HIROSHIMA MATH. J.
15 (1985), 141–143

Birational-integral extensions and differential modules

Mitsuo KANEMITSU and Ken-ichi YOSHIDA

(Received August 30, 1984)

Throughout this paper, a ring will mean a commutative Noetherian ring with identity.

Let R be a Noetherian domain and let $\bar{R}$ be the integral closure of R in its quotient field. An intermediate ring between R and $\bar{R}$ will be called a *birational-integral extension* of R. Let A be a birational-integral extension of R. We assume that A is a finite R-module. Let ${}_{A}^{+}R$ be the seminormalization of R in A. If ${}_{A}^{+}R=A$, then we say that the extension A/R is a *cuspidal type.*

In this paper, we shall prove that a cuspidal type extension is obtained by a finite chain of constant subrings of some derivations.

Let $C={}_{A}^{+}R$ and let I_C be the kernel of the canonical homomorphism

$$\Psi_C\colon C\otimes_R C \longrightarrow C.$$

Then I_C is generated by $\{\alpha\otimes 1-1\otimes\alpha/\alpha\in C\}$ and C/R is a cuspidal type extension. For any ring S, we put $S_{red}=S/nil(S)$ where $nil(S)$ denotes the nilradical of S. Let $\bar{\varphi}_A$ be a module-homomorphism of A to $(A\otimes_R A)_{red}$ over R defined by $\bar{\varphi}_A(\alpha)=\alpha\otimes 1-1\otimes\alpha$ mod $nil(A\otimes_R A)$. In [2], M. Manaresi proved that $\ker\bar{\varphi}_A={}_{A}^{w}R$ where ${}_{A}^{w}R$ is the weak normalization of R in A. In our situation, since $C={}_{C}^{+}R$, we have $C={}_{C}^{w}R$. By this result, each $\alpha\otimes 1-1\otimes\alpha$ $(\alpha\in C)$ is nilpotent and so I_C is nilpotent, say $I_C^{q+1}=(0)$ for some integer q. Then we see that the q-th order differential module $\Omega_R^q(C)=I_C/I_C^{q+1}$ of C over R is isomorphic to I_C and there exists the canonical q-th order derivation Δ_q of C over R to $\Omega_R^q(C)$ defined by $\Delta_q(\alpha)=\alpha\otimes 1-1\otimes\alpha$. We see that $\Delta_q^{-1}(0)$ is a subring of C containing R.

In the paper [1], J. Lipman introduced the following notion: For a ring S and a subring T of S, we say that

$${}_{S}^{*}T=\{\alpha\in S/\alpha\otimes 1=1\otimes\alpha \text{ in } S\otimes_T S\}$$

is the *strict closure* of T in S. If $T={}_{S}^{*}T$, then we say that T is *strictly closed* in S.

Using this notion, we have:

PROPOSITION 1. *Let R, C and Δ_q be as above, and let N be a $C\otimes_R C$-submodule of $\Omega_R^q(C)$ (for example, I_C^t, where t is an integer). Then $\Delta_q^{-1}(N)$ is strictly closed in C.*

PROOF. Since $\alpha\otimes 1\equiv 1\otimes\alpha$ mod N for any $\alpha\in\Delta_q^{-1}(N)$, $(C\otimes_R C)/N$ can be regarded as a $\Delta_q^{-1}(N)$-module. Hence there exists the canonical mapping

$$\sigma\colon C\otimes_{\Delta_q^{-1}(N)}C \longrightarrow (C\otimes_R C)/N.$$

To prove that ${}_C^*\Delta_q^{-1}(N)\subset\Delta_q^{-1}(N)$, take β in ${}_C^*\Delta_q^{-1}(N)$. Then we have $\beta\otimes 1-1\otimes\beta=0$ in $C\otimes_{\Delta_q^{-1}(N)}C$. Hence we have $\sigma(\beta\otimes 1-1\otimes\beta)=0$ in $(C\otimes_R C)/N$. $\Delta_q(\beta)\in N$ in $C\otimes_R C$. So ${}_C^*\Delta_q^{-1}(N)\subset\Delta_q^{-1}(N)$. Clearly $\Delta_q^{-1}(N)\subset{}_C^*\Delta_q^{-1}(N)$. Therefore the proof is complete. q.e.d.

