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Spaces of orderings and quadratic extensions of fields

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Let P be a preordering of a field F of finite index and $K=F(\sqrt{a})$ be a radical extension of F(i.e. a is an element of Kaplansky's radical of F). We denote by n the number of the connected components of $X(F/P)$. In [4], we showed that $n=\dim H_F(P)/P$ ([4], Theorem 2.5) and the number of connected components of $X(K/P')$ is $2n$, where $P'=\Sigma P\dot{K}^2$ ([4], Theorem 3.10).

The main purpose of this paper is to study a relation between $X(F)$ and $X(K)$, where F is a quasi-pythagorean field whose Kaplansky's radical $R(F)$ is of finite index and $K=F(\sqrt{a})$ is a quadratic extension of F. In §2, we show that if $a\in H_F$, then $X(K)$ is equivalent to $H_F(a)\oplus H_F(a)$ (Theorem 2.9). In §3, we assume that $X(F)$ is connected and show that the following results. If $a\in B_{R(F)}$, then $X(K)$ is equivalent to $X(F)$, where $B_{R(F)}$ is the set of $R(F)$-basic elements of $\dot{F}$ (Theorem 3.3). If $a\in B_{R(F)}\setminus\pm R(F)$ and $D_F\langle 1, a\rangle D_F\langle 1, -a\rangle=B_{R(F)}$, then $X(F)$ is equivalent to a group extension of $H_{X_1}(a)\oplus H_{X_1}(a)$, where the space $H_{X_1}(a)$ is defined in §3 (Theorem 3.5).

§1. Valuations on quasi-pythagorean fields

In this section, we state some results on valuations on quasi-pythagorean fields. By a field F, we shall always mean a field of characteristic different from two. We denote by $\dot{F}$ the multiplicative group of F. Let v be a valuation on F. The value group Γ will always be written multiplicatively. The objects: the valuation ring of v, the maximal ideal of v, the group of units and the residue class field of v will be denoted by A, M, U and $\bar{F}$ respectively. For a subset $B\subseteq A$, we put $\bar{B}=\{x+M\in\bar{F}|x\in B\}$.

We write v' for the composition $\dot{F}\xrightarrow{v}\Gamma\to\Gamma/\Gamma^2$. For simplicity, we also write v' for the induced homomorphism $\dot{F}/\dot{F}^2\to\Gamma/\Gamma^2$. There is a natural short exact sequence

$$1\longrightarrow U\dot{F}^2/\dot{F}^2\longrightarrow\dot{F}/\dot{F}^2\xrightarrow{v'}\Gamma/\Gamma^2\longrightarrow 1\,.$$

Since the three groups involved are all elementary 2-groups, this is a split exact sequence. We shall choose and fix a splitting λ: $\dot{F}/\dot{F}^2\to U\dot{F}^2/\dot{F}^2$. Composing λ with the natural maps $U\dot{F}^2/\dot{F}^2\cong U/U\cap\dot{F}^2\to(\bar{F})^{\cdot}/(\bar{F})^{\cdot 2}$, we get a surjective homomorphism λ': $\dot{F}/\dot{F}^2\to(\bar{F})^{\cdot}/(\bar{F})^{\cdot 2}$. By abuse of notation, the composition of this map with $\dot{F}\to\dot{F}/\dot{F}^2$ will again be denoted by λ'. Throughout this section, we assume that char $\bar{F}\neq 2$. We consider the group ring of the group Γ/Γ^2 over the Witt ring $W(\bar{F})$, denoted by $W(\bar{F})[\Gamma/\Gamma^2]$; a typical element of this ring will be written in the form $\Sigma\varphi_i[g_i]$, where $\varphi_i\in W(\bar{F})$, and $g_i\in\Gamma/\Gamma^2$.

PROPOSITION 1.1 ([6], Proposition 2.4). *Let a be an element of $\dot{F}$. The rule $a\mapsto\langle\lambda'(a)\rangle[v'(a)]\in W(\bar{F})[\Gamma/\Gamma^2]$ induces a well-defined, surjective ring homomorphism f of $W(F)$ to $W(\bar{F})[\Gamma/\Gamma^2]$. Ker f is an ideal of $W(F)$ generated by the set $\{\langle 1, -r\rangle | r\in 1+M\}$.*

PROPOSITION 1.2. *Let $a_1,\dots, a_n$ be elements of $U\dot{F}^2$. Then we have $\bar{B}=D_F\langle\lambda'(a_1),\dots,\lambda'(a_n)\rangle$, where $B=D_F\langle a_1,\dots, a_n\rangle\cap U$.*

PROOF. We first show that $\bar{B}\subseteq D_F\langle\lambda'(a_1),\dots,\lambda'(a_n)\rangle$. We may assume that $a_1,\dots, a_n\in U$. Let $x=a_1z_1^2+\cdots+a_nz_n^2$ be an element of B. If $z_i\in A$ for any i, then $\bar{x}=\bar{a}_1\bar{z}_1^2+\cdots+\bar{a}_n\bar{z}_n^2\in D_F\langle\lambda'(a_1),\dots,\lambda'(a_n)\rangle$. Next we consider the case when $z_i\notin A$ for some i. Say $v(z_1)=\min\{v(z_i)\}$ in Γ. Then $z_i/z_1\in A$ for all i, and $z_1^{-2}x=a_1+a_2(z_2/z_1)^2+\cdots+a_n(z_n/z_1)^2$. From this, we have $0=\bar{a}_1+\bar{a}_2\bar{y}_n^2+\cdots+\bar{a}_n\bar{y}_n^2$ $(y_i=z_i/z_1\in A)$, and so the form $\langle\bar{a}_1,\dots,\bar{a}_n\rangle$ is isotropic. It implies $\bar{x}\in D_F\langle\bar{a}_1,\dots,\bar{a}_n\rangle=(\bar{F})^{\cdot}$. Hence in any case we have $\bar{B}\subseteq D_F\langle\lambda'(a_1),\dots,\lambda'(a_n)\rangle$. The reverse inclusion is clear. Q.E.D.

