u_k-u_j\|_1 &= \lim_{n\to\infty} \|J_{\lambda,h(n)}v_k-J_{\lambda,h(n)}v_j\|_1\\&\leqq \|v_k-v_j\|_1\end{aligned}$$

for $k, j=1, 2,\dots$. Hence, there exists $u\in L^1(\boldsymbol{R}^d)$ such that $u_k\to u$ in $L^1(\boldsymbol{R}^d)$ as $k\to\infty$. Since $A$ is the closure of $A_0$, it follows from (4.7) that $u\in D(A)$ and $\lambda^{-1}(u-v)\in Au$. Obviously, this implies that $u+\lambda Au\ni v$ and $u=(I-\lambda A)^{-1}v$. By Proposition 2.2 (i), we also have

$$\|u_h-u_{h,k}\|_1 \leqq \|v-v_k\|_1.$$

Hence

$$\|u_h-u\|_1 \leqq \|u-u_{h,k}\|_1 + \|v-v_k\|_1.$$

Let $h=h(n)$ and let $n$ tend to the infinity. Then,

$$\limsup_{n\to\infty} \|u_{h(n)}-u\|_1 \leqq \|u-u_k\|_1 + \|v-v_k\|_1$$

for $k=1, 2,\dots$. Consequently, $u_{h(n)}\to u=(I-\lambda A)^{-1}v$ in $L^1(\boldsymbol{R}^d)$ as $n\to\infty$. Since the limit is uniquely determined by $v$, we can conclude that $u_h$ itself converges to $(I-\lambda A)^{-1}v$ in $L^1(\boldsymbol{R}^d)$ as $h\downarrow 0$. Thus the proof of the Theorem is completed.

In the above proof of the Theorem we did not use the fact that operator A satisfies $(c_2)$, although we proved it. Thus, we have the following result due to Crandall [4].

COROLLARY 4.1. *Let* $v \in L^1(\boldsymbol{R}^d)$ *and* $\lambda>0$. *Then there exists* $u \in D(A)$ *such that* $u-\lambda Au \in v$.

As we observed before, it is known that $u(t, x)=(T(t)u_0)(x)$ is an entropy solution of (CP) if $u_0 \in L^1(\boldsymbol{R}^d) \cap L^1(\boldsymbol{R}^d)$. We here show it through the product formula (0.1).

COROLLARY 4.2. *Let* $\{T(t)\}_{t\geqq 0}$ *be the semigroup determined by* (1.3). *Let* $u_0 \in L^1(\boldsymbol{R}^d) \cap L^\infty(\boldsymbol{R}^d)$ *and let* $u(t, x)=(T(t))(x)$. *Then,* $u(t, x)$ *is the entropy solution of (CP) with initial value* $u_0$.

PROOF. Set

$$u_h(t, x) = (C_h^{[t/h]}u_0)(x) \qquad \text{for} \quad (t, x) \in (0, \infty) \times \boldsymbol{R}^d.$$

Then, it follows from the Theorem that $u_h(t, \cdot)$ converges to $u(t, \cdot)$ in $L^1(\boldsymbol{R}^d)$ as $h \downarrow 0$. Using (iii) of Proposition 2.1, we see that

$$\|u_h(t, \cdot)\|_\infty \leqq \|u_0\|_\infty.$$

Therefore, $u(t, \cdot) \in L^\infty(\boldsymbol{R}^d)$ and $\|u(t, \cdot)\|_\infty \leq \|u_0\|_\infty$ for $t \geqq 0$. Furthermore, $t \to T(t)u_0$ is continuous on $[0, \infty)$ into $L^1(\boldsymbol{R}^d)$ and so condition $(a_2)$ is satisfied. It remains to check condition $(a_3)$. Let $k \in \boldsymbol{R}$ and $f \in C_0^\infty((0, \infty) \times \boldsymbol{R}^d)$ with $f \geqq 0$. Notice that

$$u_h(t+h, x) = (C_h u_h(t, \cdot))(x).$$

Hence, Proposition 3.1 implies that

$$\begin{aligned}&\int_{\boldsymbol{R}^d} h^{-1}(|u_h(t+h, x)-k|-|u_h(t, x)-k|)f(t, x)dx\\ &\leqq \int_{\boldsymbol{R}d} \operatorname{sign}(u_h(t, x)-k)\int_k^{u_h(t,x)} h^{-1}(f(t, x+h\phi'(\xi))-f(t, x))d\xi dx.\end{aligned}$$

