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Two-step methods with one off-step node

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1. Introduction

Consider the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{1.1}$$

where the function $f(x, y)$ is assumed to be sufficiently smooth. Let $y(x)$ be the solution of this problem and

$$x_n = x_0 + nh \quad (n=1, 2,\dots; h>0), \tag{1.2}$$

where h is a stepsize. Let y_1 be an approximation of $y(x_1)$ obtained by some appropriate method. We are concerned with the case where the approximations y_j $(j=2, 3,\dots)$ of $y(x_j)$ are computed by two-step methods. Conventional explicit two-step methods such as linear two-step methods [1] and pseudo-Runge-Kutta methods [1, 2] compute y_j $(j=2, 3,\dots)$ with starting values y_0 and y_1. Methods of order at most $k+2$ $(k=2, 3, 4)$ have been found for k function evaluations per step [1, 2, 3].

In this paper, to achieve the order $k+3$ at the cost of providing an additional starting value y_v, we introduce off-step nodes

$$x_{n+v} = x_0 + (n+v)h \quad (n=0, 1,\dots; 0<v<1) \tag{1.3}$$

and propose a method for computing approximations y_{n+v} and y_{n+1} $(n=1, 2,\dots)$ of $y(x_{n+v})$ and $y(x_{n+1})$ respectively. Let

$$y_{n+1} = y_n + s(y_n - y_{n-1}) + h \textstyle\sum_{j=0}^{r} p_j k_j, \tag{1.4}$$

$$y_{n+v} = y_n + b_{r0}(y_n - y_{n-1}) + b_{r1}(y_n - y_{n-1+v}) + b_{r2}(y_{n+1} - y_n) + h \textstyle\sum_{j=0}^{r-1} c_{rj} k_j, \tag{1.5}$$

where

$$k_0 = f(x_{n-1}, y_{n-1}), \quad k_1 = f(x_{n-1+v}, y_{n-1+v}), \quad k_2 = f(x_n, y_n), \tag{1.6}$$

$$k_i = f(x_n + a_i h,\ y_n + b_{i0}(y_n - y_{n-1}) + b_{i1}(y_n - y_{n-1+v}) + b_{i2}(y_{n+1} - y_n) + h \textstyle\sum_{j=0}^{i-1} c_{ij} k_j), \tag{1.7}$$

$$a_i = b_{i0} + (1-v)b_{i1} + b_{i2} + \textstyle\sum_{j=0}^{i-1} c_{ij}, \quad 0 < a_i \leqq 1 \quad (i=3, 4,\dots, r) \tag{1.8}$$

and a_i, b_{ik}, c_{ij} $(i=3, 4,\dots, r;\ k=0, 1, 2;\ j=0, 1,\dots, i-1)$, p_j $(j=0, 1,\dots, r)$ and s are constants. The method (1.4)–(1.5) is called an explicit method if $b_{i2}=0$ $(i=3, 4,\dots, r)$ and an implicit one otherwise. The explicit method requires $r-1$ function evaluations per step. The stepsize control is implemented by comparing (1.4) with the method

$$z_{n+1} = y_n + z(y_n - y_{n-1}) + h \sum_{j=0}^{r+1} w_j k_j, \tag{1.9}$$

where $k_{r+1} = f(x_{n+1}, y_{n+1})$ and w_j $(j=0, 1,\dots, r+1)$ and z are constants.

It is shown that for $r=3, 4$ there exist an explicit method (1.4) of order $r+2$, a method (1.5) of order $r+1$ and a method (1.9) of order $r+1$ with $w_{r+1}=0$ and that for $r=3, 4$ there exist an implicit method (1.4) of order $r+3$, a method (1.5) of order $r+2$ and a method (1.9) of order $r+2$. Predictors for implicit methods are constructed. The implicit method (1.4) can be used also as an explicit three- or four-step method of order $r+3$ with $r-1$ function evaluations if y_{n+1} is predicted with sufficient accuracy and the corrector is applied only once per step.