A field F is called quasi-pythagorean if $R(F)=D_F(2)$, where $R(F)$ is Kaplansky's radical of F. It was proved in [2], Corollary 2.9, that a field F is quasi-pythagorean if and only if I^2F is torsion free. A field F being quasi-pythagorean is also equivalent to the condition that $\langle 1, a\rangle\langle 1, -r\rangle=0\in W(F)$ for any $a\in\dot{F}$ and $r\in D_F(2)$. Let v be a valuation on a quasi-pythagorean field F. Then for any $a\in\dot{F}$ and $r\in D_F(2)$, $f(\langle 1, a\rangle\langle 1, -r\rangle)=0\in W(\bar{F})[\Gamma/\Gamma^2]$, namely

$$\langle 1\rangle[1]+\langle\lambda'(a)\rangle[v'(a)]-\langle\lambda'(r)\rangle[v'(r)]-\langle\lambda'(ar)\rangle[v'(ar)]=0\cdots(*).$$

PROPOSITION 1.3. *Let F be a quasi-pythagorean field and v be a valuation on F. Then $\bar{F}$ is quasi-pythagorean. Moreover if $\Gamma\neq\Gamma^2$, then $\bar{F}$ is pythagorean.*

PROOF. For any element $x\in U$, we have $D_F(2)\subseteq D_F\langle 1, x\rangle$, so $D_F(2)\subseteq D_F\langle 1,\bar{x}\rangle$ by Proposition 1.2. This implies $R(\bar{F})=D_F(2)$; hence $\bar{F}$ is quasi-pythagorean. If $\Gamma\neq\Gamma^2$, then there exists $a\in\dot{F}\backslash U\dot{F}^2$. For this a and for any element $r\in D_F(2)\cap U$, $(*)$ holds. Since $v'(a)\neq 1$ and $v'(r)=1$, $(*)$ implies $\langle 1, -\lambda'(r)\rangle=0\in W(\bar{F})$. So $\lambda'(r)=\bar{r}\in(\bar{F})^{\cdot 2}$, and we have $D_F(2)=(D_F(2)\cap U)^-=(\bar{F})^{\cdot 2}$. Hence $\bar{F}$ is pythagorean. Q.E.D.

We call $K=F(\sqrt{a})$ a non-radical (quadratic) extension if $a\notin R(F)$. For a valuation v on F, we consider the condition $1+M\subseteq R(F)$. This is equivalent to the condition that, for any non-radical extension $K=F(\sqrt{a})$, there is a unique extension of v to a valuation v' on K.

LEMMA 1.4. *Let v be a valuation on F. If $1+M\subseteq R(F)$, then the restriction $f|_{I^2F}$ of the ring homomorphism f: $W(F)\to W(\bar{F})[\Gamma/\Gamma^2]$ to I^2F is injective.*

PROOF. Let J be the ideal of $W(F)$ generated by the set $\{\langle 1, -r\rangle | r\in R(F)\}$. Since the form $\langle 1, -r\rangle$, $r\in R(F)$ is universal, it can easily be shown that J is precisely the set $\{\langle 1, -r\rangle | r\in R(F)\}$. So $I^2F\cap J=\{0\}$ by [1], Hauptsatz. On the other hand, Proposition 1.1 implies Ker $f\subseteq J$, hence Ker $f\cap I^2F=\{0\}$.

Q. E. D.

Let v be a valuation on F. If $1+M\subseteq R(F)$ and $(*)$ holds for any $a\in\dot{F}$ and $r\in D_F(2)$, then we have $\langle 1, a\rangle\langle 1, -r\rangle=0$ by Lemma 1.4, and so F is quasi-pythagorean.

PROPOSITION 1.5. *Let v be a valuation on F with $1+M\subseteq R(F)$. Then the following statements hold.*

(1) If $\Gamma=\Gamma^2$ and $\bar{F}$ is quasi-pythagorean, then F is quasi-pythagorean.

(2) If $|\Gamma/\Gamma^2|=2$ and $\bar{F}$ is pythagorean, then F is quasi-pythagorean.

(3) If $|\Gamma/\Gamma^2|\geqq 4$ and $\bar{F}$ is formally real pythagorean, then F is quasi-pythagorean.

PROOF. To show (1), we note $\dot{F}=U\dot{F}^2$ since $\Gamma=\Gamma^2$. By Proposition 1.2, $\lambda'(r)\in D_{\bar{F}}(2)=R(\bar{F})$ for any $r\in D_F(2)$. Hence $\langle 1, \lambda'(a)\rangle\langle 1, -\lambda'(r)\rangle=0\in W(\bar{F})$ and $(*)$ holds for any $a\in\dot{F}$ and $r\in D_F(2)$. Next we prove (2) and (3). First we show that if $|\Gamma/\Gamma^2|\geqq 2$ and $\bar{F}$ is formally real pythagorean, then $\bar{F}$ is quasi-pythagorean. Since $\bar{F}$ is formally real, $D_F(\infty)\subseteq U\dot{F}^2$ by [7], Lemma 3.7. So $\lambda'(r)\in D_{\bar{F}}(2)=(\bar{F})^{\cdot 2}$ and $v'(r)=1$ for any $r\in D_F(2)$. This implies $(*)$ and F is quasi-pythagorean. Now, to complete the proof, we have to show that if $|\Gamma/\Gamma^2|=2$ and $\bar{F}$ is non-real pythagorean, then F is quasi-pythagorean. In this case we have $I\bar{F}=\{0\}$ and this shows $(*)$ since $|\Gamma/\Gamma^2|=2$. Q. E. D.

