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Convergence, consistency and stability of step-by-step methods for ordinary differential equations

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1. Introduction

Consider the initial value problem

(1.1) $$y' = f(x, y) \quad (a \leqq x \leqq b), \quad y(x_0) = \eta,$$

where the function $f(x, y)$ is continuous and satisfies a Lipschitz condition with respect to y in $I \times R$, $I=[a, b]$, $R=(-\infty, \infty)$. Let $y(x)$ be the solution of this problem and let

(1.2) $$x_n = a + nh \qquad (n = 0, 1,\dots, ; h > 0),$$

where h is a stepsize. We are concerned with the case where the approximations y_j $(j=1, 2,\dots)$ of $y(x_j)$ are computed by step-by-step methods. Most of the conventional step-by-step methods such as one-step methods, linear multistep methods [1], hybrid methods [3], pseudo-Runge-Kutta methods [3, 4] and two-step methods [5] are of the form

(1.3) $$\textstyle\sum_{j=0}^{k} a_j y_{n+j} = h\Phi(x_n, y_n,\dots, y_{n+k}; h) \qquad (n = 0, 1,\dots),$$

where a_j $(j=0, 1,\dots, k)$ are real constants. Methods of this type determine y_{n+k} for given y_{n+i} $(i=0, 1,\dots, k-1)$ and require starting values y_i $(i=0, 1,\dots, k-1)$.

To achieve higher order with no increase in stepnumber, in this paper, besides the step nodes (1.2) we introduce $m-1$ sets of subsidiary nodes

(1.4) $$x_{n+v_i} = a + (n+v_i)h \qquad (n = 0, 1,\dots; i = 2, 3,\dots, m)$$

and the approximations y_{in} of $y(x_{n+v_i})$, and consider the system of difference equations

(1.5) $$\textstyle\sum_{i=1}^{m} \sum_{j=0}^{k_i} a_{ijq} y_{in+j} = h\Phi_q(x_n, y_{1n},\dots, y_{1n+k_1},\dots, y_{mn},\dots, y_{mn+k_m}; h)$$
$$(n = 0, 1,\dots, N;\ q = 1, 2,\dots, m),$$

where $v_1=0$, $y_{1n}=y_n$ $(n=0, 1,\dots)$, v_i $(i=2, 3,\dots, m)$ are nonnegative numbers and a_{ijq} $(j=0, 1,\dots, k_i;\ i, q=1, 2,\dots, m;\ k_i \geqq 1)$ are real constants. Methods of this type determine y_{in+k_i} $(i=1, 2,\dots, m)$ for given y_{in+j} $(j=0, 1,\dots, k_i-1;\ i=1, 2,\dots, m)$ and require starting values y_{ij} $(j=0, 1,\dots, k_i-1;\ i=1, 2,\dots, m)$. v_i $(i=2, 3,\dots, m)$ need not be integers and it is not required that $k_i+v_i \leqq k_1$ $(i=2, 3,\dots, m)$. The node x_{n+v} is called an off-step node if v is not an integer. Clearly the method (1.5) reduces to (1.3) when $m=1$.

Urabe's compound multistep method [9] and his implicit one-step method [10] can be considered as methods (1.5) with $m=2$, $k_1=k_2$ and $v_2=1$. Two-step methods with one and two off-step nodes have been studied by the author [6, 7, 8].

In section 2 assumptions on $\Phi_q(x, u; v)$ $(q=1, 2,\dots, m)$ are stated, the consistency condition for (1.5) is introduced and the root condition is stated for the characteristic equation defined in terms of the coefficients a_{ijq}'s.

In section 3 convergence of the method (1.5) is defined and it is shown that under certain conditions the method (1.5) is convegent if and only if it satisfies the consistency condition and the root condition.

In section 4 stability of the method (1.5) is defined as the boundedness of the effects of perturbations in $\Phi_q(x, u; v)$ $(q=1, 2,\dots, m)$ and in starting values. It is shown that under certain conditions the method (1.5) is stable if and only if it satisfies the root condition.

In section 5 an a priori error estimate of the method (1.5) is obtained and the order of the method (1.5) is defined.

2. Preliminaries

2.1. Notation

Let $\mathscr{F}$ be the set of all functions $f(x, y)$ which are continuous and satisfy Lipschitz conditions with respect to y in $I\times R$. For $f\in\mathscr{F}$ and $\eta\in R$ denote by $y(x)$ the solution of the initial value problem (1.1). This solution exists over the interval I [1]. Let v_i $(i=2, 3,\dots, m)$ be nonnegative numbers if $m>1$, h_0 be a positive number, $H=[0, h_0]$, $v_1=0$, and

$$(2.1)\qquad x_{n+v_i} = a + (n+v_i)h \quad (n = 0, 1,\dots;\ i = 1, 2,\dots, m;\ 0 < h \leqq h_0).$$

To obtain the approximations y_{in} of $y(x_{n+v_i})$, we consider the system (1.5) of difference equations. Unless stated otherwise, N denotes a positive integer such that $a+(N+\mu)h\leqq b$, where $\mu=\max_{1\leqq i\leqq m}(k_i+v_i)$.

Let

$$(2.2)\quad k = \textstyle\sum_{i=1}^{m} k_i,\quad k^* = \max_{1\leqq i\leqq m} k_i,\quad k_* = \min_{1\leqq i\leqq m} k_i,\quad \Omega = I\times R^{k+m}\times H,$$

M_p $(p=0, 1,\dots, k^*)$ be the $m\times m$ matrices with (i, j) entries a_{jk_i-pi} and let

$$(2.3)\quad \boldsymbol{u}_n = (u_{1n+k_1}, u_{2n+k_2},\dots, u_{mn+k_m})^t \qquad (n = -k^*, -k^*+1,\dots, N),$$

where $u_{iq}=a_{jqi}=0$ for $q<0$. Assume that M_0 is nonsingular. Denote by $\Phi_q(x_n, u_n; h)$

$$\Phi_q(x_n, u_{1n},\ldots, u_{1n+k_1},\ldots, u_{mn},\ldots, u_{mn+k_m}; h)$$

and let

$$\boldsymbol{\Phi}(x_n, u_n; h) = (\Phi_1(x_n, u_n; h),\ldots, \Phi_m(x_n, u_n; h))^t. \tag{2.4}$$

Then (1.5) can be written as

$$\textstyle\sum_{j=0}^{k^*} M_j \boldsymbol{y}_{n-j} = h\boldsymbol{\Phi}(x_n, y_n; h) \qquad (n = 0, 1,\ldots, N). \tag{2.5}$$

In the sequel $\sum\sum$ stands for $\sum_{i=1}^{m}\sum_{j=0}^{k_i}$ and $\|\cdot\|$ denotes the 1-norm of a m-vector or an $m\times m$ matrix.

Now we introduce the following

CONDITION A. $\Phi_q(x, u; v)$ $(q=1, 2,\ldots, m)$ are continuous in Ω and there exists a positive constant L such that

$$|\Phi_q(x, u; v) - \Phi_q(x, \tilde{u}; v)| \leqq L \textstyle\sum\sum|u_{ij}-\tilde{u}_{ij}| \tag{2.6}$$
$$\text{for all } (x, u, v), (x, \tilde{u}, v)\in\Omega \quad (q = 1, 2,\ldots, m).$$

THEOREM 1. *Suppose that Condition A is satisfied. Then there exists a positive number* h_1 $(h_1\leqq h_0)$ *such that for any* $x\in I$ *and* $u_{ij}\in R$ $(j=0, 1,\ldots, k_i-1;$ $i=1, 2,\ldots, m)$ the system of equations

$$\textstyle\sum\sum a_{ijq}u_{ij} = h\Phi_q(x, u; h)\ (q = 1, 2,\ldots, m) \quad \textit{for}\ h \leqq h_1 \tag{2.7}$$

has a unique solution u_{ik_i} $(i=1, 2,\ldots, m)$.