2. Preliminaries

Let

$$y_{n+1} = y_n + s(y_n - y_{n-1}) + h \sum_{j=0}^{r} p_j k_{jn} \quad (r=3, 4), \tag{2.1}$$

$$y_{n+v} = y_n + b_{r0}(y_n - y_{n-1}) + b_{r1}(y_n - y_{n-1+v}) + b_{r2}(y_{n+1} - y_n) + h \sum_{j=0}^{r-1} c_{rj} k_{jn}, \tag{2.2}$$

$$t_{n+1} = u(y_n - y_{n-1}) + h \sum_{j=0}^{r+1} v_j k_{jn}, \tag{2.3}$$

$$y^*_{n+1} = y_n + t(y_n - y_{n-1}) + h\left[\sum_{i=0}^{3} q_i k_{2n-3+i} + \sum_{j=0}^{2} q_{j+4} k_{rn-3+j} + \delta_{4r} \sum_{j=0}^{1} q_{7+j} k_{3n-2+j}\right], \tag{2.4}$$

$$z_{n+1} = y_{n+1} + t_{n+1}, \tag{2.5}$$

where δ_{ij} is Kronecker's delta, $v=a_r$ and

$$k_{0n} = k_{2n-1}, \quad k_{1n} = k_{rn-1}, \quad k_{r+1n} = k_{2n+1}, \quad k_{2n} = f(x_n, y_n), \tag{2.6}$$
$$k_{in} = f(x_n + a_i h,\ y_n + b_{i0}(y_n - y_{n-1}) + b_{i1}(y_n - y_{n-1+v}) + b_{i2}(y_{n+1} - y_n) + h \sum_{j=0}^{i-1} c_{ij} k_{jn}) \quad (3 \leqq i \leqq r),$$

$$a_i = b_{i0} + (1-v)b_{i1} + b_{i2} + \sum_{j=0}^{i-1} c_{ij}, \quad 0 < a_i \leqq 1. \tag{2.7}$$

From (2.1) and (2.2) we have

$$y_{n+1+\sigma} = (s+w+1)y_{n+\sigma} - (s+w+sw)y_{n-1+\sigma} + swy_{n-2+\sigma} + h\Phi_\sigma(x_n, y_{n-2}, y_{n-1}, y_n, y_{n+1}, y_{n+2}, y_{n-1+v}, y_{n+v}; h) \quad (\sigma=1, v), \tag{2.8}$$

where $w = -b_{r1}$. Hence the method (2.1)–(2.2) is stable if

$$-1 < s < 1, \quad -1 < w < 1. \tag{2.9}$$

Denote by $y(x)$ the solution of (1.1) and let

$$a_0 = -1,\ a_1 = a_r - 1, \quad a_2 = 0, \quad a_{r+1} = 1, \tag{2.10}$$

$$y(x) + s(y(x) - y(x-h)) + h\sum_{j=0}^{r} p_j y'(x+a_jh) - y(x+h) = \sum_{j=1}^{8} S_j(h^j/j!)y^{(j)}(x) + O(h^9), \tag{2.11}$$

$$u(y(x) - y(x-h)) + h\sum_{j=0}^{r+1} v_j y'(x+a_jh) = \sum_{j=1}^{8} U_j(h^j/j!)y^{(j)}(x) + O(h^9), \tag{2.12}$$

$$\begin{aligned} y(x) &+ t(y(x) - y(x-h)) + h\sum_{j=0}^{3} q_j y'(x+(j-3)h) \\ &+ h\sum_{j=0}^{2} q_{j+4} y'(x+(a_r+j-3)h) \\ &+ \delta_{4r} h \sum_{j=0}^{1} q_{7+j} y'(x+(a_3+j-2)h) - y(x+h) \\ &= \sum_{j=1}^{8} T_j(h^j/j!)y^{(j)}(x) + O(h^9), \end{aligned} \tag{2.13}$$

$$\begin{aligned} y(x) &+ \sum_{j=0}^{1} b_{ij}(y(x) - y(x+a_jh)) + b_{i2}(y(x+h) - y(x)) \\ &+ h\sum_{j=0}^{i-1} c_{ij} y'(x+a_jh) - y(x+a_ih) = \sum_{j=1}^{8} e_{ij}(h^j/j!)y^{(j)}(x) + O(h^9). \end{aligned} \tag{2.14}$$