LEMMA 1.6. *Let F be a non-real quasi-pythagorean field. Then there is no valuation on F such that $|\Gamma/\Gamma^2|\geqq 4$.*

PROOF. Suppose on the contrary that there is a valuation on F with $|\Gamma/\Gamma^2|\geqq 4$. Let a, r be elements of $\dot{F}$ such that $v'(a)\neq 1$, $v'(r)\neq 1$ and $v'(ar)\neq 1$. Then $(*)$ is not valid. Since F is non-real quasi-pythagorean, $1+M\subseteq R(F)=\dot{F}$, and so $\langle 1, a\rangle\langle 1, -r\rangle\neq 0\in W(F)$. This contradicts the fact $r\in R(F)$. Q. E. D.

Combining Proposition 1.3, Proposition 1.5 and Lemma 1.6, we have the following theorem.

THEOREM 1.7. *Let v be a valuation on F with $1+M\subseteq R(F)$. Then F is* *quasi-pythagorean if and only if one of the following statements holds.*

(1) $\Gamma=\Gamma^2$ *and* $\bar{F}$ *is quasi-pythagorean.*

(2) $|\Gamma/\Gamma^2|=2$ *and* $\bar{F}$ *is pythagorean.*

(3) $|\Gamma/\Gamma^2|\geqq 4$ *and* $\bar{F}$ *is formally real pythagorean.*

PROOF. By Proposition 1.3 and Proposition 1.5, it suffices to show that if F is quasi-pythagorean and $|\Gamma/\Gamma^2|\geqq 4$, then $\bar{F}$ is formally real. Lemma 1.6 implies F is formally real, and v is compatible with the weak preordering $R(F)=D_F(\infty)$. Hence $\bar{F}$ is formally real by [7], Proposition 3.8. Q.E.D.

EXAMPLE 1.8. Let k be a non-real pythagorean field and $k((x))$ be the power series field in one variable x over k. Then $k((x))$ is quasi-pythagorean by Theorem 1.7, (2) but is not pythagorean because x is not a square. The field $k((x))((y))$ is not quasi-pythagorean by Lemma 1.6.

§2. Spaces of orderings

In this section, we shall study an equivalence of finite spaces of orderings. Let (X_1, G_1) and (X_2, G_2) be finite spaces of orderings in the terminology of [8] or [9]. A morphism φ of (X_1, G_1) to (X_2, G_2) is a group homomorphism $\varphi\colon \chi(G_1)\to\chi(G_2)$ which carries X_1 into X_2. A morphism φ is called an equivalence if $\varphi\colon \chi(G_1)\cong\chi(G_2)$ and $\varphi(X_1)=X_2$. Two spaces (X_1, G_1) and (X_2, G_2) are called equivalent (denoted $(X_1, G_1)\sim(X_2, G_2)$) if there exists such an equivalence. Let $X_1=\{\sigma_1,\dots,\sigma_n\}$ and V_1 be an n-dimensional vector space over $\boldsymbol{Z}_2=\boldsymbol{Z}/2\boldsymbol{Z}$. Let $\{e_1,\dots,e_n\}$ be a basis of V_1 and let W_1 be a subspace of V_1 generated by the set $\{e_{i_1}+e_{i_2}+e_{i_3}+e_{i_4}|\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}=1\}$. Since X_1 is finite, the group homomorphism $f_{X_1}\colon V_1\to\chi(G_1)$, defined by $f_{X_1}(e_i)=\sigma_i$, $i=1,\dots,n$, is surjective.

PROPOSITION 2.1. *In the above situation, we have* $\operatorname{Ker} f_{X_1}=W_1$.

PROOF. It is clear that $\operatorname{Ker} f_{X_1}\supseteq W_1$. For the reverse inclusion, it is sufficient to show that if $\sigma_{i_1}\cdots\sigma_{i_m}=1$, then $e_{i_1}+\cdots+e_{i_m}\in W_1$. The proof proceeds by induction on m. We may assume $m\geqq 6$ and $\sigma_{i_1},\dots,\sigma_{i_{m-1}}$ are linearly independent. Consider the subspace Y of X_1 generated by $\sigma_{i_1},\dots,\sigma_{i_{m-1}}$. By [8], Basic Lemma 3.1, it must consist of more than $\sigma_{i_1},\dots,\sigma_{i_m}$. Thus there exists an ordering σ_j which is the product of at least 3 and at most $m-3$ of $\sigma_{i_1},\dots,\sigma_{i_{m-1}}$. We may assume $\sigma_j=\sigma_{i_1}\cdots\sigma_{i_s}$ $(3\leqq s\leqq m-3)$. Then $\sigma_j\sigma_{i_{s+1}}\cdots\sigma_{i_m}=1$, and by inductive assumption, $e_j+e_{i_1}+\cdots+e_{i_s}$ and $e_j+e_{i_{s+1}}+\cdots+e_{i_m}$ are elements of W_1. Thus we have $e_{i_1}+\cdots+e_{i_m}\in W_1$. Q.E.D.