PROOF. Denote $\boldsymbol{u}_0$ in (2.3) by $\boldsymbol{v}$ and let

$$\theta_q(\boldsymbol{v}) = \Phi_q(x, u_{10},\ldots, u_{1k_1-1}, v_1,\ldots, u_{mk_m-1}, v_m; h) \qquad (q = 1, 2,\ldots, m),$$
$$\boldsymbol{\theta}(\boldsymbol{v}) = (\theta_1(\boldsymbol{v}), \theta_2(\boldsymbol{v}),\ldots, \theta_m(\boldsymbol{v}))^t.$$

Then (2.7) can be written as

$$M_0\boldsymbol{v} + \textstyle\sum_{j=1}^{k^*} M_j\boldsymbol{u}_{-j} = h\boldsymbol{\theta}(\boldsymbol{v}). \tag{2.8}$$

For any $\boldsymbol{v}^{(0)}\in R^m$ consider the iteration

$$M_0\boldsymbol{v}^{(n+1)} + \textstyle\sum_{j=1}^{k^*} M_j\boldsymbol{u}_{-j} = h\boldsymbol{\theta}(\boldsymbol{v}^{(n)}) \qquad (n=0, 1,\ldots).$$

Then

$$M_0(\boldsymbol{v}^{(n+1)} - \boldsymbol{v}^{(n)}) = h\{\boldsymbol{\theta}(\boldsymbol{v}^{(n)}) - \boldsymbol{\theta}(\boldsymbol{v}^{(n-1)})\} \qquad (n=1, 2,\ldots),$$

and by condition A with $K=mL\ \|M_0^{-1}\|$

$$\|v^{(n+1)} - v^{(n)}\| \leqq hK\|v^{(n)} - v^{(n-1)}\| \qquad (n=1, 2,\dots).$$

Choose h_1 $(0<h_1\leqq h_0)$ so that $\rho=Kh<1$ if $0<h\leqq h_1$ and we have

$$\|v^{(n+1)} - v^{(n)}\| \leqq \rho^n\|v^{(1)} - v^{(0)}\| \qquad (n=0, 1,\dots) \quad \text{for } h \leqq h_1.$$

Hence for any positive integer p

$$\|v^{(n+p)} - v^{(n)}\| \leqq \rho^n\|v^{(1)} - v^{(0)}\|/(1-\rho) \qquad (n=0, 1,\dots) \quad \text{for } h \leqq h_1.$$

Thus $\{v^{(n)}\}$ is a Cauchy sequence and there exists $v^*=\lim_{n\to\infty} v^{(n)}$. By continuity of $\boldsymbol{\Phi}$ v^* is a solution of (2.8).

Suppose that $\tilde{v}$ is also a solution of (2.8). Then we have

$$M_0(\tilde{v}-v^*) = h\{\theta(\tilde{v})-\theta(v^*)\},$$

so that $\|\tilde{v}-v^*\|\leqq\rho\|\tilde{v}-v^*\|$. Since $0\leqq\rho<1$, it follows that $\|\tilde{v}-v^*\|=0$ and the solution of (2.7) is unique. This completes the proof.

Let

$$(2.9) \qquad T_q(x;h) = \sum\sum a_{ijq}y(x+(j+v_i)h) - h\Phi_q(x, y(x+v_1h),\dots, y(x+(k_m+v_m)h); h) \quad (q = 1, 2,\dots, m),$$

$$(2.10) \qquad \boldsymbol{T}(x;h) = (T_1(x;h), T_2(x;h),\dots, T_m(x;h))^t,$$

$$(2.11) \qquad \varphi_{ij}(x) = \textstyle\sum_{q=0}^{k_i} a_{jqi}x^q \qquad (i, j = 1, 2,\dots, m).$$

Denote by $A(x)$ the matrix with the (i, j) entry $\varphi_{ij}(x)$ and by $\phi_{ij}(x)$ $(i, j=1, 2,\dots, m)$ the cofactor of $\varphi_{ij}(x)$. Put

$$(2.12) \qquad A(x) = [\boldsymbol{\varphi}_1(x), \boldsymbol{\varphi}_2(x),\dots, \boldsymbol{\varphi}_m(x)],$$

$$(2.13) \qquad \rho(x) = \det A(x) = \textstyle\sum_{j=0}^{k} b_jx^j.$$

Then $b_k=\det M_0\neq 0$ by the assumption.

Let E be the operator such that

$$(2.14) \qquad Ex = x + h, \qquad Ey_{in} = y_{in+1}.$$

Then (1.5) can be rewritten as

$$(2.15) \qquad \textstyle\sum_{j=1}^{m} \varphi_{ij}(E)y_{jn} = h\Phi_i(x_n, y_n; h) \qquad (i = 1, 2,\dots, m).$$

Eliminating y_{jn} $(j\neq i)$ from the left side of (2.15), we have

$$(2.16) \qquad \textstyle\sum_{j=0}^{k} b_jy_{in+j} = h\Psi_i(x_n, y_n; h) \qquad (i = 1, 2,\dots, m),$$

where

$$(2.17)\qquad \Psi_i(x_n, y_n; h) = \textstyle\sum_{j=1}^{m} \phi_{ji}(E)\Phi_j(x_n, y_n; h)$$
$$(n = 0, 1,\ldots, N - k;\ i = 1, 2,\ldots, m).$$

Let

$$(2.18)\qquad R_i(x; h) = \textstyle\sum_{j=0}^{m} \phi_{ji}(E)T_j(x; h) \qquad (i = 1, 2,\ldots, m),$$

$$(2.19)\qquad \boldsymbol{R}(x; h) = (R_1(x+k_1h; h),\ldots, R_m(x+k_mh; h))^t,$$

$$(2.20)\qquad \boldsymbol{\Psi}(x_n, y_n; h) = (\Psi_1(x_{n+k_1}, y_{n+k_1}; h),\ldots, \Psi(x_{n+k_m}, y_{n+k_m}; h))^t,$$

where $\Psi_i(x_n, y_n; h)=0$ for $n<0$ and $R_i(x; h)=0$ for $x<a$. Then (2.16) can be rewritten as

$$(2.21)\qquad \textstyle\sum_{j=0}^{k} b_j\boldsymbol{y}_{n+j} = h\boldsymbol{\Psi}(x_n, y_n; h) \qquad (n = -k^*, -k^* + 1,\ldots, N - k).$$

Let $\{y_{in}\}$ $(n=0, 1,\ldots, N+k_i;\ i=1, 2,\ldots, m)$ be the solution of (1.5) such that

$$(2.22)\qquad y_{in} \longrightarrow \eta \qquad (j = 0, 1,\ldots, k_i - 1;\ i = 1, 2,\ldots, m) \text{ as } h \longrightarrow 0$$

and let

$$(2.23)\qquad e_{ij} = y_{ij} - y(x_{j+v_i}) \qquad (j = 0, 1,\ldots, N + k_i;\ i = 1, 2,\ldots, m).$$

Then we have

$$(2.24)\quad \textstyle\sum_{j=0}^{k^*} M_j\boldsymbol{e}_{n-j} = h\boldsymbol{\Phi}(x_n, y(x_n)+e_n; h)$$
$$- h\boldsymbol{\Phi}(x_n, y(x_n); h) - \boldsymbol{T}(x_n; h) \quad (n = 0, 1,\ldots, N),$$

$$(2.25)\quad \textstyle\sum_{j=0}^{k} b_j\boldsymbol{e}_{n+j} = h\boldsymbol{\Psi}(x_n, y(x_n)+e_n; h) - h\boldsymbol{\Psi}(x_n, y(x_n); h)$$
$$- \boldsymbol{R}(x_n; h) \qquad (n = -k^*, -k^* + 1,\ldots, N - k).$$

2.2. Conditions

We introduce the following five conditions:

CONDITION B. If $f\equiv 0$, then $\Phi_q\equiv 0$ $\quad (q=1, 2,\ldots, m)$.

CONDITION C1. $\Sigma\Sigma a_{ijq}=0$ $\quad (q=1, 2,\ldots, m)$.

CONDITION C2. $\Psi_q(x, y,\ldots, y; 0)=\rho'(1)f(x, y)$ $\quad (q=1, 2,\ldots, m)$.