Then we have

$$(-1)^{k-1}s + k\sum_{j=0}^{r} a_j^{k-1}p_j - 1 = S_k \quad (k=1, 2, \ldots, 8), \tag{2.15}$$

$$(-1)^{k-1}u + k\sum_{j=0}^{r+1} a_j^{k-1}v_j = U_k, \tag{2.16}$$

$$\begin{aligned} (-1)^{k-1}t &+ k\sum_{i=0}^{3} (i-3)^{k-1}q_i + k\sum_{j=0}^{2} (a_r - 3 + j)^{k-1} q_{4+j} \\ &+ \delta_{4r} k \sum_{i=0}^{1} (a_3 - 2 + i)^{k-1} q_{7+i} - 1 = T_k, \end{aligned} \tag{2.17}$$

$$(-1)^{k-1}b_{i0} - a_1^k b_{i1} + b_{i2} + k\sum_{j=0}^{i-1} a_j^{k-1} c_{ij} - a_i^k = e_{ik}, \tag{2.18}$$

Let

$$k_{in}^* = y'(x_n + a_ih) \quad (i=0, 1, 2, r+1), \tag{2.19}$$

$$\begin{aligned} k_{in}^* &= f(x_n + a_ih,\ y(x_n) + \sum_{j=0}^{1} b_{ij}(y(x_n) - y(x_n + a_jh)) + b_{i2}(y(x_{n+1}) - y(x_n)) \\ &\quad + h\sum_{j=0}^{i-1} c_{ij}k_{jn}^*) \quad (3 \leqq i \leqq r), \end{aligned} \tag{2.20}$$

$$g(x) = f_y(x, y(x)), \tag{2.21}$$

$$T(x_n) = y(x_n) + s(y(x_n) - y(x_{n-1})) + h\sum_{j=0}^{r} p_j k_{jn}^* - y(x_{n+1}), \tag{2.22}$$

$$R(x_n) = u(y(x_n) - y(x_{n-1})) + h\sum_{j=0}^{r+1} v_j k_{jn}^*, \tag{2.23}$$

$$\begin{aligned} T^*(x_n) &= y(x_n) + t(y(x_n) - y(x_{n-1})) + h\sum_{i=0}^{3} q_i k_{2n-3+i}^* \\ &+ \sum_{j=0}^{2} q_{4+j} k_{rn-3+j}^* + \delta_{4r} h \sum_{i=0}^{1} q_{7+i} k_{3n-2+i}^* - y(x_{n+1}), \end{aligned} \tag{2.24}$$

(2.25) $T_v(x_n) = y(x_n) + \sum_{i=0}^{1} b_{ri}(y(x_n) - y(x_n + a_i h)) + b_{r2}(y(x_{n+1}) - y(x_n))$
$+ h \sum_{j=0}^{r-1} c_{rj} k_{jn}^* - y(x_{n+v})$,

(2.26) $F_{k+1} = \sum_{i=3}^{r} e_{ik} p_i,\ G_{k+2} = \sum_{i=3}^{r} a_i e_{ik} p_i,\ H_{k+2} = c_{43} e_{3k} p_4,$
$K_{k+1} = \sum_{i=3}^{r} e_{ik} v_i,\ M_{k+2} = \sum_{i=3}^{r} a_i e_{ik} v_i,\ N_{k+2} = c_{43} e_{3k} v_4,$
$J_{k+1} = e_{3k}(q_7 + q_8),\ L_{k+1} = \sum_{j=3}^{r} c_{rj} e_{jk} \quad (k=5, 6, 7)$,

(2.27) $A_i = a_i(a_i+1),\ B_i = (a_i - a_1)A_i,\ C_i = (a_i - a_3)B_i,\ D_i = a_i(a_i - 1)$,
$E_i = (a_i + 1)D_i \quad (i=1, 2, 3, 4)$,