We denote by $\bar{f}_{X_1}$ the isomorphism $V_1/W_1\to\chi(G_1)$ which is induced by f_{X_1}.

PROPOSITION 2.2. *Let* (X_1, G_1) *and* (X_2, G_2) *be finite spaces of orderings. Then the following statements are equivalent*:

(1) $(X_1, G_1) \sim (X_2, G_2)$.

(2) *There exists a bijection* $f: X_1 \to X_2$ *which satisfies the condition that* $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$ *if and only if* $f(\sigma_1)f(\sigma_2)f(\sigma_3)f(\sigma_4) = 1$.

PROOF. The assertion (1)$\Rightarrow$(2) is clear. For (2)$\Rightarrow$(1), let $n = |X_1| = |X_2|$, and V_1 (resp. V_2) be an n-dimensional vector space with a basis $\{e_1, \ldots, e_n\}$ (resp. $\{e'_1, \ldots, e'_n\}$). Let W_1 (resp. W_2) be a subspace of V_1 (resp. V_2) generated by the set $\{e_{i_1} + e_{i_2} + e_{i_3} + e_{i_4} | \sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4} = 1\}$ (resp. $\{e'_{i_1} + e'_{i_2} + e'_{i_3} + e'_{i_4} | f(\sigma_{i_1})f(\sigma_{i_2})f(\sigma_{i_3}) f(\sigma_{i_4}) = 1\}$). Then the isomorphism $h: V_1 \to V_2$, defined by $h(e_i) = e'_i$, induces an isomorphism $\bar{h}: V_1/W_1 \to V_2/W_2$ by the assumption (2). By Proposition 2.1, two morphisms $\bar{f}_{X_i}: V_i/W_i \to \chi(G_i)$, $i = 1, 2$, are isomorphisms and so there exists an isomorphism $\varphi: \chi(G_1) \to \chi(G_2)$ such that $\varphi \bar{f}_{X_1} = \bar{f}_{X_2} h$. It is clear that $\varphi(X_1) = X_2$, so the assertion (2)$\Rightarrow$(1) is proved. Q.E.D.

Let P be a preordering of F. We denote by $X(F)$ the space of all orderings of F and by $X(F/P)$ the subspace of all orderings σ with $P(\sigma) \supseteq P$, where $P(\sigma)$ is the positive cone of σ. For a subset Y of $X(F)$, we denote by $Y^\perp$ the preordering $\cap P(\sigma)$, $\sigma \in Y$. For a form $f = \langle a_1, \ldots, a_n \rangle$ over F, if there exist $p_1, \ldots, p_n \in P \cup \{0\}$ such that $a_1p_1 + \cdots + a_np_n = b$ and $(p_1, \ldots, p_n) \neq (0, \ldots, 0)$, then we say that the form f represents b over P. We put $D_F(f/P) = \{b \in \dot{F} | f$ represents b over $P\}$. The topological structure of $X(F)$ is determined by Harrison sets $H_F(a) = \{\sigma \in X(F) | a \in P(\sigma)\}$ as its subbasis, where a ranges over $\dot{F}$, An arbitrary open set in $X(F)$ is thus a union of sets of the form $H_F(a_1, \ldots, a_r) = H_F(a_1) \cap \cdots \cap H_F(a_r)$. We write $H_F(a_1, \ldots, a_n/P) = H_F(a_1, \ldots, a_n) \cap X(F/P)$, where $a_i \in \dot{F}$. We put $H_F = \{x \in \dot{F} | D_F(\langle\langle x \rangle\rangle) D_F(\langle\langle -x \rangle\rangle) = \dot{F}\}$ and $H_F(P) = \{x \in \dot{F} | D_F(\langle\langle x \rangle\rangle/P) D_F(\langle\langle -x \rangle\rangle/P) = \dot{F}\}$.

Let $K = F(\sqrt{a})$ be a radical extension of F. We denote by ε and N the inclusion map $F \to K$ and the norm map $K \to F$ respectively. If P is of finite index, then there is a short exact sequence

$$1 \longrightarrow \dot{F}/H_F(P) \xrightarrow{\bar{\varepsilon}} \dot{K}/H_K(P') \xrightarrow{\bar{N}} \dot{F}/H_F(P) \longrightarrow 1$$

where $P' = \Sigma P \dot{K}^2$ and $\bar{\varepsilon}$, $\bar{N}$ are induced maps of ε and N respectively ([4], Theorem 3.10). We generalize this as the following theorem.

THEOREM 2.3. *Let P be a preordering of F of finite index, and $K = F(\sqrt{a})$ be a quadratic extension of F with $a \in H_F$ and $a \notin -P$. Then the sequence*

$$1 \longrightarrow \dot{F}/H_F(T) \xrightarrow{\bar{\varepsilon}} \dot{K}/H_K(P') \xrightarrow{\bar{N}} \dot{F}/H_F(T) \longrightarrow 1$$

is exact, where $T = D_F(\langle 1, a \rangle/P)$ and $P' = \Sigma P \dot{K}^2$.

For the proof of Theorem 2.3, we need some lemmas. First we note that $P' \cap F = D_F(\langle\langle a \rangle\rangle / P)$ and $P' = (X')^{\perp}$ where $X' = \{\tau \in X(K) | \text{the restriction of } \tau \text{ to } F \text{ belongs to } H_F(a/P)\}$ ([4], Lemma 3.1). The proofs of the following Lemma 2.4 and Corollary 2.5 are similar to those of [4], Lemma 3.4 and Corollary 3.5, and will be omitted.