CONDITION R. The modulus of no root of $\rho(\zeta)=0$ exceeds 1 and the roots of modulus 1 are simple.

CONDITION C2′. $\Phi_q(x, y,\ldots, y; 0)=\sum\sum(j+v_i)a_{ijq}f(x, y)$ $\quad (q=1, 2,\ldots, m)$.

We say that the method (1.5) is *consistent* if Conditions C1 and C2 are satisfied.

LEMMA 1. *Suppose that Conditions A, C1 and C2 are satisfied. Then there exist $r_i(h)$ $(i=1, 2,\dots, m)$ such that*

$$(2.26)\qquad |R_i(x;\, h)| \leqq hr_i(h),\ r_i(h) \longrightarrow 0 \quad as \quad h \longrightarrow 0 \quad (i = 1, 2,\dots, m).$$

PROOF. Since $\sum_{j=1}^{m} \varphi_{ij}(1)=0$ $(i=1, 2,\dots, m)$ by Condition C1, it follows that $\rho(1)=\det A(1)=0$. Let L_1 be a Lipschitz constant of $f(x, y)$, $G_0=\max_{x\in I}|y'(x)|$ and

$$w(h) = \max_{x,x+t\in I,|t|\leqq h} |f(x+t,\, y(x)) - f(x,\, y(x))|.$$

Then, by continuity of $f(x, y)$, $w(h)\to 0$ as $h\to 0$.

For $x\in I$ and $x+h\in I$

$$y(x+h) - y(x) - hy'(x) = \int_0^h [y'(x+t) - y'(x)]dt.$$

Since

$$|f(x+t,\, y(x+t)) - f(x,\, y(x))| \leqq |f(x+t,\, y(x+t)) - f(x+t,\, y(x))| + |f(x+t,\, y(x)) - f(x,\, y(x))| \leqq w(h) + hL_1G_0 \qquad (0\leqq t\leqq h),$$

for some θ $(|\theta|\leqq 1)$ we have

$$(2.27)\qquad y(x+h) = y(x) + hy'(x) + \theta h[w(h)+hL_1G_0].$$

Let

$$(2.28)\quad s_i(h) = \max_{x\in J} |\Psi_i(x,\, y(x+v_1h),\dots,\, y(x+(k_1+v_1+k-k_i)h),\dots,\, y(x+(k_m+v_m+k-k_i)h);\, h) - \Psi_i(x,\, y(x),\dots,\, y(x);\, 0)| \quad (i = 1, 2,\dots, m),$$

where $J=[a,\, b-(\mu+k-k_*)h]$. As $\Psi_i(x, u; v)$ is continuous in its arguments by (2.17) and Condition A, it follows that $s_i(h)\to 0$ as $h\to 0$. Since by (2.18), (2.9), (2.15) and (2.16)

$$(2.29)\qquad R_i(x;\, h) = \sum_{j=0}^{k} b_j y(x+(j+v_i)h) - \Psi_i(x,\, y(x+v_1h),\dots,\, y(x+(k_m+v_m+k-k_i)h);\, h),$$

by (2.27), (2.28) and Conditions C1 and C2 we have

$$|R_i(x;\, h)| \leqq h\sum_{j=0}^{k}|b_j|[w((j+v_i)h) + (j+v_i)hL_1G_0] + hs_i(h) \quad (i = 1, 2,\dots, m).$$

Hence there exist $r_i(h)$ $(i=1, 2,\dots, m)$ satisfying (2.26).

LEMMA 2. *Condition C2 follows from Conditions C1 and C2′.*

PROOF. As has been shown in the proof of Lemma 1, $\rho(1)=0$ follows from

Condition C1. Setting $x_n=x$ and $y_{n+j}=y$ $(j=0, 1,\dots, k_q;\ q=1, 2,\dots, m)$ in (2.17), and letting $h\to 0$, we have

$$\Psi_i(x, y,\dots, y; 0) = \sum_{j=1}^{m} \phi_{ji}(1)\Phi_j(x, y,\dots, y; 0).$$

Denote by δ_{ij} the Kronecker's delta. Then since

$$\sum_{j=1}^{m} \phi_{jq}(x)\varphi_{ji}(x) = \delta_{qi}\rho(x) \qquad (i, q = 1, 2,\dots, m),$$

we have

$$\sum_{j=1}^{m} \phi_{jq}(1)\varphi'_{ji}(1) = \delta_{qi}\rho'(1) - \sum_{j=1}^{m} \phi'_{jq}(1)\varphi_{ji}(1).$$

Hence by Condition C1

$$\sum_{j=1}^{m} \phi_{ji}(1)\left[\sum_{q=1}^{m} \varphi'_{jq}(1) + \sum_{q=1}^{m} \nu_q\phi_{jq}(1)\right]$$
$$= \rho'(1) - \sum_{j=1}^{m} \phi'_{ji}(1) \sum_{q=1}^{m} \varphi_{jq}(1) + \sum_{q=1}^{m} \nu_q\rho(1)\delta_{qi} = \rho'(1).$$

Since by Condition C2′

$$\Phi_j(x, y,\dots, y; 0) = \sum_{i=1}^{m} \{\varphi'_{ji}(1) + \nu_i\varphi_{ji}(1)\} f(x, y),$$

Condition C2 follows.

REMARK. From this proof it is seen that Condition C2 follows also from Condition C1 and the condition

$$\Phi_q(x, y,\dots, y; 0) = \sum\sum j a_{ijq} f(x, y) \qquad (q = 1, 2,\dots, m).$$

It is also seen that, if Condition A is satisfied and $e_n=o(h)$ $(n=-k^*, -k^*+1,\dots, N)$, then Conditions C1 and C2 are satisfied by (2.24), (2.25) and (2.29). Under Condition C1, Condition C2 coincides with Condition C2′ if $m=1$.

2.3. Systems of difference equations

Let $\{U_n^{(i)}\}$ be the set of k^* solutions of the homogeneous matrix difference equation

$$\sum_{j=0}^{k^*} M_j U_{n-j} = 0 \qquad (n = 0, 1,\dots, N) \tag{2.30}$$

satisfying the initial conditions

$$U_{-i}^{(i)} = I, \quad U_{-j}^{(i)} = 0 \qquad (j \neq i;\ i, j = 1, 2,\dots, k^*). \tag{2.31}$$

Then the solution $\{z_n\}$ of the system of difference equations

$$\sum_{j=0}^{k^*} M_j z_{n-j} = c_n \qquad (n = 0, 1,\dots, N) \tag{2.32}$$

can be written as

(2.33) $$z_n = \sum_{i=1}^{k^*} U_n^{(i)} z_{-i} + \sum_{j=0}^{n} E_{jn} c_j \qquad (n = -k^*, -k^*+1, \dots, N),$$

where

(2.34) $$E_{jn} = U_{n-j-1}^{(1)} M_0^{-1} \qquad (j = 0, 1, \dots, n;\ n = 0, 1, \dots, N).$$

Now consider the system of difference equations

(2.35) $$\sum\sum a_{ijqn} z_{in+j} = c_{qn} \qquad (n = 0, 1, \dots, N;\ q = 1, 2, \dots, m),$$

where

(2.36) $$a_{ijqn} = a_{ijq} - hb_{ijqn} \quad (j = 0, 1, \dots, k_i;\ i, q = 1, 2, \dots, m;\ n = 0, 1, \dots, N),$$

(2.27) $$|c_{qn}| \leqq C, \quad |b_{ijqn}| \leqq L,$$

c_{qn}, b_{ijqn} and C are constants. Let M_{pn} $(p=0, 1, \dots, k^*)$ be the matrices with (i, j) entries a_{jk_j-pin} and let $c_n=(c_{1n}, c_{2n}, \dots, c_{mn})^t$, where $b_{jqin}=0$ for $q<0$. Then (2.35) can be rewritten as