(2.28) $d = 2a_1 + 1,\ d_1 = 5a_1^2 - a_1 - 2,\ d_2 = 6a_1^2 - a_1 - 3,$
$m = 3a_3^2 d_1 - a_3(3a_1+1)(a_1-1) - d_2,$
$u_1 = 7a_3^3 - 5(a_1+1)a_3^2 + (a_1-4)a_3 + 2a_1,$

(2.29) $m_i = a_i d_1 - a_1 d_2,\ w_i = 3a_i^2 + 2a_i(1-a_1) - a_1,$
$r_i = a_i(3a_i+4) - 2a_1(2a_i+3),\ u_i = 2a_i + a_1(4a_i - 3) - 5a_1^2 \quad (i=3, 4)$,

(2.30) $X = a_1 + a_3,\quad Y = a_1 a_3,\ U = a_1 + a_3 + a_4,\ V = a_1 a_3 + a_1 a_4 + a_3 a_4,$
$W = a_1 a_3 a_4,$

(2.31) $R_1 = 3 + 5X + 10Y,\ R_2 = 2 + 3X + 5Y,\ R_3 = 22 - 27X + 35Y,$
$R_4 = 27 - 35X + 50Y,\ R_5 = 10 + 14X + 21Y,$
$R_6 = 130 - 154X + 189Y,\ R_7 = 15 + 20X + 28Y,$
$R_8 = 225 - 260X + 308Y.$

Choosing $e_{ik}=0$ $(k=1, 2, 3, 4;\ i=3, 4)$, we have

(2.32) $b_{i0} - a_1 b_{i1} + b_{i2} + \sum_{j=0}^{i-1} c_{ij} = a_i,$
$-b_{i0} - a_1^2 b_{i1} + b_{i2} + 2\sum_{j=0}^{i-1} a_j c_{ij} = a_i^2,$
$-b_{i0} - a_1^2(2a_1+3)b_{i1} + 5b_{i2} + 6\sum_{j=1}^{i-1} A_j c_{ij} = a_i^2(2a_i+3)$,
$db_{i0} + a_1^3(a_1+2)b_{i1} + (7-10a_1)b_{i2} + 12B_3 c_{i3} = a_i^2 r_i,$
$A_1^3 b_{i1} - 4d_1 b_{i2} + 2R_1 B_3 c_{i3} - u_i A_i^2 = 2de_{i5},$
$4(a_1-1)^2 b_{i2} + 2w_3 B_3 c_{i3} - B_i^2 = e_{i6} + 2(1-a_1)e_{i5},$
$A_1 E_1^2 b_{i1} + 2mB_3 c_{i3} - A_i E_i m_i = d_1 e_{i6} + 2(1-a_1)(3a_1^2-1)e_{i5},$
$u_1 B_3 c_{i3} - (a_i-1)B_i^2 = e_{i7} + (1-2a_1)e_{i6} \quad (r=4)$,

(2.33) $T(x) = \sum_{j=1}^{8} S_j(h^j/j!)y^{(j)}(x) + \sum_{k=5}^{7} (h^{k+1}/k!)[F_{k+1} g(x) y^{(k)}(x)$
$+ G_{k+2} h g'(x) y^{(k)}(x) + H_{k+2} h g^2(x) y^{(k)}(x) + O(h^2)]$,

(2.34) $R(x) = \sum_{j=1}^{8} U_j(h^j/j!)y^{(j)}(x) + \sum_{k=5}^{7} (h^{k+1}/k!)[K_{k+1}g(x)y^{(k)}(x)$
$+ M_{k+2}hg'(x)y^{(k)}(x) + N_{k+2}hg^2(x)y^{(k)}(x) + O(h^2)]$,

(2.35) $T^*(x) = \sum_{j=1}^{8} T_j(h^j/j!)y^{(j)}(x) + \sum_{j=5}^{7} (h^{k+1}/k!)[J_{k+1}g(x)y^{(k)}(x) + O(h)]$,

(2.36) $T_\nu(x) = \sum_{j=1}^{8} e_{rj}(h^j/j!)y^{(j)}(x) + \sum_{k=5}^{7} (h^{k+1}/k!)[L_{k+1}g(x)y^{(k)}(x) + O(h)]$.