LEMMA 2.4. *In the situation of Theorem* 2.3, *let* σ *and* τ *be arbitrary orderings of* $H_F(a/P)$ *and* σ_i, τ_i $(i=1, 2)$ *be the extensions to* K *of* σ, τ *respectively. Then* $\{\sigma_1, \sigma_2, \tau_1, \tau_2\}$ *is not a fan of index* 8.

For an ordering τ of K, we denote by $\bar{\tau}$ the ordering of K with the positive cone $P(\tau)^-$, where the bar means the conjugation of K over F. For a subset $B \subseteq X'$, we also write $\bar{B} = \{\bar{\tau} | \tau \in B\}$.

COROLLARY 2.5. *In the situation of Theorem* 2.3, *let* Y *be a connected component of* $X' = X(K/P')$. *Then* $Y \cap \bar{Y} = \phi$.

In [9], Marshall introduced the notion of direct sum of spaces of orderings ([9], Definition 2.6). Let $(X_i, G/\Delta_i)$ $i=1,\dots, k$ be subspaces of (X, G), and suppose $X = \cup X_i$, and that the product $\Pi[X_i] = \chi(G)$ is a direct product. Then (X, G) is called the direct sum of the subspaces $(X_i, G/\Delta_i)$, $i=1,\dots, k$ and written as $X = X_1 \oplus \cdots \oplus X_k$.

LEMMA 2.6. *In the situation of Theorem* 2.3, *let* Z *be a fan of index* 8 *in* $H_F(a/P)$. *Then* $Y = \{\tau \in X(F) | \tau|_F \in Z\}$, *the extension of* Z *to* K, *is a direct sum of two fans* Y_1, Y_2 *of index* 8 *such that* $Y_i|_F = Z$, $i=1, 2$.

PROOF. We put $P_0 = Z^{\perp}$ and $P'_0 = Y^{\perp}$. By [4], Corollary 3.3, the sequence

$$1 \longrightarrow \dot{F}/D_F(\langle\langle a \rangle\rangle / P_0) \xrightarrow{\bar{\varepsilon}} \dot{K}/P'_0 \xrightarrow{\bar{N}} \dot{F}/P_0$$

is exact. From the facts $Z \subseteq H_F(a/P)$ and $a \in H_F$, it follows that $D_F(\langle\langle a \rangle\rangle / P_0) = P_0$ and $\operatorname{Im} \bar{N} = P_0 D_F \langle 1, -a \rangle / P_0 = \dot{F}/P_0$. So the exactness of the sequence implies that $\dim \dot{K}/P'_0 = 6$. Let $Y = Y_1 \oplus \cdots \oplus Y_n$ be the decomposition of Y to the connected components. Since $|Y| = 8$, the following two cases can occur.

Case 1. $n=2$ and Y_i, $i=1, 2$ are fans of index 8.

Case 2. $n=3$ and $|Y_1| = |Y_2| = 1$, $|Y_3| = 6$.

In the case 2, Y_3 must contain a fan of index 8 that is an extension of two orderings of F. This contradicts Lemma 2.4. Thus $n=2$ and Y_i, $i=1, 2$ are fans of index 8. By Lemma 2.4, we have $Y_i|_F = Z$, $i=1, 2$. Q. E. D.

PROPOSITION 2.7. *In the situation of Theorem* 2.3, *we have* $H_F(T) = H_K(P') \cap \dot{F}$.

PROOF. First we show the inclusion $H_F(T) \subseteq H_K(P') \cap \dot{F}$. Let x be an element of F such that $x \notin H_K(P')$. We must show $x \notin H_F(T)$. We note that P' is of finite index by [4], Corollary 3.3. Since $x \notin H_K(P')$, there exists a fan of index 8, $\{\tau_1, \tau_2, \tau_3, \tau_4\}$, in $X(K/P')$ such that $\tau_1, \tau_2 \in H_K(-x/P')$ and $\tau_3, \tau_4 \in H_K(x/P')$ by [4], Proposition 2.4. Let σ_i, $i=1,\dots, 4$ be the restrictions of τ_i to F. Then it is clear that $\sigma_1\sigma_2\sigma_3\sigma_4=1$ and orderings $\sigma_1,\dots, \sigma_4$ are distinct by Lemma 2.4. Thus $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is a fan of index 8, where $\sigma_1, \sigma_2 \in H_F(-x/T)$ and $\sigma_3, \sigma_4 \in H_F(x/T)$. This implies $x \notin H_F(T)$ by [4], Proposition 2.4.

Next we show the reverse inclusion $H_F(T) \supseteq H_K(P') \cap F$. Let x be an element of F with $x \notin H_F(T)$. We must show $x \notin H_K(P')$. Since $x \notin H_F(T)$, there exists a fan of index 8, $Z=\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, in $X(F/T)$ such that $\sigma_1, \sigma_2 \in H_F(-x/T)$ and $\sigma_3, \sigma_4 \in H_F(x/T)$. By Lemma 2.6, the extension of Z to K is a direct sum of two fans Z_1, Z_2 of index 8 such that $Z_i|_F=Z$, $i=1, 2$. From the facts $Z_i \cap H_K(x/P') \neq \phi$ and $Z_i \cap H_K(-x/P') \neq \phi$ $(i=1, 2)$, it follows that $x \notin H_K(P')$ by [4], Proposition 2.4. Q. E. D.

The proof of the following proposition is similar to that of [4], Proposition 3.9, and will be omitted.

PROPOSITION 2.8. *In the situation of Theorem* 2.3, $N(H_K(P')) \subseteq H_F(T)$.