Set $f(t, x)=0$ for $x \in \boldsymbol{R}^d$ and $t \leqq 0$. Integrating both sides of the above inequality over $0<t<\infty$ and using a change of variables, we have

$$\begin{aligned}(4.8)\quad &\int_0^\infty\int_{\boldsymbol{R}^d} |u_h(t, x)-k|h^{-1}(f(t-h, x)-f(t, x))dxdt\\ &\leqq \int_0^\infty\int_{\boldsymbol{R}^d} \operatorname{sign}(u_h(t, x)-k)\int_k^{u_h(t,x)} h^{-1}(f(t, x+h\phi'(\xi))-f(t, x))d\xi dxdt.\end{aligned}$$

Let $\{h(n)\}$ be a null sequence such that $u_{h(n)}(t, x)$ converges a.e. to $u(t, x)$ as $n \to \infty$. Put $h=h(n)$ in (4.8) and let $n$ tend to the infinity in the resultant inequality. Then the Lebesgue convergence theorem yields

$$-\int_0^\infty\int_{\boldsymbol{R}^d}|u(t,x)-k|f_t(t,x)dxdt$$
$$\leqq\int_0^\infty\int_{\boldsymbol{R}^d}\operatorname{sign}(u(t,x)-k)\int_k^{u(t,x)}\phi'(\xi)f_x(t,x)d\xi dxdt,$$

from which the inequality (1.1) follows. Q.E.D.

As mentioned before, it is proved in [9] that there exists a unique entropy solution $u$ of $(CP)$ even if initial value $u_0$ lies in $L^\infty(\boldsymbol{R}^d)$. By virtue of the hyperbolic nature of $(CP)$, the Theorem can be used to construct for $u_0\in L^\infty(\boldsymbol{R}^d)$ the entropy solution $u$ of $(CP)$ via the iteration scheme (0.1). In fact, we have the following corollary, which precisely gives an answer to the problem proposed by Giga and Miyakawa [7]. In the remainder part of this paper, let $C_hu$ be the function defined by (0.2) for $u\in L^\infty(\boldsymbol{R}^d)$ and $h>0$.

COROLLARY 4.3. *Let $u_0\in L^\infty(\boldsymbol{R}^d)$. Then there exists a function $u(t,\cdot)$ on $[0,\infty)$ into $L^\infty(\boldsymbol{R}^d)$ such that, for $r>0$ and $T>0$,*

$$\lim_{h\downarrow 0}\int_{|x|\leqq r}|u(t,x)-(C_h^{[t/h]}u_0)(x)|dx=0, \tag{4.9}$$

*uniformly in $t\in[0,T]$ and the function $u$ is an entropy solution of $(CP)$ with initial value $u_0$.*

For the proof, we first show a few properties of the operator $C_h$ on $L^\infty(\boldsymbol{R}^d)$ which reflect the hyperbolic nature of the problem. (See also Lemma 2.1 and Lemma 2.2 in [7].)

PROPOSITION 4.1. *Let $h>0$. Then:*

(i) *$C_h$ is an operator on $L^\infty(\boldsymbol{R}^d)$ into itself and*

$$\|C_h\|_\infty\leq\|u\|_\infty\qquad\text{for}\quad u\in L^\infty(\boldsymbol{R}^d). \tag{4.10}$$

(ii) *If $u,v\in L^\infty(\boldsymbol{R}^d)$ and $M\geqq\sup\{|\phi'(\xi)|;|\xi|\leqq\max(\|u\|_\infty,\|v\|_\infty)\}$, then*

$$\int_{|x|>r}|(C_hu)(x)-(C_hv)(x)|dx\leqq\int_{|x|\leqq r+hM}|u(x)-v(x)|dx \tag{4.11}$$

*for any $r>0$.*

PROOF. The assertion (i) can be shown in the same way as the proof of Proposition 2.1 (iii). Let $u,v\in L^\infty(\boldsymbol{R}^d)$ and $m=\max(\|u\|_\infty,\|v\|_\infty)$. Let $M\geqq\sup\{|\phi'(\xi)|;|\xi|\leqq m\}$ and $r>0$. Then, by (iii) of Proposition 1.1,

$$\int_{|x|\leqq r}|(C_h u)(x)-(C_h v)(x)|dx$$
$$\leqq \int_{|x|\leqq r}\left\{\int_{\boldsymbol{R}}|F(u(x-h\phi'(\xi)),\, \xi)-F(v(x-h\phi'(\xi)),\, \xi)|d\xi\right\}dx$$
$$=\int_{|x|\leqq r}\left\{\int_{|\xi|\leqq m}|F(u(x-h\phi'(\xi)),\, \xi)-F(v(x-h\phi'(\xi)),\, \xi)|d\xi\right\}dx.$$

Hence, the application of Fubini's theorem yields

$$\int_{|x|\leqq r}|(C_h u)(x)-(C_h v)(x)|dx$$
$$\leqq \int_{|\xi|\leqq m}\left\{\int_{|x|\leqq r}|F(u(x-h\phi'(\xi)),\, \xi)-F(v(x-h\phi'(\xi)),\, \xi)|dx\right\}d\xi$$
$$\leqq \int_{|\xi|\leqq m}\left\{\int_{|x|\leqq r+hM}|F(u(x),\, \xi)-F(v(x),\, \xi)|dx\right\}d\xi$$
$$=\int_{|x|\leqq r+hM}\left\{\int_{|\xi|\leqq m}|F(u(x),\, \xi)-F(v(x),\, \xi)|dx\right\}d\xi.$$

We now apply (iii) and (iv) of Proposition 1.1 to get (4.10). Q. E. D.