Let

(2.37) $P_1(x) = 5x^2 - 1,\ P_2(x) = 5x^2 + x - 1,\ P_3(x) = 35x^2 - 89x + 49,$
$P_4(x) = 25x^2 - 60x + 31,\ P_5(x) = 21x^2 + 7x - 4,$
$P_6(x) = 189x^2 - 497x + 284,$

(2.38) $Q_0(x) = P_1(x)P_3(x) + P_2(x)P_4(x),\ Q_1(x) = P_5(x)P_4(x) - P_1(x)P_6(x),$
$Q_2(x) = P_2(x)P_6(x) + P_3(x)P_5(x),$

(2.39) $g_1 = 1862a_4^3 - 8736a_4^2 + 12922a_4 - 5641,$
$l_1 = 24598a_4^3 - 150192a_4^2 + 294938a_4 - 187657,$
$g_i = 399a_i^2 - 1071a_i + 577,\ l_i = 5271a_i^2 - 24087a_i + 26177 \quad (i=3, 4).$

The choice $S_i=0$ $(i=1, 2,\ldots, 5)$ yields

(2.40) $\sum_{j=0}^{r} p_i = 1 - s,\ 2\sum_{j=1}^{r} (a_j+1)p_j = 3 - s,\ 6\sum_{j=1}^{r} A_jp_j = 5 + s,$
$12\sum_{j=3}^{r} B_jp_j = 7 - 10a_1 - ds,\ 60C_4p_4 = R_4 + R_1s,$
$60a_4C_4p_4 + R_2s - R_3 = 10S_6,$
$420a_4^2C_4p_4 - R_5s - R_6 = 60S_7 + 70(1-X)S_6,$
$-(R_5+7a_4R_2)s - R_6 + 7a_4R_3 = 60S_7 + 70(1-U)S_6 \quad (r=4),$
$(R_7+2a_4R_5)s - R_8 + 2a_4R_6 = 105S_8 + 120(1-U)S_7 - 140(U-V)S_6.$

Setting $U_j=0$ $(1 \leqq j \leqq r+1)$, we have

(2.41) $\sum_{i=0}^{r+1} v_i = -u,\ \sum_{j=1}^{r+1} (a_j+1)v_j = -u/2,\ A_1v_1 + \sum_{j=3}^{r+1} A_jv_j = u/6,$
$12\sum_{j=3}^{r+1} B_jv_j = -du,\ 60\sum_{j=4}^{r+1} C_jv_j - R_1u = 12U_5,$
$60\sum_{j=4}^{r+1} a_jC_jv_j + R_2u = 10U_6 + 12(1-X)U_5,$
$420\sum_{j=4}^{r+1} a_j^2C_jv_j - R_5u = 60U_7 + 70(1-X)U_6 - 84(X-Y)U_5,$
$-(24+35U+56V+105W)u = 60U_7 - 70UU_6 \quad (r=4),$
$(35+48U+70V+112W)u = 105U_8 - 120UU_7 - 140(1-V)U_6.$

Choosing $T_i=0$ $(1 \leqq i \leqq r+3)$, $q_4+q_5+q_6=0$ and $q_7=-q_8$, we have

(2.42) $\sum_{j=0}^{3} q_j = 1 - t,\ \sum_{j=0}^{2} (j-3)q_j - q_4 + q_6 + \delta_{4r}q_8 = (1+t)/2,$