Now we shall prove the exactness of the sequence

$$1 \longrightarrow \dot{F}/H_F(T) \xrightarrow{\bar{\varepsilon}} \dot{K}/H_K(P') \xrightarrow{\bar{N}} \dot{F}/H_F(T) \longrightarrow 1$$

in Theorem 2.3. By Proposition 2.7, $\bar{\varepsilon}$ is injective and by Proposition 2.8, $\bar{N}$ is well-defined. $\bar{N}$ is surjective since $a \in H_F$ and it is clear that $\operatorname{Im}\bar{\varepsilon} \subseteq \operatorname{Ker}\bar{N}$. It remains to prove that $\operatorname{Im}\bar{\varepsilon} \supseteq \operatorname{Ker}\bar{N}$. For this, we have only to show that $\dim \dot{K}/H_K(P') \leqq 2\dim \dot{F}/H_F(T)$. By [4], Corollary 3.3, the sequence

$$1 \longrightarrow \dot{F}/T \longrightarrow \dot{K}/P' \longrightarrow \dot{F}/T \longrightarrow 1$$

is exact, and so $\dim \dot{K}/P' = 2\dim \dot{F}/T$. Thus it suffices to show that $\dim H_K(P')/P' \geqq 2\dim H_F(T)/T$. The number n of connected components of $X(F/T)$ equals $\dim H(T)/T$ by [4], Theorem 2.5. Let $X_1,\dots, X_n$ be the connected components of $X(F/T)$. By [4], Proposition 2.4, there exist $a_i \in H_F(T)$, $i=1,\dots, n$ such that $X_i=H_F(a_i/T)$. Let Y_i, $i=1,\dots, n$ be the extensions of X_i to K. Then $Y_i= H_K(a_i/P')$ and each Y_i is a full subspace of $X(K/P')$ since $a_i \in H_K(P')$. It is clear that the sets Y_i, $i=1,\dots, n$ are pairwise disjoint. By Corollary 2.5, Y_i is not connected for any i, and hence the number of connected components of $X(K/P')$ is at least $2n$. Thus, it follows from [4], Theorem 2.5 that $\dim H_K(P')/P' \geqq 2\dim H_F(T)/T$ and the proof of Theorem 2.3 is completed.

Let n be the number of the connected components of $X(F/T)$; then that of $X(K/P')$ equals $2n$ by Theorem 2.3. Moreover we have the following theorem.

THEOREM 2.9. *In the situation of Theorem* 2.3, *the space* $X(K/P')$ *is equivalent to* $X(F/T)\oplus X(F/T)$.

PROOF. Since $N(\sqrt{a})\in H_F\subseteq H_F(T)$, there exists an element $g\in \dot{F}$ such that $g\sqrt{a}\in H_K(P')$ by Theorem 2.3. We put $Y_1=H_K(g\sqrt{a}/P')$ and $Y_2=H_K(-g\sqrt{a}/P')$. Then Y_1 and Y_2 are full subspaces of $X(K/P')$ and we have $X(K/P')=Y_1\cup Y_2$ (disjoint). We shall show $Y_1\sim X(F/T)$. For $\sigma\in Y_1$, we denote by $f(\sigma)\in X(F/T)$ the restriction of σ to F. It is clear that the mapping f: $Y_1\to X(F/T)$ is bijective, and that if $\sigma_1\sigma_2\sigma_3\sigma_4=1$ ($\sigma_i\in Y_1$), then $f(\sigma_1)f(\sigma_2)f(\sigma_3)f(\sigma_4)=1$. Conversely let $Z=\{f(\sigma_1), f(\sigma_2), f(\sigma_3), f(\sigma_4)\}$ be a fan of index 8 in $X(F/T)$. Then the extension of Z to K is a direct sum of two fans Z_1, Z_2 of index 8 and $Z_i|_F=Z$, $i=1, 2$ by Lemma 2.6. We may assume that $Z_i\subseteq Y_i$, $i=1, 2$ since Y_i, $i=1, 2$ are full subspaces of $X(K/P')$. Hence $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}=Y_1$, and so $\sigma_1\sigma_2\sigma_3\sigma_4=1$. This shows that f satisfies the condition of Proposition 2.2, (2) and we have $Y_1\sim X(F/T)$. Similarly, $Y_2\sim X(F/T)$. Q.E.D.

EXAMPLE 2.10. We give an example of a quadratic extension K of a field F such that K is S.A.P. and F is not S.A.P. Let F be a quasi-pythagorean field whose Kaplansky's radical $R(F)$ is of finite index. Let X_i, $i=1,\dots, n$ be the connected components of $X(F)$. Suppose $|X_1|>1$ and $|X_i|=1$ for $i=2,\dots, n$. By [4], Proposition 2.4, there exists $a\in H_F$ such that $H_F(a)=\cup X_i$, $i=2,\dots, n$. Put $K=F(\sqrt{a})$. Then by Theorem 2.9, K is S.A.P., but F is not S.A.P. because $|X_1|>1$.

§3. Quadratic extensions of quasi-pythagorean fields

In [9], Definition 3.6, a space of orderings (X, G) is called a group extension of (X', G') if G' is a subgroup of G and $X=\{\sigma\in\chi(G)|\,\sigma|_{G'}\in X'\}$. We call (X, G) an n-dimensional group extension of (X', G') if $\dim G/G'=n$. Let P be a preordering of a field F. We say $x\in\dot{F}$ is P-rigid if $D_F(\langle 1, x\rangle/P)=P\cup xP$. If $\dot{F}\neq P\cup -P$ we will say $x\in\dot{F}$ is P-basic if either x or $-x$ is not P-rigid. In case $\dot{F}=P\cup -P$, we consider all elements of $\dot{F}$ to be P-basic. We denote by B_P the set of P-basic elements of $\dot{F}$.

