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Codimension of Jacobian ideals and (R_n) conditions for complete intersections

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1. Introduction

In this paper we are interested in studying an estimation of the codimension of Jacobian ideals in a formal power series ring over a field of characteristic 0, and as an application we shall obtain a criterion on the (R_n) conditions for complete intersections.

Let R be an excellent regular local ring containing $\boldsymbol{Q}$ and let A be a factor ring R/I of R where I is generated by a regular sequence $\{f_1, f_2,\dots, f_r\}$. Recall that A satisfies Serre's condition (R_n) if and only if $A_{\mathfrak{p}}$ is regular for any $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) \leqq n$, or A is non-singular in codimension n. In particular, A is reduced if and only if A satisfies (R_0), and normal if and only if A satisfies (R_1), for A is automatically Cohen-Macaulay and so satisfies (S_1) and (S_2). It is clear that after a field extension a generic choice of generators of I, say $g_1, g_2,\dots, g_r$, have the property that $R/(g_1, g_2,\dots, g_{r-i})$ satisfies (R_{n+i}) if A satisfies (R_n). We will prove in Theorem (3.1) that such generators can be chosen without any field extension. More precisely, we can take $g_i = f_i + n_{i,i+1}f_{i+1} + \cdots + n_{i,r}f_r$ where $n_{i,j}$ $(1 \leqq i < j \leqq r)$ are integers.

The important ground for proving this theorem exists in the evaluation of the height of Jacobian ideals, which will be discussed in section 2 and the main result will be stated in Theorem (2.1).

Our method depends crucially on the good behavior of derivations and on the Jacobian criterion for regularity, the validity of which is admitted only in case of characteristic 0. Therefore we naturally assume in this paper that every ring contains the field $\boldsymbol{Q}$ of rational numbers.

2. Codimension of Jacobian ideals

In this section we always denote $R = k[[X_1, X_2,\dots, X_m]]$ where k is a field of characteristic 0. If $f_1, f_2,\dots, f_r$ are non-units in R and $1 \leqq s \leqq r$, then we denote by $J_s(f_1, f_2,\dots, f_r)$ the ideal generated by all the $s \times s$ minors of the Jacobian matrix $(\partial f_i/\partial X_j)_{1 \leqq i \leqq r,\ 1 \leqq j \leqq m}$.

For the present we concentrate upon the subject — how to estimate the height of $J_s(f_1, f_2,\dots, f_r)$. The main fact is the following

THEOREM (2.1) $\mathrm{ht}(J_s(f_1, f_2,\dots, f_r)) \geqq \mathrm{ht}((J_s(f_1, f_2,\dots, f_r), f_1,\dots, f_s)) - s+1$. *Here it should be noted that the unit ideal R has infinite height.*

REMARK. By Krull's principal ideal theorem it is obvious that
$\mathrm{ht}(J_s(f_1, f_2,\dots, f_r)) \geqq \mathrm{ht}((J_s(f_1, f_2,\dots, f_r), f_1,\dots, f_s)) - s$.

PROOF. For simplicity we denote $J_s(f_1, f_2,\dots, f_r)$ by J. We may consider only the case $s < \mathrm{ht}(J, f_1, f_2,\dots, f_s) < \infty$. Suppose that $\mathrm{ht}(J) = \mathrm{ht}(J, f_1, f_2, \dots, f_s) - s$. Then there would exist a prime ideal $\mathfrak{p}$ which contains J such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(J, f_1, f_2,\dots, f_s) - s$. Since $\mathrm{ht}((J, f_1,\dots, f_s) + \mathfrak{p}/\mathfrak{p}) \leqq s \leqq \dim(R) - \mathrm{ht}(J, f_1,\dots, f_s) + s = \dim(R/\mathfrak{p})$, there would be another prime ideal $\mathfrak{P}$ which contains $(J, f_1,\dots, f_s) + \mathfrak{p}$ and $\mathrm{ht}(\mathfrak{P}/\mathfrak{p}) = s$. Now let S be the $\mathfrak{P}R_{\mathfrak{P}}$-adic completion of $R_{\mathfrak{P}}$ and let K be a coefficient field of S. Notice that $\dim(S/\mathfrak{p}S) = s$ and $S/\mathfrak{p}S$ is reduced because of the excellence of R. If we denote the total quotient ring of $S/\mathfrak{p}S$ by L, then there is an S-module homomorphism ψ of $D_k(S)$ to $D_K(S/\mathfrak{p}S) \otimes_S L$ so that the following diagram is commutative:

$$\begin{array}{ccc} S & \longrightarrow & S/\mathfrak{p}S \\ {\scriptstyle d_k}\downarrow & & \downarrow{\scriptstyle d_K \otimes 1} \\ D_k(S) & \xrightarrow{\psi} & D_K(S/\mathfrak{p}S) \otimes_S L, \end{array}$$

where $D_k(S)$ (resp. $D_K(S/\mathfrak{p}S)$) denotes the universally finite module of differentials of S (resp. $S/\mathfrak{p}S$) over k (resp. K) and d_k (resp. d_K) is the universal derivation of S (resp. $S/\mathfrak{p}S$) over k (resp. K). (For the definition see [3].) Thus we obtain a mapping $\Lambda^s\psi\colon \Lambda^s D_k(S) \to \Lambda^s(D_K(S/\mathfrak{p}S) \otimes_S L)$.

Now we shall prove that $\{f_1, f_2,\dots, f_s\}$ never forms a system of parameters of $S/\mathfrak{p}S$. (Recall that $\dim(S/\mathfrak{p}S) = s$.) In fact, if it was a system of parameters, then $S/\mathfrak{p}S$ would be a finite extension of the formal power series ring $T := K[[f_1, f_2,\dots, f_s]]$ and hence $D_K(S/\mathfrak{p}S) \otimes_S L = \sum_{i=1}^{s} L d_K f_i$. Note that the K-derivation $\partial/\partial f_i$ on T can be extended to the K-derivation on L. Denote it by δ_i $(1 \leqq i \leqq s)$. Then ψ could be given by the following equality:

$$\psi(d_k x) = \textstyle\sum_{i=1}^{s} (\delta_i x) d_K f_i \qquad (x \in S).$$

In particular, $(\Lambda^s\psi)(d_k f_1 \Lambda d_k f_2 \Lambda \cdots \Lambda d_k f_s)$

$$= \det((\delta_i f_j)_{i,j=1,2,\dots,s})\,(d_K f_1 \Lambda d_K f_2 \Lambda \cdots \Lambda d_K f_s)$$
$$= d_K f_1 \Lambda d_K f_2 \Lambda \cdots \Lambda d_K f_s \neq 0.$$

On the other hand since $\mathfrak{p} \supset J$, the following holds:

$$d_k f_1 \Lambda d_k f_2 \Lambda \cdots \Lambda d_k f_s$$
$$= \textstyle\sum_{1 \leqq i_1,\dots,i_s \leqq m} (\partial f_1/\partial X_{i_1}) \cdots (\partial f_s/\partial X_{i_s}) d_k X_{i_1} \Lambda \cdots \Lambda d_k X_{i_s} \in \mathfrak{p}(\Lambda^s D_k(S)).$$

Therefore we have $(\Lambda^s\psi)(d_kf_1\Lambda d_kf_2\Lambda\cdots\Lambda d_kf_s)=0$. This contradiction shows that $\{f_1, f_2,\dots, f_s\}$ does not generate a parameter ideal of $S/\mathfrak{p}S$.

Hence we have $s>\mathrm{ht}(((f_1, f_2,\dots, f_s)+\mathfrak{p})S/\mathfrak{p}S)=\mathrm{ht}((f_1,\dots, f_s)+\mathfrak{p}/\mathfrak{p})=\mathrm{ht}((J, f_1,\dots, f_s)+\mathfrak{p}/\mathfrak{p})$ and finally we get the inequality: $\mathrm{ht}((J, f_1,\dots, f_s)+\mathfrak{p})<s+\mathrm{ht}(\mathfrak{p})$. This contradicts the choice of $\mathfrak{p}$.

REMARK (2.2) Let $s=r=1$ in Theorem (2.1). Then the theorem shows that $\mathrm{ht}(f, \partial f/\partial X_1, \partial f/\partial X_2,\dots, \partial f/\partial X_m)=\mathrm{ht}(\partial f/\partial X_1,\dots, \partial f/\partial X_m)$ for any non-unit f of R. This is rather well known since f is always integral over $(\partial f/\partial X_1, \partial f/\partial X_2,\dots, \partial f/\partial X_m)$ by [4; Satz (5.2)]. Thus Theorem (2.1) can be considered as a kind of generalization of [4; Satz (5.2)].