$3q_0 + q_1 - (a_1-1)q_4 + a_1q_6 + \delta_{4r}(a_3-1)q_8 = (5+t)/12,$

$-2q_0 - D_1q_4 + A_1q_6 + \delta_{4r}D_3q_8 = (9+t)/12,$

$-E_1q_4 + (a_1+2)A_1q_6 + \delta_{4r}E_3q_8 = (251+19t)/120,$

$(a_1+2)A_1q_6 + \delta_{4r}E_3q_8 = [448+16t-a_1(251+19t)]/120,$

$\delta_{4r}2520(a_3-a_4)(a_3-a_4+1)E_3q_8 - g_rt - l_r = 60T_7,$

$-g_1t - l_1 + 840(a_3-a_4)(a_3-a_4+1)(7a_3+14a_4-34)E_3q_8$

$= 105T_8 \quad (r=4).$

3. Explicit methods

In this section we set $b_{i2}=0$ $(i=3, 4)$ and show the following

THEOREM 1. *For* $r=3, 4$ *there exist an explicit method* (2.1) *of order* $r+2$ *and a method* (2.2) *of order* $r+1$ *which embed a method* (2.5) *of order* $r+1$.

3.1. Case r=3

Choosing $S_i=0$ $(i=1, 2,\dots, 5)$, $U_j=e_{3j}=0$ $(j=1, 2, 3, 4)$ and $b_{31}=v_4=0$, we have

(3.1) $db_{30} = a_3^2r_3,\ 6A_1c_{31} - b_{30} = a_3^2(2a_3+3),$

$-2c_{30} + 2(a_3-1)c_{31} - b_{30} = a_3^2,\ c_{30} + c_{31} + c_{32} + b_{30} = a_3,$

(3.2) $P_1(a_3)s = -P_4(a_3),\ 12A_3p_3 = 17 - 10a_3 - ds,\ A_1p_1 + A_3p_3 = (5+s)/6,$

$a_3p_1 + p_2 + (a_3+1)p_3 = (3-s)/2,\quad p_0 + p_1 + p_2 + p_3 = 1 - s,$

(3.3) $12A_3v_3 = -du,\ A_1v_1 + A_3v_3 = u/6,\ a_3v_1 + v_2 + (a_3+1)v_3 = -u/2,$

$v_0 + v_1 + v_2 + v_3 = -u,$

(3.4) $10S_6 = R_2s - R_3,\ 12U_5 = -R_1u,\ 10U_6 = [R_2+(1-X)R_1]u,$

$2de_{35} = -u_3A_3^2.$

For any given $u\neq 0$ and $a_3\neq 1/2$ such that $5a_3^2\neq 1$, other constants are determined uniquely. The method is stable if and only if $(15-\sqrt{65})/10<a_3<1$. For example the choice $a_3=(6-\sqrt{5})/5$ yields $s=0$.

3.2. Case r=4

Setting $S_i=0$ $(i=1, 2,\dots, 6)$, $U_j=e_{ij}=0$ $(j=1, 2,\dots, 5;\ i=3, 4)$ and $v_5=0$, we have

(3.5) $A_1^3b_{31} = u_3A_3^2,\ db_{30} + a_1^3(a_1+2)b_{31} = a_3^2r_3,$

$$-b_{30} - a_1^2(2a_1+3)b_{31} + 6A_1c_{31} = a_3^2(2a_3+3),$$
$$-b_{30} - a_1^2b_{31} - 2c_{30} + 2a_1c_{31} = a_3^2,$$
$$b_{30} - a_1b_{31} + c_{30} + c_{31} + c_{32} = a_3,$$

(3.6) $A_1^3b_{41} + 2R_1B_3c_{43} = u_4A_4^2,\ db_{40} + a_1^3(a_1+2)b_{41} + 12B_3c_{43} = a_4^2r_4,$
$$-b_{40} - a_1^2(2a_1+3)b_{41} + 6A_1c_{41} + 6A_3c_{43} = a_4^2(2a_4+3),$$
$$-b_{40} - a_1^2b_{41} - 2c_{40} + 2a_1c_{41} + 2a_3c_{43} = a_4^2,$$
$$b_{40} - a_1b_{41} + c_{40} + c_{41} + c_{42} + c_{43} = a_4,$$