Proof of Corollary 4.3. Let $u_0 \in L^\infty(\boldsymbol{R}^d)$ and $M=\sup\{|\phi'(\xi)|;\ |\xi|\leqq \|u_0\|_\infty\}$. For each $r>0$ and $T>0$, define a function $u_0^{r,T}$ on $\boldsymbol{R}^d$ by

$$u_0^{r,T}(x)=\begin{cases} u_0(x) & \text{if } |x|\leqq r+TM, \\ 0 & \text{otherwise.}\end{cases}$$

Obviously, $u_0^{r,T}\in L^1(\boldsymbol{R}^d)\cap L^\infty(\boldsymbol{R}^d)$ and $\|u_0^{r,T}\|_\infty \leqq \|u_0\|_\infty$ for $r>0$ and $T>0$. Therefore, (4.9) implies that

$$\|C_h^n u_0\|_\infty \leqq \|u_0\|_\infty \quad \text{and} \quad \|C_h^n u_0^{r,T}\|_\infty \leqq \|u_0^{r,T}\|_\infty \leqq \|u^0\|_\infty \tag{4.12}$$

for $r$, $T$, $h>0$ and $n=1, 2, \ldots$. Hence, using (ii) of Proposition 4.1 inductively, we have

$$\int_{|x|<r}|(C_h^{[t/h]}u_0)(x)-(C_h^{[t/h]}u_0^{r,T})(x)|dx \tag{4.13}$$
$$\leqq \int_{|x|\leqq r+[t/h]hM}|u_0(x)-u_0^{r,T}(x)|dx=0$$

for $t\in[0, T]$ and $r$, $T$, $h>0$.

Let $\{T(t)\}_{t\geqq 0}$ be the semigroup on $L^1(\boldsymbol{R}^d)$ constructed through (1.3). Since $u_0^{r,T}\in L^1(\boldsymbol{R}^d)$, the Theorem implies that

$$\sup\nolimits_{t\in[0,T]}\int_{|x|>r}|(C_h^{[t/h]}u_0^{r,T})(x)-(T(t)u_0^{r,T})(x)|dx \longrightarrow 0 \quad as \quad h\downarrow 0$$

for $r$, $T>0$. Hence we infer from (4.12) that

$$\sup_{t\in[0,T]}\int_{|x|<r}|(C_h^{[t/h]}u_0)(x)-(T(t)u_0^{r,T})(x)|dx \longrightarrow 0 \quad as \quad h\downarrow 0 \tag{4.14}$$

for $r$, $T>0$. Therefore, in view of (4.11), we see that there exits a function $u(t,\cdot)$ on $[0, \infty)$ into $L^\infty(\boldsymbol{R}^d)$ which satisfies (4.9) and $\|u(t,\cdot)\|_\infty \leqq \|u_0\|_\infty$. Furthermore, (4.14) implies that for each fixed $r$, $T>0$ and each $t\in[0, T]$,

$$u(t, x) = (T(t)u_0^{r,T})(x) \tag{4.15}$$

for a.a. $x\in\boldsymbol{R}^d$ with $|x|<r$. Since $T(0)u_0^{r,T}=u_0^{r,T}$ and $T(t)u_0^{r,T}$ is continuous in $t\in[0, \infty)$ with respect to the norm $\|\cdot\|_1$, we see that $(a_3)$ holds for the function $u(t, x)$.

Let $k\in\boldsymbol{R}$ and $f\in C_0^\infty((0, \infty)\times\boldsymbol{R}^d)$ with $f\geqq 0$. Then Corollary 4.2 states that the inequality (1.1) holds for the function $u^{r,T}(t, x)=(T(t)u_0^{r,T})(x)$. Choose $r>0$ and $T>0$ so that the support of $f$ is contained in the set $(0, T)\times\{x\in\boldsymbol{R}^d; |x|\leqq r\}$. Then (4.15) implies $u(t, x)=u^{r,T}(t, x)$ for $(t, x)$ belonging to the support of $f$, and consequently the inequality (1.1) holds for the function $u$. Thus the function $u(t, x)$ is an entropy solution of $(CP)$ with the initial value $u_0$. Q. E. D.

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