Throughout this section, we assume that F is a formally real quasi-pythagorean field and $X(F)$ is a finite connected space. Then we have $\mathrm{gr}(X(F))\neq\{1\}$ by [8], Theorem 4.7. Also we have $B_{R(F)}=\cap \operatorname{Ker}\alpha$, $\alpha\in\mathrm{gr}(X(F))$ by [10], Theorem 6.6. Let X_1 be the set of all restrictions $\sigma|_{B_{R(F)}}$, $\sigma\in X(F)$. Then $(X_1, B_{R(F)})$ is a space of orderings by [8], Theorem 4.8 and $X(F)$ is an n-dimensional group extension of $(X_1, B_{R(F)})$, where $n=\dim\mathrm{gr}(X(F))$. For $\alpha\in\mathrm{gr}(X(F))$, let X_α be the set of all restrictions $\sigma|_{\operatorname{Ker}\alpha}$, $\sigma\in X(F)$. Then the same arguments hold for the case $(X_\alpha, \operatorname{Ker}\alpha)$; so $(X_\alpha, \operatorname{Ker}\alpha)$ is a space of orderings and $X(F)$ is a 1-dimensional group extension of $(X_\alpha, \operatorname{Ker}\alpha)$.

LEMMA 3.1. *Let a be an element of $\dot{F}\backslash B_{R(F)}$. Then $X(F)$ is equivalent to a 1-dimensional group extension of $H_F(a)$.*

PROOF. By the assumption $a \notin B_{R(F)}$, there exists $\alpha \in \mathrm{gr}(X(F))$ such that $\alpha(a) = -1$. Let f: $H_F(a) \to X_\alpha$ be the map defined by $f(\sigma) = \sigma|_{\mathrm{Ker}\,\alpha}$. It is clear that f is bijective and satisfies the condition that if $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$ and $\sigma_i \in H_F(a)$, then $f(\sigma_1)f(\sigma_2)f(\sigma_3)f(\sigma_4) = 1$. Conversely let $\{f(\sigma_1), f(\sigma_2), f(\sigma_3), f(\sigma_4)\}$ be a fan of index 8 in X_α. Then $\{\sigma_i, \alpha\sigma_i | i = 1, \ldots, 4\}$ is a fan of index 16 and so $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$. By Proposition 2.2, $H_F(a)$ is equivalent to X_α. Thus $X(F)$ is equivalent to a 1-dimensional group extension of $H_F(a)$. Q.E.D.

LEMMA 3.2. *Let $K = F(\sqrt{a})$ be a quadratic extension of F where $a \in \dot{F} \smallsetminus B_{R(F)}$. Then for any fan Y in $H_F(a)$, the extension of Y to K is also a fan.*

PROOF. We put $P = Y^\perp$ and $P' = \Sigma P\dot{K}^2$. Then by [4], Corollary 3.3, the sequence

$$1 \longrightarrow \dot{F}/P \xrightarrow{\bar{\varepsilon}} \dot{K}/P' \xrightarrow{\bar{N}} \dot{F}/P$$

is exact. From the fact $a \notin B_{R(F)}$, it follows that $-a$ is $R(F)$-rigid, and so $\dim \mathrm{Im}\cdot(\bar{N}) = \dim(D_F\langle 1, -a\rangle P/P) = 1$. Hence we have $n = \dim \dot{F}/P + 1$, where $n = \dim \dot{K}/P'$. Now the fact $|X(K/P')| = 2|Y| = 2^{n-1}$ implies that $X(K/P')$ is a fan. Q.E.D.

THEOREM 3.3. *Let $K = F(\sqrt{a})$ be a quadratic extension of F, where $a \in \dot{F} \smallsetminus B_{R(F)}$. Then $X(K)$ is equivalent to $X(F)$.*

PROOF. We fix an ordering $\sigma \in X(K)$ and put $\beta = \sigma\bar{\sigma}$, where the bar means the conjugation of K over F. Then $\beta(\sqrt{a}) = -1$ and $\beta = \tau\bar{\tau}$ for any $\tau \in X(K)$ by Lemma 3.2. Hence $\beta\tau = \bar{\tau}$ and this shows that $\beta \in \mathrm{gr}(X(K))$. $X(K)$ is equivalent to a 1-dimensional group extension of $(X_\beta, \mathrm{Ker}\,\beta)$ and X_β is equivalent to $H_K(\sqrt{a})$. So it is sufficient to show that $H_K(\sqrt{a})$ is equivalent to $H_F(a)$ by Lemma 3.1. Let f: $H_K(\sqrt{a}) \to H_F(a)$ be the map defined by $f(\sigma) = \sigma|_F$. Then f is a bijection and satisfies the condition that $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$ if and only if $f(\sigma_1)f(\sigma_2)f(\sigma_3)f(\sigma_4) = 1$ by Lemma 3.2. Hence $H_K(\sqrt{a}) \sim H_F(a)$ by Proposition 2.2. Q.E.D.