(3.7) $n_1s = n_2,\ 60C_4p_4 = R_4 + R_1s,\ B_3p_3 + B_4p_4 = (17-10a_4-ds)/12,$
$$A_1p_1 + A_3p_3 + A_4p_4 = (5+s)/6,$$
$$a_4p_1 + p_2 + (a_3+1)p_3 + (a_4+1)p_4 = (3-s)/2,$$
$$p_0 + p_1 + p_2 + p_3 + p_4 = 1 - s,$$

(3.8) $60C_4v_4 = R_1u,\ B_3v_3 + B_4v_4 = -du/12,\ A_1v_1 + A_3v_3 + A_4v_4 = u/6,$
$$a_4v_1 + v_2 + (a_3+1)v_3 + (a_4+1)v_4 = -u/2,$$
$$v_0 + v_1 + v_2 + v_3 + v_4 = -u,$$

(3.9) $60S_7 = -(R_5+7a_4R_2)s - R_6 + 7a_4R_3,\ 10U_6 = (R_2+a_4R_1)u,$
$$60U_7 = -(24+35U+56V+105W)u + 70UU_6,\ e_{36} = -B_3^2,$$
$$e_{46} = 2w_3B_3c_{43} - B_4^2,$$

where

(3.10) $$n_1 = 2a_3P_1(a_4) + P_2(a_4),\ n_2 = P_3(a_4) - 2a_3P_4(a_4).$$

For any given $u \neq 0$, c_{43}, a_3 and $a_4 \neq 1/2$ such that $a_3 \neq a_4$ and $n_1 \neq 0$ other constants are determined uniquely. For instance we have $s=0$ and $60S_7 = -317/550$ for $a_3 = 19/22$ and $a_4 = 2/5$.

4. Implicit methods

In this section we show the following

THEOREM 2. *For $r=3, 4$ there exist an implicit method* (2.1) *of order $r+3$, a method* (2.2) *of order $r+2$ and a method* (2.5) *of order $r+2$.*

This method can be used also as an explicit method if the corrector is applied only once per step.

4.1. Case r=3

Choosing $S_i=0$ $(i=1, 2,\ldots, 6)$, $U_j=e_{3j}=0$ $(j=1, 2,\ldots, 5)$, $q_4+q_5+q_6=0$ and $b_{31}=0$, we have (3.2) and

(4.1) $4d_1b_{32} = -u_3A_3^2,\ db_{30} + (17-10a_3)b_{32} = a_3^2r_3,$

$-2b_{30} + 5b_{32} + 6D_3c_{31} = a_3^2(2a_3+3),$

$-b_{30} + b_{32} - 2c_{30} + 2a_1c_{31} = a_3^2,$

$b_{30} + b_{32} + c_{30} + c_{31} + c_{32} = a_3,$

(4.2) $15a_3^4 - 36a_3^3 + 14a_3^2 + 9a_3 - 4 = 0,$

(4.3) $120(2-a_3)(1-a_3)v_4 = R_1u,\ B_3v_3 + B_4v_4 = -du/12,$

$A_1v_1 + A_3v_3 + 2v_4 = u/6,$

$v_0 + v_1 + v_2 + v_3 + v_4 = -u,$

(4.4) $120E_3q_6 = 699 - 251a_3 + (35-19a_3)t,\ -E_1q_4 + E_3q_6 = (251+19t)/120,$

$-2q_0 - D_1q_4 + D_3q_6 = (9+t)/12,$

$3q_0 + q_1 - (a_1-1)q_4 + a_1q_6 = (5+t)/12,$

$-3q_0 - 2q_1 - q_2 - q_4 + q_6 = (1+t)/2,\ q_0 + q_1 + q_2 + q_3 = 1 - t,$

(4.5) $60S_7 = -R_5s - R_6,\ 10U_6 = (15a_3^2+a_3-3)u,$

$60U_7 = (210a_3^3-147a_3^2-63a_3+32)u,$

$e_{36} = 4(a_1-1)^2b_{32} - B_3^2,\ 60T_7 = -g_3t - l_3.$

For any given $u\neq 0$, t and a_3 satisfying (4.2), other constants are determined uniquely. For instance the choice $a_3=0.7809341293$ yields $s=0.2974663081$.