(2.38) $$\sum_{j=0}^{k^*} M_{jn} z_{n-j} = c_n \qquad (n = 0, 1, \dots, N).$$

LEMMA 3. *There exist matrices* $V_n^{(i)}$ $(i=1, 2, \dots, k^*;\ n=-k^*, -k^*+1, \dots, N)$ *and* F_{jn} $(j=0, 1, \dots, n;\ n=0, 1, \dots, N)$ *and positive constants c, d and* h_2 $(h_2 \leqq h_1)$ *such that for* $h \leqq h_2$ *the system* (2.38) *has a solution* $\{z_n\}$, *which can be expressed as*

(2.39) $$z_n = \sum_{i=1}^{k^*} V_n^{(i)} z_{-i} + \sum_{j=0}^{n} F_{jn} c_j \quad (n = -k^*, -k^*+1, \dots, N),$$

where

(2.40) $$\|V_n^{(i)} - U_n^{(i)}\| \leqq ch \quad (i = 1, 2, \dots, k^*,\ n = 0, 1, \dots, k-1) \text{ for } h \leqq h_2,$$
$$V_{-j}^{(i)} = U_{-j}^{(i)} \quad (i, j = 1, 2, \dots, k^*),$$

(2.41) $$\|F_{qn} - E_{qn}\| \leqq dh \quad (q = 0, 1, \dots, n;\ n = 0, 1, \dots, k-1) \text{ for } h \leqq h_2.$$

PROOF. Set $M_{pn}=M_p-hK_{pn}$ $(p=0, 1, \dots, k^*)$. Then $\|K_{pn}\| \leqq mL$. Let $K=mL\|M_0^{-1}\|$ and h_2 $(h_2 \leqq h_1)$ be a positive number such that $2Kh_2 \leqq 1$. Then M_{0n}^{-1} exists for $h \leqq h_2$ because $h\|M_0^{-1}K_{0n}\| \leqq 1/2$. Put $M_{0n}^{-1}=M_0^{-1}+hD_n$ $(n=0, 1, \dots, N)$. Then $\|D_n\| \leqq 2\|M_0^{-1}\|K$ $(n=0, 1, \dots, N)$. Setting

$$N_j = M_0^{-1} M_j, \quad N_{jn} = M_{0n}^{-1} M_{jn}, \quad N_{jn} = N_j + hL_{jn} \qquad \text{for} \quad h \leqq h_2$$
$$(j = 1, 2, \dots, k^*;\ n = 0, 1, \dots, N),$$

we have

$$\|L_{jn}\| \leqq \|D_n\| (\|M_j\| + h_2 mL) + K \qquad (j = 1, 2, \dots, k^*;\ n = 0, 1, \dots, N).$$

Let $\{V_n^{(i)}\}$ $(n=-k^*, -k^*+1,\dots, N;\ i=1, 2,\dots, k^*)$ be the set of k^* solutions of the matrix difference equation $\sum_{j=0}^{k^*} M_{jn}V_{n-j}=0$ $(n=0, 1,\dots, N)$ satisfying the initial conditions $V_{-i}^{(i)}=I$, $V_{-j}^{(i)}=0$ $(j\neq i;\ i, j=1, 2,\dots, k^*)$. Then the solution of (2.38) is given by (2.39), where

$$F_{jn} = V_{n-j-1}^{(1)}M_{0n}^{-1} \qquad (j = 0, 1,\dots, n;\ n = 0, 1,\dots, N).$$

Let for $n=0, 1,\dots, N$

$$(2.42)\quad V_n^{(i)}=U_n^{(i)}+hG_{in} \quad (i=1, 2,\dots, k^*),\ F_{qn}=E_{qn}+hH_{qn} \quad (q=0, 1,\dots, n).$$

Then we shall show that for $n=0, 1,\dots, k-1$ there exist constants c_n and d_n such that

$$(2.43)\quad \|G_{in}\| \leqq c_n\ (i = 1, 2,\dots, k^*),\ \|H_{qn}\| \leqq d_n \quad (q = 0, 1,\dots, n) \quad \text{for} \quad h \leqq h_2.$$

Since $G_{j0}=-L_{j0}$ $(j=1, 2,\dots, k^*)$ and $H_{00}=D_0$, there exist c_0 and d_0 satisfying (2.43) for $n=0$. Assume that (2.43) holds for $n=0, 1,\dots, p-1$ $(p<k)$. Then since

$$G_{jp} = -\sum_{i=1}^{k^*} [L_{ij}V_{p-i}^{(j)}+N_iG_{jp-i}] \qquad (j = 1, 2,\dots, k^*),$$
$$H_{qn} = V_{n-q-1}^{(1)}D_n + G_{1n-q-1}M_0^{-1} \qquad (q = 0, 1, \dots, n),$$

there exist constants c_p and d_p satisfying (2.43). Hence for some constants c and d (2.40) and (2.41) hold. Thus the lemma is proved.

Consider the system of difference equations

$$(2.44)\qquad \sum_{j=0}^{k^*} M_jz_{n-j} = h\boldsymbol{\Theta}(z_n) \qquad (n = 0, 1,\dots, N),$$

where $\boldsymbol{\Theta}(z_n)=(\Theta_1(z_n),\dots, \Theta_m(z_n))^t$. Then we have the following

COROLLARY. *Suppose that for some constant* C_0

$$(2.45)\qquad |\Theta_q(z_n)| \leqq C_0 + L\sum\sum|z_{in+j}| \qquad (n = 0, 1,\dots, N;\ q = 1, 2,\dots, m).$$

Then the system (2.45) *has a solution* $\{z_n\}$ *for* $h<h_2$, *and there exist constants* A_0 *and* A_1 *such that*

$$(2.46)\quad \|z_n\| \leqq C_0A_0h + A_1 \sum_{j=1}^{k^*} \|z_{-j}\| \qquad (n = 0, 1,\dots, k-1) \quad \textit{for} \quad h \leqq h_2.$$

PROOF. By (2.45) there exist constants b_{ijqn} and d_{qn} such that

$$\Theta_q(z_n) = \sum\sum b_{ijqn}z_{in+j} + d_{qn} \qquad (n = 0, 1,\dots, N;\ q = 1, 2,\dots, m),$$

where $|b_{ijqn}|\leqq L$ and $|d_{qn}|\leqq C_0$. Let $\boldsymbol{d}_n=(d_{1n}, d_{2n},\dots, d_{mn})^t$. Then (2.44) can be written as $\sum_{j=0}^{k^*} M_{jn}z_{n-j}=h\boldsymbol{d}_n$. By Lemma 3 this system has a solution $\{z_n\}$ for $h\leqq h_2$, which can be expreesed as

$$z_n = \sum_{i=1}^{k^*} V_n^{(i)} z_{-i} + h \sum_{q=0}^{n} F_{qn} d_q \qquad (n=0, 1,\dots, k-1).$$

From this (2.46) follows. This completes the proof.

Let $\{u_n^{(i)}\}$ $(n=0, 1,\dots; i=0, 1,\dots, k-1)$ be the set of k solutions of the difference equation

(2.47) $$\sum_{j=0}^{k} b_j u_{n+j} = 0 \qquad (n = 0, 1,\dots)$$

satisfying the initial conditions $u_j^{(i)}=\delta_{ij}$ $(i, j=0, 1,\dots, k-1)$, and let $u_j^{(k-1)}=0$ for $j<0$. If Condition R is satisfied, then there exists a constant G [2] such that

(2.48) $$|u_n^{(i)}| \leqq G \qquad (n = 0, 1,\dots; i = 0, 1,\dots, k - 1), \quad 2kG \geqq 1.$$

Eliminating z_{jn} $(j \neq i)$ from (2.35), we have

(2.49) $$\sum_{j=0}^{k} b_j z_{in+j} = h \sum_{j=0}^{k} d_{ijn} z_{in+j} + g_{in} \qquad (n = 0, 1,\dots, N - k: i = 1, 2,\dots, m),$$

where d_{ijn}'s are polynomials in h with bounded coefficients,

(2.50) $$g_{in} = \sum_{j=1}^{m} \phi_{ji}(E) c_{jn} + h \sum_{j=1}^{m} \sigma_{jin}(E; h) c_{jn},$$