LEMMA (2.3) *Let R as above and let $f_1, f_2,\dots, f_r$ be non-units of R such that* $\mathrm{ht}(f_1, f_2,\dots, f_r)=s$. *Then the following conditions are equivalent.*

(a) *$R/(f_1, f_2,\dots, f_r)$ satisfies (R_n).*

(b) $\mathrm{ht}((J_s(f_1, f_2,\dots, f_r), f_1, f_2,\dots, f_r)/(f_1, f_2,\dots, f_r)) \geqq n+1$.

PROOF. This is clear from the Jacobian criterion for regularity: For a prime ideal $\mathfrak{P}$ of R, $(R/(f_1, f_2,\dots, f_r))_{\mathfrak{P}}$ is regular if and only if $\mathrm{rank}((\partial f_i/\partial X_j \bmod (\mathfrak{P}))_{1\leqq i\leqq r,\ 1\leqq j\leqq m})=s$. (See [2; (29.A)].)

Combining Theorem (2.1) with this lemma we have the following

COROLLARY (2.4) *Let R as above and let $f_1, f_2,\dots, f_r$ be a regular sequence of R. If $R/(f_1, f_2,\dots, f_r)$ satisfies (R_n) for some $n\geqq 0$, then* $\mathrm{ht}(J_r(f_1, f_2,\dots, f_r))\geqq n+2$.

PROOF. By the previous lemma the inequality:

$$\mathrm{ht}(J_r(f_1, f_2,\dots, f_r), f_1, f_2,\dots, f_r) \geqq n+r+1$$

holds and the consequence is obvious from Theorem (2.1).

Corollary (2.5) gives some additional information as a special case of corollary (2.4).

COROLLARY (2.5) *Let R and $f_1, f_2,\dots, f_r$ be as in Corollary (2.4). If $R/(f_1, f_2,\dots, f_r)$ is an isolated singularity, then*

$$\mathrm{ht}(J_r(f_1, f_2,\dots, f_r)) = m-r+1.$$

PROOF. Applying Corollary (2.4) in case $n=\dim(R/(f_1, f_2,\dots, f_r))-1=m-r-1$, we obtain $\mathrm{ht}(J_r(f_1, f_2,\dots, f_r))\geqq m-r+1$. On the other hand since $(\partial f_i/\partial X_j)_{i,j}$ is a matrix of size $r\times m$, [1; §6 Theorem 3] shows $\mathrm{ht}(J_r(f_1, f_2,\dots, f_r))\leqq m-r+1$.

3. (R_n) conditions for complete intersections

The purpose of this section is to prove the following

THEOREM (3.1) *Let R be an excellent regular local ring containing $\boldsymbol{Q}$ and let $f_1, f_2,\dots, f_r$ be a regular sequence of R. Assume that $R/(f_1, f_2,\dots, f_r)$ satisfies (R_n) for some $n \geqq 0$. Then there exist integers $n_2, n_3,\dots, n_r$ such that $R/(f_1+n_2f_2, f_2+n_3f_3,\dots, f_{r-1}+n_rf_r)$ satisfies (R_{n+1}).*

COROLLARY (3.2) *Let R be as above and let I be an ideal of R generated by a regular sequence. If R/I satisfies (R_n) for some $n \geqq 0$, then there exists a minimal generating set $f_1, f_2,\dots, f_r$ of I such that each $A_i:=R/(f_1, f_2,\dots, f_{r-i})$ satisfies (R_{n+i}). In particular if A_0 is reduced, then A_1 is normal. If A_0 is an isolated singularity, then so is any A_i.*

REMARK (3.3) In order to improve (R_n) conditions it is necessary to replace f_i with $f_i+n_{i+1}f_{i+1}$ as in Theorem (3.1). For instance let $I=(XY, ZW)\subset R=k[[X, Y, Z, W]]$. Then R/I is reduced, hence satisfies (R_0). Although $R/(XY)$ does not satisfy (R_1), $R/(XY+ZW)$ is certainly normal. Also remark that the same example shows that Theorem (3.1) fails provided $\mathrm{char}(k)>0$. (Take $\mathrm{char}(k)=2$, $f_1=XY$ and $f_2=XY+ZW$.)