Next, we recall the following

DEFINITION. Let S be a ring and let T be a subring of S. Let $i\colon T\to S$ be the inclusion mapping. We say that the extension S/T is *epimorphic* if, for any pair of ring homomorphisms f_1 and $f_2\colon S\to D$, $f_1 i=f_2 i$ implies $f_1=f_2$.

Then the following results are well known (see [3]).

PROPOSITION 2. *Let S be a ring and let T be a subring of S. Then we have:*
(i) ${}_S^*T=S$ *if and only if the extension S/T is epimorphic.*
(ii) *S/T is epimorphic and integral if and only if $S=T$.*

For convenience, let $C=C_0$ and $C_1=\Delta_q^{-1}(0)$.

PROPOSITION 3. *In the above notation, if $C_0\supsetneqq R$, then $C_0\supsetneqq C_1\supset R$.*

PROOF. If $C=C_0=C_1$, then we have ${}_C^*R=C$. Hence C/R is epimorphic by Proposition 2, (i). Therefore C/R is epimorphic and integral. By Proposition 2, (ii), we have $C_0=C=R$. This is a contradiction. q.e.d.

Repeating this process, we have:

THEOREM 4. *There exists a finite sequence of subrings C_i of C such that*

$$C=C_0\supset C_1\supset\cdots\supset C_d=R,$$

where C_i is the constant subring of the canonical high order derivation of C_{i-1} for every $i=1,\dots,d$.

PROOF. Let P be a minimal element of $Ass_R(C/R)$. Then the residue R_P-module C_P/R_P is an Artinian module. Hence, by Proposition 3, there exists a sequence of intermediate rings between C_P and R_P such that

$$R_P=C'_g\subset\cdots\subset C'_0=C_P,$$

where C'_{i+1} is the constant subring of C'_i with respect to the canonical high order derivation over R_P, $\Delta'_i\colon C'_i\to\Omega_{R_P}^{q'_i}(C'_i)$. Let $C_0=C\supset\cdots\supset C_g$ be the sequence where C_{i+1} is the constant subring of the canonical high order derivation over R, Δ_i: $C_i \to \Omega_R^{q_i}(C_i)$. By the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\Delta_0} & \Omega_R^{q_0}(C) \\ \downarrow & & \downarrow \\ C_P & \xrightarrow{\Delta_0'} & \Omega_{R_P}^{q_0}(C_P) \end{array},$$

we have $(C_1)_P \subset C_1'$. Since $(C_i)_P \subset C_i'$ holds for $i=0, 1,\dots, g$ by induction on i, we have $R_P = C_g' \supset (C_g)_P \supset R_P$. Hence $(C_g)_P = R_P$. Let $\mathfrak{c}(C/R)$ be the conductor ideal of R in C. Since $P \supset \mathfrak{c}(C/R)$ and $P \not\supset \mathfrak{c}(C_g/R)$, we have $\mathfrak{c}(C_g/R) \supsetneqq \mathfrak{c}(C/R)$. Repeating this process we arrive at our conclusion. q. e. d.

Since $I_C^{q+1} = (0)$, there exists a sequence of ideals

$$I_C \supset I_C^2 \supset \cdots \supset I_C^q \supset I_C^{q+1} = (0).$$

For the canonical high order derivations Δ_i: $C \to I_C/I_C^{i+1} = \Omega_R^i(C)$, let $B_{i+1} = \Delta_i^{-1}(0)$. Then $B_{i+1} = \Delta_q^{-1}(I_C^{i+1})$, and B_{i+1} is the constant subring of Δ_i/B_i in B_i. Since Δ_i/B_i is a first order derivation, we have the following:

THEOREM 5. *There exists a sequence of constant subrings of some derivations and an integer g such that*

$$C = C_0' \supset C_1' \supset \cdots \supset C_g' = R.$$

We ask the converse of this theorem. Let d: $A \to \Omega_R(A)$ be the canonical derivation and put $B = d^{-1}(0)$. Is the extension A/B a cuspidal type extension?

References

[1] J. Lipman, Stable ideals and Arf rings, Amer. J. Math., **93** (1971), 649–685.

[2] M. Manaresi, Some properties of weakly normal varieties, Nagoya Math. J., **77** (1980), 61–74.

[3] P. Samuel, Les epimorphismes d'anneaux, Séminaire d'algebres commutative dirigé par P. Samuel, Secretariat Math., Paris, 1968.

Department of Mathematics
Aichi University of Education
and
Department of Applied Mathematics
Okayama University of Science