$\sigma_{jin}(x; h)$'s are polynomials in x and h with bounded coefficients. Hence there exist constants d, g and g_i $(i=1, 2,\dots, m)$ such that

(2.51) $$|d_{ijn}| \leqq d \quad \text{for} \quad h \leqq h_2 \qquad (j = 0, 1,\dots, k),$$

(2.52) $$|g_{in}| \leqq g_i \leqq gC \quad \text{for} \quad h \leqq h_2 \qquad (n=0, 1,\dots, N-k; i=1, 2,\dots, m).$$

LEMMA 4. *Let $\{z_{in}\}$ be the solution of (2.49) and suppose that*

(2.53) $$|z_{ij}| \leqq Z_i \qquad (j = 0, 1,\dots, k - 1; i = 1, 2,\dots, m).$$

Let the constants d and g_i $(i=1, 2,\dots, m)$ satisfy (2.51) and (2.52) respectively. Then there exists a positive constant h_3 $(h_3 \leqq h_2)$ such that

(2.54) $$|z_{in}| \leqq K_i e^{nhL^*} \qquad (n = 0, 1,\dots, N + k_i; i = 1, 2,\dots, m) \quad \textit{for} \quad h \leqq h_3,$$

where

(2.55) $$K_i=2G[kZ_i+B(N+k_i)g_i] \quad (i=1, 2,\dots, m), \; L^*=2B(k+1)dG, \; B=|b_k^{-1}|.$$

PROOF. Put $w_{in}=h \sum_{j=0}^{k} d_{ijn} z_{in+j} + g_{in}$. Then z_{in} can be expressed as

$$z_{in} = \sum_{j=0}^{k-1} z_{ij} u_n^{(j)} + b_k^{-1} \sum_{j=0}^{n-k} w_{ij} u_{n-j-1}^{(k-1)} \qquad (n=0, 1,\dots, N+k_i).$$

Hence

$$z_{in} = \sum_{j=0}^{k-1} z_{ij} u_n^{(j)} + b_k^{-1} \sum_{j=0}^{n-k} g_{ij} u_{n-j-1}^{(k-1)} + hb_k^{-1} \sum_{j=0}^{n} \left(\sum_{q=0}^{\min(k,j)} d_{iqj-q}\, u_{n-1-j+q}^{(k-1)}\right) z_{ij},$$

from which we have

$$(1-hd)|z_{in}| \leqq G \sum_{j=0}^{k-1} |z_{ij}| + BG(n-k+1)g_i + hB(k+1)dG \sum_{j=0}^{n-1} |z_{ij}| \quad \text{for} \quad h \leqq h_2.$$

Choosing h_3 $(0<h_3 \leqq h_2)$ so that $2dh_3 \leqq 1$, we have

$$|z_{in}| \leqq K_i + hL^* \sum_{j=0}^{n-1} |z_{ij}| \quad (n = 0, 1,\ldots, N + k_i) \qquad \text{for} \quad h \leqq h_3.$$

We shall show that

$$|z_{in}| \leqq K_i(1+hL^*)^n \tag{2.56}$$

holds for $n=0, 1,\ldots, N+k_i$. For $j=0, 1,\ldots, k-1$ we have by (2.53), (2.48) and (2.55)

$$|z_{ij}| \leqq Z_i \leqq 2kGZ_i \leqq K_i \leqq K_i(1+hL^*)^j.$$

Assume that (2.56) is valid for $n=0, 1,\ldots, p-1$. Then

$$|z_{ip}| \leqq K_i + hL^*K_i \sum_{j=0}^{p-1} (1+hL^*)^j \leqq K_i(1+hL^*)^p.$$

Hence (2.56) holds for $n=0, 1,\ldots, N+k_i$ and (2.54) follows.

3. Convergence

The method (1.5) is called *convergent* if for any $f \in \mathscr{F}$ and $\eta \in R$

$$\max_{0 \leqq n \leqq N+k_i, 1 \leqq i \leqq m} |e_{in}| \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0 \tag{3.1}$$

for all $x \in (a, b)$, all q $(1 \leqq q \leqq m)$ and all solutions $\{y_{in}\}$ of (1.5) satisfying (2.22), where

$$h = (x-a)/(N+k_q+v_q) \tag{3.2}$$

and N is a positive integer such that

$$h \leqq h_1, \quad a + (N+\mu)h \leqq b. \tag{3.3}$$

THEOREM 2. *The method* (1.5) *is convergent if Conditions A, C1, C2 and R are satisfied.*

PROOF. By (2.17) and Condition A $\Psi_i(x, u; v)$ $(i=1, 2,\ldots, m)$ satisfy Lipschitz conditions with respect to u with a Lipschitz constant L_0. By Lemma 1 there exist $r_i(h)$ $(i=1, 2,\ldots, m)$ that satisfy (2.26). Let

$$\boldsymbol{r}(h) = (r_1(h), r_2(h),\dots, r_m(h))^t,$$
$$\boldsymbol{\Theta}(e_j) = \boldsymbol{\Psi}(x_j, y(x_j) + e_j; h) - \boldsymbol{\Psi}(x_j, y(x_j); h),$$

and for any $x \in (a, b)$ and q $(1 \leqq q \leqq m)$ choose N so that (3.2) and (3.3) are satisfied. Then by (2.25) we have

$$\boldsymbol{e}_n = \sum_{j=0}^{k-1} \boldsymbol{e}_{j-k^*} u_{n+k^*}^{(j)} + b_k^{-1} \sum_{j=-k^*}^{n-k} [h\boldsymbol{\Theta}(\boldsymbol{e}_j) - \boldsymbol{R}(x_j; h)] u_{n-1-j}^{(k-1)}$$
$$(n = -k^*, -k^*+1,\dots, N),$$

$$\|\boldsymbol{\Theta}(\boldsymbol{e}_j)\| \leqq mL_0 \sum_{q=k_*-k^*}^{k} \|\boldsymbol{e}_{j+q}\| \qquad (j = -k^*, -k^*+1,\dots),$$
$$\|\boldsymbol{R}(x_j; h)\| \leqq h\|\boldsymbol{r}(h)\|.$$

Hence it follows that

$$(1-hd)\|\boldsymbol{e}_n\| \leqq G \sum_{j=0}^{k-1} \|\boldsymbol{e}_{j-k^*}\| + BG(n-k+k^*)h\|\boldsymbol{r}(h)\|$$
$$+ hdG(k+k^*-k_*) \sum_{j=-k^*}^{n-1} \|\boldsymbol{e}_j\|,$$

where $d = mBL_0$. Choosing h_3 $(0 < h_3 \leqq h_2)$ so that $2dh_3 \leqq 1$, and setting

$$K^* = 2G \sum_{j=0}^{k-1} \|\boldsymbol{e}_{j-k^*}\| + 2BG(b-a)\|\boldsymbol{r}(h)\|, \tag{3.4}$$

$$L^* = 2d(k+k^*-k_*)G, \tag{3.5}$$

we have

$$\|\boldsymbol{e}_n\| \leqq K^* + hL^* \sum_{j=-k^*}^{n-1} \|\boldsymbol{e}_j\| \qquad \text{for} \quad h \leqq h_3.$$

It can be shown by induction that

$$\|\boldsymbol{e}_n\| \leqq K^*(1+hL^*)^{n+k^*} \qquad (n = -k^*, -k^*+1,\dots, N),$$

so that

$$\|\boldsymbol{e}_n\| \leqq K^* e^{(b-a)L^*} \qquad (n = -k^*, -k^*+1,\dots, N). \tag{3.6}$$

By Conditions A and C1 from (2.9) we have for some constant C^*

$$|T_i(x; h)| \leqq C^* h \quad (i = 1, 2,\dots, m) \qquad \text{for} \quad h \leqq h_3.$$

By Corollary to Lemma 3 for some constants C_1 and C_2

$$\|\boldsymbol{e}_q\| \leqq C_1 \sum_{j=0}^{k^*} \|\boldsymbol{e}_{-j}\| + C_2 h \qquad (q = 0, 1,\dots, k-k^*-1) \qquad \text{for} \quad h \leqq h_3.$$

Suppose that (2.22) is satisfied. Then since $y(x_{j+\nu_i}) \to \eta$ $(j = 0, 1,\dots. k_i - 1;$ $i = 1, 2,\dots, m)$ as $h \to 0$, we have $\|\boldsymbol{e}_{-j}\| \to 0$ $(j = 0, 1,\dots, k^*)$, so that by (3.6) and (3.4) $\max_{-k^* \leqq m \leqq N} \|\boldsymbol{e}_n\| \to 0$ as $h \to 0$. Hence the method (1.5) is convergent.