The following lemma will be necessary to prove Theorem (3.1).

LEMMA (3.4) *Let R be a Cohen-Macaulay local ring which contains $\boldsymbol{Q}$ and let J be an ideal of R and $f_1, f_2,\dots, f_r \in R$. Suppose that $\mathrm{ht}(J)\geqq a$ and $\mathrm{ht}(J, f_1, f_2,\dots, f_r)\geqq a+r-1$ for some non-negative integer a. Then there exist integers $n_2, n_3,\dots, n_r$ such that*

$$\mathrm{ht}(J, f_1+n_2f_2, f_2+n_3f_3,\dots, f_{r-1}+n_rf_r) \geqq a+r-1.$$

PROOF. Take a regular sequence $g_1, g_2,\dots, g_a$ in J. Considering $R/(g_1, g_2,\dots, g_a)$ instead of R, we may assume $a=0$. We proceed the proof by induction on r. If $r=1$, then there is nothing to prove.

Assume $r=2$, hence $\mathrm{ht}(J, f_1, f_2)\geqq 1$. It is sufficient to prove that for any large n, f_1+nf_2 does not belong to any minimal prime ideal of R which contains J. However it is clear from the following fact: If $f_1+nf_2\in\mathfrak{p}$ for some integer n and a minimal prime ideal $\mathfrak{p}$ of R containing J, then $f_1+(n+m)f_2\notin\mathfrak{p}$ for any $m>0$, since $f_2\notin\mathfrak{p}$.

Next we assume $r\geqq 3$. Since $\mathrm{ht}(J, f_1, f_2,\dots, f_r)\geqq r-1$, Krull's principal ideal theorem shows that $\mathrm{ht}(J, f_1, f_2,\dots, f_{r-1})\geqq r-2$. By the induction hypothesis there are integers $n_2, n_3,\dots, n_{r-1}$ satisfying $\mathrm{ht}(J, f_1+n_2f_2,\dots, f_{r-2}+n_{r-1}f_{r-1})\geqq r-2$. If we denote $J'=(J, f_1+n_2f_2,\dots, f_{r-2}+n_{r-1}f_{r-1})$, then we have

$\mathrm{ht}(J')\geqq r-2$ and $\mathrm{ht}(J', f_{r-1}, f_r)\geqq r-1$. Hence by the case $r=2$, there exists an integer n_r such that $\mathrm{ht}(J', f_{r-1}+n_r f_r)\geqq r-1$.

PROOF of THEOREM (3.1) We should first notice that R/I satisfies (R_n) if and only if $\hat{R}/I\hat{R}$ satisfies (R_n) for any ideal I, for R is excellent. Therefore we may assume that R is a complete regular local ring, thus we may denote $R=k[[X_1, X_2,\dots, X_m]]$ where k is a field. By Corollary (2.4) the condition (R_n) requires the inequality: $\mathrm{ht}(J_r(f_1, f_2,\dots, f_r))\geqq n+2$. And Lemma (2.3) shows that $\mathrm{ht}(J_r(f_1, f_2,\dots, f_r), f_1, f_2,\dots, f_r)\geqq n+r+1$. Now applying the previous lemma, we know the existence of integers $n_2, n_3,\dots, n_r$ such that

$$\mathrm{ht}(J_r(f_1, f_2,\dots, f_r), f_1 + n_2 f_2,\dots, f_{r-1} + n_r f_r) \geqq n + r + 1.$$

If we denote $g_i=f_i+n_{i+1}f_{i+1}$ $(1\leqq i\leqq r-1)$ and $g_r=f_r$, then it is clear that $J_{r-1}(g_1, g_2,\dots, g_{r-1})\supset J_r(g_1, g_2,\dots, g_r)=J_r(f_1, f_2,\dots, f_r)$. Hence we have

$$\mathrm{ht}(J_{r-1}(g_1, g_2,\dots, g_{r-1}), g_1, g_2,\dots, g_{r-1}) \geqq n + r + 1.$$

This shows that $R/(g_1, g_2,\dots, g_{r-1})$ satisfies (R_{n+1}) by Lemma (2.3).

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