Now we consider the case $K = F(\sqrt{a})$, $a \in B_{R(F)}$. If $a \in R(F)$, then $X(K) \sim X(F) \oplus X(F)$ by Theorem 2.9. In the rest of this section, we assume that $a \in B_{R(F)} \smallsetminus \pm R(F)$ and $D_F\langle 1, -a\rangle D_F\langle 1, a\rangle = B_{R(F)}$ ($D_F\langle 1, -a\rangle D_F\langle 1, a\rangle \subseteq B_{R(F)}$ always holds by [8], Lemma 4.9). We note that $X(F)$ is not a fan since $B_{R(F)} \neq \pm R\,(F)$.By [5], Theorem 3.4, there exists a valuation v on F such that v is compatible with $R(F)$ and $X(\bar{F})$ is not connected. Moreover $X(F)$ is equivalent to an n-dimensional group extension of $X(\bar{F})$, where $n = \dim \Gamma/\Gamma^2 = \dim \mathrm{gr}$

$(X(F))$ (see [5], Proposition 1.1). So $(X_1, B_{R(F)})$ is equivalent to $X(\bar{F})$. The bijective map $f: X_1 \to X(\bar{F})$ is defined as follows; for $\tau \in X_1, f(\tau) = \bar{\sigma}$, where τ is the restriction of σ to $B_{R(F)}$. We put $H_{X_1}(a) = \{\sigma \in X_1 | \sigma(a) = 1\}$. Then $f(H_{X_1}(a)) = H_{\bar{F}}(\lambda'(a))$, and so $H_{X_1}(a)$ is equivalent to $H_{\bar{F}}(\lambda'(a))$. By Theorem 1.7, $\bar{F}$ is a formally real pythagorean field.

LEMMA 3.4. *In the above situation, the following statements hold.*

(1) $U\dot{F}^2 = B_{R(F)}$.

(2) $\lambda'(a) \in H_{\bar{F}}$ (i.e. $D_{\bar{F}}\langle 1, \lambda'(a)\rangle D_{\bar{F}}\langle 1, -\lambda'(a)\rangle = \bar{F}$).

PROOF. The inclusion $U\dot{F}^2 \supseteq B_{R(F)}$ follows from [7], Proposition 4.10. For the reverse inclusion $U\dot{F}^2 \subseteq B_{R(F)}$, it suffices to show that $U \subseteq \operatorname{Ker} \alpha$ for any $\alpha \in \operatorname{gr}(X(F))$. Since v is compatible with $R(F)$, we have $1+M \subseteq R(F) \cap U \subseteq \operatorname{Ker} \alpha \cap U$, and $\operatorname{Ker} \alpha \cap U/1+M$ is a subgroup of $(\bar{F})^{\cdot}$ of index at most 2. We consider $\bar{\alpha} = \operatorname{Ker} \alpha \cap U/1+M$ as an element of $\chi((\bar{F})^{\cdot}/(\bar{F})^{\cdot 2})$; then it is easy to see that $\bar{\alpha} \in \operatorname{gr}(X(\bar{F}))$, and so we have $\operatorname{Ker} \alpha \cap U/1+M = (\bar{F})^{\cdot}$ since $\operatorname{gr}(X(\bar{F})) = 1$. It implies that $U \subseteq \operatorname{Ker} \alpha$ and the assertion (1) is proved. Now the assertion (2) follows from Proposition 1.2. Q.E.D.

THEOREM 3.5. *Let F be a formally real quasi-pythagorean field which has a finite space of orderings $X(F)$. Let $K = F(\sqrt{a})$ be a quadratic extension of F where a is an element of $B_{R(F)} \setminus \pm R(F)$ such that $D_F\langle 1, a\rangle D_F\langle 1, -a\rangle = B_{R(F)}$. Then $X(K)$ is equivalent to an n-dimensional group extension of $H_{X_1}(a) \oplus H_{X_1}(a)$ where $n = \dim \operatorname{gr}(X(F))$.*

PROOF. The valuation v can be uniquely extended to a valuation $\tilde{v}$ on K, as we noted before Lemma 1.4. We denote by $\tilde{\Gamma}$ and $\bar{K}$ the value group and the residue field of $\tilde{v}$ respectively. The facts $a \notin R(F)$ and $(1+M)U^2 \subseteq R(F)$ imply that $\lambda'(a) \notin (\bar{F})^{\cdot 2}$, and so $\bar{K} = \bar{F}(\sqrt{\lambda'(a)})$ is a quadratic extension of $\bar{F}$. Since $[K: F] \geqq [\tilde{\Gamma}: \Gamma][\bar{K}: \bar{F}]$, we have $\tilde{\Gamma} = \Gamma$. We put $Y = \{\sigma \in X(K) | \tilde{v}$ is compatible with $\sigma\}$. Then $|Y| = 2^n |X(\bar{K})| = 2^{n+1} |H_{\bar{F}}(\lambda'(a))|$. Since $a \in \bigcap \operatorname{Ker} \alpha$, $\alpha \in \operatorname{gr}(X(F))$, we have $2^n |H_{X_1}(a)| = |H_F(a)|$ and so $|X(K)| = 2|H_F(a)| = 2^{n+1} |H_{X_1}(a)|$. As is noted before Lemma 3.4, $H_{X_1}(a)$ is equivalent to $H_{\bar{F}}(\lambda'(a))$. Thus $|X(K)| = 2^{n+1} |H_{\bar{F}}(\lambda'(a))|$. This shows that $|Y| = |X(K)|$, hence $\tilde{v}$ is compatible with $D_K(\infty)$, the weak preordering of K. By Lemma 3.4, $\lambda'(a) \in H_{\bar{F}}$, so $X(\bar{K})$ is equivalent to $H_{\bar{F}}(\lambda'(a)) \oplus H_{\bar{F}}(\lambda'(a))$ by Theorem 2.9. Now the assertion follows from $H_{X_1}(a) \sim H_{\bar{F}}(\lambda'(a))$. Q.E.D.

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