By Lemma 2 we have the following

COROLLARY. *The method* (1.5) *is convergent if Conditions A, C1, C2′ and R are satisfied.*

THEOREM 3. *If Condition B is satisfied and the method* (1.5) *is convergent, then Condition R is satisfied.*

PROOF. Consider the initial value problem $y'=0$, $y(a)=0$. Then by Condition B the method (1.5) reduces to

$$\sum\sum a_{ijq}y_{in+j} = 0 \qquad (n = 0, 1,\dots, N;\ q = 1, 2,\dots, m). \tag{3.7}$$

Let $\{y_{in}\}$ be the solution of (3.7) satisfying

$$y_{ij} \longrightarrow 0 \qquad (j = 0, 1,\dots, k_i - 1;\ i = 1, 2,\dots, m) \quad \text{as} \quad h \longrightarrow 0. \tag{3.8}$$

Suppose that $\zeta_0 = re^{i\varphi}$ ($r>1$, $0 \leqq \varphi < 2\pi$) is a root of $\rho(\zeta)=0$. Then since

$$\det[\boldsymbol{\varphi}_1(\zeta_0), \boldsymbol{\varphi}_2(\zeta_0),\dots, \boldsymbol{\varphi}_m(\zeta_0)] = 0, \tag{3.9}$$

there exist constants c_i ($i=1, 2,\dots, m$) such that $\sum_{j=1}^m c_j\boldsymbol{\varphi}_j(\zeta_0)=0$ with $c_q=1$ for some q ($1 \leqq q \leqq m$). Hence $\sum_{j=1}^m c_j\varphi_j(\zeta_0)\zeta_0^n=0$ ($n=0, 1,\dots, N+k_i$; $i=1, 2,\dots, m$) and $\{c_i\zeta_0^n\}$ is a solution of (3.7). Since $\{\bar{c}_i\bar{\zeta}_0^n\}$ is also a solution of (3.7),

$$y_{in} = h\,\mathrm{Re}\,(c_i\zeta_0^n) \qquad (n = 0, 1,\dots, N + k_i;\ i = 1, 2,\dots, m) \tag{3.10}$$

is a solution of (3.7) satisfying (3.8). Choose h and N so that (3.2) and (3.3) are satisfied, and put $M=N+k_q$. Then since the method (1.5) is convergent,

$$y_{qM} = (x-a)\cos M\varphi(r^M/(M+v_q)) \longrightarrow 0 \quad \text{as} \quad M \longrightarrow \infty,$$

so that $\cos M\varphi \to 0$ as $M \to \infty$ because $r>1$. But then $|\sin M\varphi| \to 1$ as $M \to \infty$ and we have $\sin\varphi=0$, because

$$|\cos(M+1)\varphi - \cos(M-1)\varphi| = 2|\sin M\varphi|\,|\sin\varphi|.$$

It follows that $\varphi=0$ or π and so $|\cos M\varphi|=1$. This is a contradiction. Hence the modulus of no root of $\rho(\zeta)=0$ exceeds 1.

Next suppose that $\zeta_0=e^{i\varphi}$ ($0 \leqq \varphi < 2\pi$) is a multiple root of $\rho(\zeta)=0$. Let $A_j(x)$ ($j=1, 2,\dots, m$) be $A(x)$ with $\boldsymbol{\varphi}_j(x)$ replaced by $\boldsymbol{\varphi}'_j(x)$. Then

$$\rho'(\zeta_0) = \sum_{j=1}^m \det A_j(\zeta_0) = 0. \tag{3.11}$$

We consider first the case $\operatorname{rank} A(\zeta_0)=m-1$. Assuming that $\boldsymbol{\varphi}_i(\zeta_0)$ ($i=1, 2,\dots, m-1$) are linearly independent, we have for some c_i ($i=1, 2,\dots, m-1$) $\boldsymbol{\varphi}_m(\zeta_0) = -\sum_{i=1}^{m-1} c_i\boldsymbol{\varphi}_i(\zeta_0)$, and by (3.11)

$$\det[\boldsymbol{\varphi}_1(\zeta_0),\dots, \boldsymbol{\varphi}_{m-1}(\zeta_0), \textstyle\sum_{i=1}^m c_i\boldsymbol{\varphi}'_i(\zeta_0)] = 0, \tag{3.12}$$

where $c_m=1$. From (3.9) and (3.12) it follows that

$$\det[\varphi_1(\zeta_0),\dots,\varphi_{m-1}(\zeta_0),\ \textstyle\sum_{i=1}^m c_i\{n\varphi_i(\zeta_0)\zeta_0^{-1}+\varphi_i'(\zeta_0)\}]=0.$$

Since

$$n\varphi_i(\zeta)\zeta^{n-1}+\varphi_i'(\zeta)\zeta^n=(\varphi_i(\zeta)\zeta^n)',$$

for some constants a_i $(i=1, 2,\dots, m-1)$ we have

$$\textstyle\sum_{i=1}^{m-1} a_i\varphi_i(\zeta_0)\zeta_0^n+\sum_{j=1}^m c_j(\varphi_j(\zeta_0)\zeta_0^n)'=0.$$

Hence

$$\text{(3.13)}\quad y_{in}=h\operatorname{Re}(a_i\zeta_0^n+nc_i\zeta_0^{n-1})\ (i=1, 2,\dots, m-1),\ y_{mn}=hn\cos(n-1)\varphi$$

is a solution of (3.7) satisfying (3.8). For any $x\in(a, b)$ let $h=(x-a)/(M+v_m)$ and $M=N+k_m$. Then since the method (1.5) is convergent,

$$y_{mM}=[(x-a)M/(M+v_m)]\cos(M-1)\varphi\longrightarrow 0\qquad\text{as } M\longrightarrow\infty.$$

As has been shown, this is impossible.

We consider next the case $\operatorname{rank} A(\zeta_0)<m-1$. In this case it follows that $\det A_j(\zeta_0)=0$ $(j=1, 2,\dots, m)$. From $\det A_m(\zeta_0)=0$ and (3.9) we have (3.13) with $a_i=1$ and $c_i=0$ $(i=1, 2,\dots, m-1)$, and this also leads to a contradiction. Hence the root of $\rho(\zeta)=0$ of modulus 1 must be simple.

THEOREM 4. *If Conditions A and B are satisfied and the method* (1.5) *is convergent, then Conditions C1 and C2 are satisfied.*

PROOF. Consider the initial value problem $y'=0$, $y(a)=1$. Then by Condition B (1.5) reduces to (3.7). For any $x\in(a, b)$ and q $(1\leqq q\leqq m)$ choose h and N so that (3.2) and (3.3) are satisfied. Let $\{y_{in}\}$ be the solution of (3.7) satisfying $y_{ij}=1$ $(j=0, 1,\dots, k_i;\ i=1, 2,\dots, m)$. Then since the method (1.5) is convergent, $y_{in}\to 1$ $(n=0, 1,\dots, N+k_i;\ i=1, 2,\dots, m)$ as $h\to 0$. Hence Condition C1 follows from (3.7), and we have $\rho(1)=0$ as has been shown in the proof of Lemma 1. By Theorem 3 Condition R is satisfied, so that $\rho'(1)\neq 0$. Let

$$g_j(x, y)=\Psi_j(x, y,\dots, y; 0)/\rho'(1)\qquad(j=1, 2,\dots, m).$$

Then by Condition A $g_j(x, y)\in\mathscr{F}$ $(j=1, 2,\dots, m)$.