4.2. Case r=4

Setting $S_i=0$ $(i=1, 2,\dots, 7)$, $U_j=e_{ij}=0$ $(j=1, 2,\dots, 6;\ i=3, 4)$, $q_4+q_5+q_6=0$, $q_7=-q_8$ and $b_{41}=0$, we have (3.7) and

(4.6) $4(a_4-2)^2b_{32} = B_3^2,\ A_1^3b_{31} - 4d_1b_{32} = u_3A_3^2,$

$db_{30} + a_1^3(a_1+2)b_{31} + (17-10a_4)b_{32} = a_3^2r_3,$

$-b_{30} - a_1^2(2a_1+3)b_{31} + 5b_{32} + 6D_4c_{31} = a_3^2(2a_3+3),$

$-b_{30} - a_1^2b_{31} + b_{32} - 2c_{30} + 2a_1c_{31} = a_3^2,$

$b_{30} - a_1b_{31} + b_{32} + c_{30} + c_{31} + c_{32} = a_3,$

(4.7) $2mB_3c_{43} = A_4E_4m_4,\ 4d_1b_{42} - 2R_1B_3c_{43} = -u_4A_4^2,$

$db_{40} + (17-10a_4)b_{42} + 12B_3c_{43} = a_4^2r_4,$

$-b_{40} + 5b_{42} + 6A_1c_{41} + 6A_3c_{43} = a_4^2(2a_4+3),$

$-b_{40} + b_{42} - 2c_{40} + 2a_1c_{41} + 2a_3c_{43} = a_4^2,$

$b_{40} + b_{42} + c_{40} + c_{41} + c_{42} + c_{43} = a_4,$

(4.8) $14Q_0(a_4)a_3^2 + 2Q_1(a_4)a_3 - Q_2(a_4) = 0,$

(4.9) $120(a_3-1)(1-a_4)(2-a_4)v_5 = (R_2+a_4R_1)u,$

$C_4v_4 + 2(1-a_3)(2-a_4)v_5 = R_1u/60,\ \sum_{j=3}^{5} B_jv_j = -(2a_4-1)u/12,$

$A_1v_1 + A_3v_3 + A_4v_4 + A_5v_5 = u/6,\ a_4v_1 + v_2 + \sum_{j=3}^{5}(a_j+1)v = -u/2,$

$\sum_{j=0}^{5} v_j = -u,$

(4.10) $(a_3-a_4)(a_3-a_4+1)E_3q_8 = (g_4t+l_4)/2520,$

$E_4q_6 + E_3q_8 = [699-251a_4+(35-19a_4)t]/120,$

$-E_1q_4 + E_4q_6 + E_3q_8 = (251+19t)/120,$

$-2q_0 - D_1q_4 + D_4q_6 + D_3q_8 = (9+t)/12,$

$3q_0 + q_1 - (a_1-1)q_4 + a_1q_6 + (a_3-1)q_8 = (5+t)/12,$

$-3q_0 - 2q_1 - q_2 - q_4 + q_6 + q_8 = (1+t)/2,$

$q_0 + q_1 + q_2 + q_3 = 1 - t,$

(4.11) $105S_8 = (R_7+2a_4R_5)s - R_8 + 2a_4R_6,$

$60U_7 = -(24+35U+56V+105W)u,$

$105U_8 = (35+48U+70V+112W)u + 120UU_7,$

$e_{47} = u_1B_3c_{43} - (a_4-1)B_4^2,$

$315T_8 = (7a_3+14a_4-34)(g_4t+l_4)-3(g_1t+l_1),\ e_{37} = -(a_3-1)B_3^2.$

For any given $u\neq 0$, t, $a_3\neq 1$ and $a_4\neq 1/2$ satisfying (4.8) such that $m\neq 0$, $d_1\neq 0$ and $a_3\neq a_4$, other constants are determined uniquely. For instance the choice $a_3=0.4574042350$ and $a_4=0.8812655341$ yields $s=0$.

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