Suppose that there exist q $(1\leqq q\leqq m)$, $\tilde{x}\in(a, b)$ and $\tilde{y}\in R$ such that $g_q(\tilde{x}, \tilde{y})\neq f(\tilde{x}, \tilde{y})$. Let $y(x)$ be the solution of $y'=f(x, y)$ satisfying $y(\tilde{x})=\tilde{y}$ and let $y(a)=\eta$. For any $x\in(a, b)$ choose h and N so that (3.2) and (3.3) are satisfied and put $M=N+k_q$. Let $z(x)$ be the solution of $z'=g_q(x, z)$ satisfying $z(a)=\eta$. Let $\{y_{in}\}$ be the solution of (1.5) satisfying (2.22) and let $\{z_{in}\}$ be the solution of

(3.14) $\sum_{j=0}^{k} b_j z_{qn+j} = h\rho'(1) g_q(x_n, z_{qn}) \qquad (n = 0, 1, \dots, M - k)$

satisfying $z_{qj} = y_{gj}$ $(j = 0, 1, \dots, k-1)$.

Let

(3.15) $$d_{qn} = y_{qn} - z_{qn} \qquad (n = 0, 1, \dots, M),$$

(3.16) $$e(h) = \max_{0 \leqq n \leqq N+k_i, 1 \leqq i \leqq m} |e_{in}|,$$

(3.17) $$C(d_n) = \Psi_q(x_n, y_{1n}, \dots, y_{mn+l}; h) - \Psi_q(x_n, z_{qn}, \dots, z_{qn}; 0),$$
$$s(h) = \max_{x \in J} |\Psi_q(x, y(x+v_1 h), \dots, y(x+(l+v_m)h); h) - \Psi_q(x, y(x+v_q h), \dots, y(x+v_q h); 0)|,$$

where $l = k + k_m - k_q$, $J = [a, b - (\mu + k - k_q)h]$. Then

$$\sum_{j=0}^{k} b_j d_{qn+j} = hC(d_n) \qquad (n = 0, 1, \dots, M - k).$$

By Condition A $\Psi_q(x, u; v)$ is continuous in its arguments, so that $s(h) \to 0$ as $h \to 0$. From (3.17) it follows that

$$|C(d_n)| \leqq s(h) + L_0 \sum_{i=1}^{m} \sum_{j=0}^{k+k_i-k_q} (|e_{in+j}| + |d_{qn}|) \leqq d(h) + C|d_{qn}|,$$

where

$$C = \{m(k+1-k_q) + k\} L_0, \quad d(h) = s(h) + Ce(h).$$

By the same argument as in the proof of Lemma 4, we have

$$|d_{qn}| \leqq G \sum_{j=0}^{k-1} |d_{qj}| + hB(n-k+1)d(h) + hBCG \sum_{j=0}^{n-k} |d_{qj}|,$$

which can be written as

(3.18) $$|d_{qn}| \leqq K^* + hL^* \sum_{j=0}^{n-k} |d_{qj}| \qquad (n = 0, 1, \dots, M),$$

where $K^* = BG(b-a)d(h)$, $L^* = BGC$, because $d_{qj} = 0$ $(j = 0, 1, \dots, k-1)$. From (3.18) we obtain $|d_{qn}| \leqq K^* e^{nhL^*}$ $(n = 0, 1, \dots, M)$ and $d_{qM} \to 0$ as $M \to \infty$.

By Theorem 2 $z_{qM} \to z(x)$ and $y_{qM} \to y(x)$ as $M \to \infty$, so that $y(x) = z(x)$ for all $x \in (a, b)$. But

$$y'(\tilde{x}) = f(\tilde{x}, \tilde{y}) \neq g_q(\tilde{x}, \tilde{y}) = z'(\tilde{x}).$$

This is a contradiction. Hence

(3.19) $$f(x, y) = g_j(x, y) \qquad (j = 1, 2, \dots, m)$$

is valid in $(a, b) \times R$, and by continuity of $f(x, y)$ and $g_j(x, y)$ (3.19) is valid in

$I \times R$. Thus Condition C2 is satisfied.

By Theorems 2, 3 and 4 we have

THEOREM 5. *Suppose that Conditions A and B are satisfied. Then the method* (1.5) *is convergent if and only if Conditions C1, C2 and R are satisfied.*

4. Stability

For any $f \in \mathscr{F}$ let $\{u_{in}\}$ and $\{v_{in}\}$ be the solutions of

$$(4.1) \quad \sum\sum a_{ijq}u_{in+j} = h\Phi_q(x_n, u_n; h) + h\rho_{qn} \quad (n = 0, 1,\dots, N;\ q = 1, 2,\dots, m),$$

$$(4.2) \quad \sum\sum a_{ijq}v_{in+j} = h\Phi_q(x_n, v_n; h) + h\sigma_{qn}.$$

The method (1.5) is called *stable* if there exist positive constants h^* and M such that

$$(4.3) \quad |u_{in}-v_{in}| \leqq M\varepsilon \quad (n = 0, 1,\dots, N + k_i;\ i = 1, 2,\dots, m) \quad \text{for} \quad h \leqq h^*,$$

whenever

$$(4.4) \quad |u_{ij}-v_{ij}| \leqq \varepsilon \quad (j = 0, 1,\dots, k_i - 1;\ i = 1, 2,\dots, m),$$

$$(4.5) \quad |\rho_{qn}-\sigma_{qn}| \leqq \varepsilon \quad (n = 0, 1,\dots, N;\ q = 1, 2,\dots, m).$$

THEOREM 6. *If Condition B is satisfied and the method* (1.5) *is stable, then Condition R is satisfied.*

PROOF. Choose $f=0$, $\rho_{qn}=\sigma_{qn}=0$ $(n=0, 1,\dots, N;\ q=1, 2,\dots, m)$, $v_{ij}=0$ $(j=0, 1,\dots, k_i-1;\ i=1, 2,\dots, m)$ and $\varepsilon>0$. Then by Condition B, (4.4) and (4.2)

$$(4.6) \quad \sum\sum a_{ijq}u_{in+j} = 0 \quad (n = 0, 1,\dots, N;\ q = 1, 2,\dots, m),$$

$$(4.7) \quad |u_{ij}| \leqq \varepsilon \quad (j = 0, 1,\dots, k_i - 1;\ i = 1, 2,\dots, m),$$

$$(4.8) \quad v_{in} = 0 \quad (n = 0, 1,\dots, N + k_i;\ i = 1, 2,\dots, m).$$

Suppose that $\zeta_0=re^{i\varphi}$ $(r>1;\ 0\leqq\varphi<2\pi)$ is a root of $\rho(\zeta)=0$. Then by the same argument as in the proof of Theorem 2,

$$u_{in} = \delta\,\mathrm{Re}\,(c_i\zeta_0^n) \qquad (n = 0, 1,\dots, N + k_i;\ i = 1, 2,\dots, M;\ c_q = 1)$$

is a solution of (4.6) satisfying (4.7) if $\delta>0$ is chosen so that $\delta|c_ir^{k_i-1}|\leqq\varepsilon$ $(i=1, 2,\dots, m)$. As the method is stable, there exist h^* and M such that

$$|u_{in}-v_{in}| = |u_{in}| \leqq M\varepsilon \quad (n = 0, 1,\dots, N + k_i;\ i = 1, 2,\dots, m) \quad \text{for} \quad h \leqq h^*,$$

so that

$$|\delta\,\mathrm{Re}\,(c_i\zeta_0^{N_i})| \leqq M\varepsilon \qquad (i = 1, 2,\dots, m).$$

where $N_i = N + k_i\ (i = 1, 2,\dots, m)$. Since $c_q = 1$, we have

$$|\cos N_q\varphi| \leqq M\varepsilon/|\delta r^{N_q}| \longrightarrow 0 \quad \text{as} \quad N_q \longrightarrow \infty.$$

which is impossible as has been shown in the proof of Theorem 3. Hence the modulus of no root of $\rho(\zeta)=0$ can exceed 1.

Assume next that $\zeta_0 = e^{i\varphi}$ $(0 \leqq \varphi < 2\pi)$ is a multiple root of $\rho(\zeta)=0$. In the case rank $A(\zeta_0) = m-1$ by the same argument as in the proof of Theorem 3

$$u_{in} = \delta\,\mathrm{Re}\,(a_i\zeta_0^n + nc_i\zeta_0^{n-1}) \quad (i = 1, 2,\dots, m-1), \quad u_{mn} = \delta n\cos(n-1)\varphi$$

is a solution of (4.6) satisfying (4.7) if $\delta > 0$ is chosen so that $\delta[|a_i| + |c_i|(k_i - 1)] \leqq \varepsilon$ $(i = 1, 2,\dots, m-1)$ and $\delta(k_m - 1) \leqq \varepsilon$. Hence

$$|\cos(N_m - 1)\varphi| \leqq M\varepsilon/(\delta N_m) \longrightarrow 0 \quad \text{as} \quad N_m \longrightarrow \infty,$$

which is impossible. In the same way the case rank $A(\zeta_0) < m-1$ leads to a contradiction. Hence the root of $\rho(\zeta)=0$ of modulus 1 must be simple.

THEOREM 7. *The method* (1.5) *is stable if Conditions A and R are satisfied.*

PROOF. Let

$$d_{in} = u_{in} - v_{in} \qquad (n = 0, 1,\dots, N + k_i;\ i = 1, 2,\dots, m),$$

$$\Theta_q(d_n) = \Phi_q(x_n, u_n; h) - \Phi_q(x_n, v_n; h) + \sigma_{qn} - \rho_{qn} \quad (n = 0, 1,\dots, N).$$

Then

$$\sum\sum a_{ijq} d_{in+j} = h\Theta_q(d_n) \qquad (q = 1, 2,\dots, m), \tag{4.9}$$

and by Condition A and (4.5) we have

$$|\Theta_q(d_n)| \leqq \varepsilon + L\sum\sum|d_{in+j}|.$$

By Corollary to Lemma 3 there exists a constant K_1 such that

$$|d_{ij}| \leqq K_1\varepsilon \qquad (j = 0, 1,\dots, k-1;\ i = 1, 2\dots, m) \quad \text{for} \quad h \leqq h_2.$$

Hence by Lemma 4

$$|d_{in}| \leqq K_i^* e^{(b-a)L^*} \quad (n = 0, 1,\dots, N + k_i;\ i = 1, 2,\dots, m) \quad \text{for} \quad h \leqq h_3,$$

where

$$K_i^* = 2G\{kK_1 + B(b-a)g\}\varepsilon, \quad L^* = 2B(k+1)dG.$$

Thus the method (1.5) is stable.

From Theorems 6 and 7 we have

THEOREM 8. *Suppose that Conditions A and B are satisfied. Then the method* (1.5) *is stable if and only if Condition R is satisfied.*

Combining this with Theorem 5, we have the following

COROLLARY. *Suppose that Conditions A and B are staisfied. Then the method* (1.5) *is convergent if and only if it is consistent and stable.*

5. Error estimate

In this section an a priori error estimate is obtained.

THEOREM 9. *Suppose that Conditions A and R are satisfied and that there esist positive constants* K_1, K_2, p_i *and* q_i $(i=1, 2,\dots, m)$ *such that*

(5.1) $$|T_i(x; h)| \leqq K_1 h^{p_i+1} \ (i = 1, 2,\dots, m) \quad \text{for} \quad h \leqq h_3,$$

(5.2) $$|e_{ij}| \leqq K_2 h^{q_i} \ (j = 0, 1,\dots, k_i - 1;\ i = 1, 2,\dots, m) \quad \text{for} \quad h \leqq h_3.$$

Let $p=\min_{1 \leqq i \leqq m} p_i, \quad q=\min_{1 \leqq i \leqq m} q_i$
and assume that one of the following three conditions is satisfied:

(*a*) $p_i=p,\ q_i=q \quad (i=1, 2,\dots, m)$.
(*b*) M_j $(j=0, 1,\dots, k^*)$ *are all upper triangular matrices and* $p_1 \leqq p_2 \leqq \cdots \leqq p_m$, $q_1 \leqq q_2 \leqq \cdots \leqq q_m$.
(*c*) M_j $(j=0, 1,\dots, k^*)$ *are all lower triangular matrices and* $p_1 \geqq p_2 \geqq \cdots \geqq p_m$, $q_1 \geqq q_2 \geqq \cdots \geqq q_m$.

Then for some constant K

(5.3) $$|e_{in}| \leqq K h^{t_i} \quad (n = 0, 1,\dots, N + k_i;\ i = 1, 2,\dots, m) \quad \text{for} \quad h \leqq h_3,$$

where

(5.4) $$t_i = \min(p_i, q_i, p+1, q+1) \quad (i = 1, 2,\dots, m).$$

PROOF. By Lemma 3 we have

(5.5) $$\boldsymbol{e}_n = \sum_{i=1}^{k^*} V_n^{(i)} \boldsymbol{e}_{-i} + \sum_{q=0}^{n} F_{qn} \boldsymbol{T}(x_q; h) \quad (n = -k^*, -k^* + 1, \dots, k - 1).$$

Suppose first that condition (a) is satisfied. Then by (5.5) for some constant K_3

$$\|\boldsymbol{e}_j\| \leqq K_3 h^s \quad (j = -k^*, -k^* + 1,\dots, k - 1),$$

where $s=\min(p+1, q)$. By (2.18) there exists a constant K_4 such that $|R_i(x; h)| \leqq K_4 h^{r_i+1}$ $(i=1, 2,\dots, m)$, where $r_i=\min(p_i, p+1)$. By (3.4) (3.5) and (3.6) we have (5.3).

Next suppose that condition (b) is satisfied. By (2.30) and (2.34) $U_n^{(i)}$ $(i=1, 2,\ldots, k^*; n=0, 1,\ldots, k-1)$ and E_{qn} $(q=0, 1,\ldots, n; n=0, 1,\ldots, k-1)$ are upper triangular matrices. Hence the i-th component of $\sum_{j=1}^{k^*} U_n^{(j)}\boldsymbol{e}_{-j}$ is of order h^{q_i} and each component of $h\sum_{j=1}^{k^*} G_{jn}\boldsymbol{e}_{-j}$ is of order h^{q+1}, so that the i-th component of $\sum_{j=1}^{k^*} V_n^{(j)}\boldsymbol{e}_{-j}$ is of order h^{s_i}, where $s_i=\min(q_i, q+1)$. Similarly the i-th component of $\sum_{q=0}^{n} F_{qn}\boldsymbol{T}(x_q; h)$ is of order h^{r_i+1}. Hence the i-th components of $\boldsymbol{e}_j$ $(j=-k^*, -k^*+1,\ldots, -k^*+k-1)$ are of order h^{n_i}, where $n_i=\min(s_i, r_i+1)$. Since M_j $(j=0, 1,\ldots, k^*)$ are upper triangular, so are $A(x)$ and its cofactor matrix $(\phi_{ji}(x))$. Hence by (2.18) $R_i(x; h)$ is of order h^{p_i+1}. As $h\sum_{j=1}^{m}\sigma_{jin}(E; h)T_j(x_n, h)$ is of order h^{p+2}, g_{in} in (2.49) is of order h^{r_i+1}. Hence by Lemma 4 we have (5.3).

The case where condition (c) is satisfied is treated similarly and the proof is complete.

Since $\{y_{in}\}$ $(i\neq 1)$ are subsidiary approximations and our aim is to obtain $\{y_{1n}\}$, the order of the method (1.5) is defined to be the greatest integer p such that

$$\max_{0\leqq n\leqq N+k_1}|e_{1n}| = O(h^p) \qquad \text{for} \quad h\leqq h_1$$

for sufficiently smooth $f(x, y)$ and $e_{ij}=0$ $(j=0, 1,\ldots, k_i-1; i=1, 2,\ldots, m)$.

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