HIROSHIMA MATH. J.
16 (1986), 327–352

Generalized J-integral and three-dimensional fracture mechanics II —Surface crack problems—

Dedicated to Professor Sigeru Mizohata on his sixtieth birthday

Kohji OHTASUKA

(Received August 19, 1985)

Introduction

The Griffith energy balace theory in fracture mechanics has been reformulated in terms of J-integrals, so that the fundamental relationship:

the energy release rate is expressed by a J-integral

holds for crack grwoth problems. As stated in the previous paper [12], this relationship is valid only for simple (two-dimensional) models of fracture. Thus, in [12], we proposed generalized J-integrals (abbreviated to GJ-integrals) and established the relationship

(∗) *the energy release rate is expressed by a GJ-integral*

for a three-dimensional fracture problem in which the crack is strictly contained in the material and the stress is free on the crack surface. Here we note that the arguments in [12] are also applicable to the two-dimensional problems (see [11]).

The GJ-integral introduced in [12] is expressed as the sum of a surface integral P and a volume integral R. The term P corresponds to the original J-integral. An integral corresponding to the term R was also considered by Destuynder and Djaoua [1] for a two-dimensional problem, in which it was shown that the energy release rate can be as well expressed only by this integral. Moreover, in more general theory (cf. [13], [14]), the term R appears to play a leading part.

Another feature of J-integral is that it represents the singularity of the elastic field (see [1], [11]). For the problem considered in [12], it was shown that the singularity appears only on the edge of the crack and the GJ-integral vanishes if the singularity does not appear.

In this paper we treat the case where the crack intersects the surface of the material, and moreover the crack itself is pressurized. In practical problems, such crack arises in various cases such as pressure vessels containing surface cracks (see e.g. Kikuchi, Miyamoto and Sakaguchi [6]), and there is much practical interest in estimating safety of such cracked structure. For two-dimensional pressurized crack, there have been some attempts to modify the J-integral and to obtain a relation like (*); see e.g. Karlsson and Bäcklund [5], Chen and Wu [3].

The main purpose of this paper is to give an expression of GJ-integral and establish the relationship (*) for such three-dimensional pressurized surface crack problem. It turns out that the GJ-integral consists of three terms P, $R^{(1)}$ and $R^{(2)}$; P and $R^{(1)}$ correspond to P and R in [12] and the term $R^{(2)}$ reflects the pressure near the edge of the crack. We shall see that only the singularity on the edge of the crack (hence not that on the intersection of the crack and the surface of the material) contributes to the GJ-integral.

The whole arguments are based on functional analysis as in [12]. We need the density and the trace theorems for Sobolev spaces on a domain with a cut. But the usual density and trace theorems (see e.g. Nečas [10]) are not applicable to our domain, so in section 2, we formulate and prove these theorems. Recently a trance theorem for a two-dimensional domain with a cut has been given in Grisvard [4], but the method used in [4] is different from ours.

Throughout the paper the letters C, C_0, $C_1, \cdots$, will be used to denote various positive constants, which are not necessarily the same even in a single formula.

The author wishes to thank Dr. T. Miyakawa for his helpful suggestions for improvements of the manuscript. The author also would like to acknowledge the continuing guidance and encouragment of Professor Fumi-Yuki Maeda.

1. Quasi-static formulation of the problem

1.1. Let $\mathscr{B}$ be a three-dimensional elastic body which occupies in its non-deformed state a bounded domain G in $\boldsymbol{R}^3$ with smooth boundary. We assume that the body $\mathscr{B}$ contains a crack $\mathscr{C}_t$ parametrized by t in a fixed interval $[0, T]$. Each crack $\mathscr{C}_t$ is a surface crack, that is, it intersects the boundary of $\mathscr{B}$, and is assumed to be represented by a surface $\Sigma(t)$ in $\boldsymbol{R}^3$ in its non-deformed state. The deformation of the elastic body under consideration is described in terms of a family of elliptic boundary value problems with parameter t defined on the domains $\Omega(t)=G-\Sigma(t)$ (see subsection 1.2 below).

We assume that $\Sigma(t)$ extends smoothly along a C^∞-surface Π in $\boldsymbol{R}^3$ which is transversal to ∂G and divides G into two domains $G^\pm$ as follows:

(SC1) $G=G^+ \cup G^- \cup G^0$ with $G^0=G \cap \Pi$ and $G^+ \cap G^0=G^- \cap G^0=G^+ \cap G^-=\emptyset$;

(SC2) $L=\Pi \cap \partial G$ is a C^∞-curve.

We assume that the surfaces $\Sigma(t)$ are of the form

$$\Sigma(t) = \Lambda(t) \cap G^0,$$

where $\Lambda(t)$, $t \in [0, T]$, are relatively open subsets of Π such that

(SC3) the closure $\bar{\Lambda}(t)$ is a two dimensional C^∞-submanifold of Π with C^∞-

boundary $\partial\Lambda(t)$ intersecting transversally with L at two points $\lambda_1(t)$ and $\lambda_2(t)$;

(SC4) $\Lambda=\Lambda(0)\subset\Lambda(t)\subset\Lambda(t')$ for $0\leqq t\leqq t'$, and the limit $\lim_{t\to 0} t^{-1}|\Sigma(t)-\Sigma(0)|$ exists and is non-zero, where $|\cdot|$ denotes the surface area.

(SC5) There are smooth diffeomorphisms ϕ_t: $\partial\Lambda\to\partial\Lambda(t)$, $t\in[0, T]$, such that the map $(x, t)\to\phi_t(x)$ is smooth from $\partial\Lambda\times[0, T]$ into Π.

Hereafter we use the notation:

$\Gamma(t)=\partial G-\bar{\Sigma}(t)$; $\partial_1\Sigma(t)=\partial\Lambda(t)\cap G$; $\partial_2\Sigma(t)=\bar{\Lambda}(t)\cap\partial G$; $\Sigma=\Sigma(0)$, $\Omega=\Omega(0)$, $\Gamma=\Gamma(0)$, $\lambda_i=\lambda_i(0)$, $\partial_1\Sigma=\partial_1\Sigma(0)$; $\Gamma_\pm=\partial G^\pm-(G^0\cup L)$; and $\partial_1 Q=\partial Q\cap G$, $\partial_2 Q=\bar{Q}\cap\partial G$, for any set Q in $\boldsymbol{R}^3$.

In case $t=0$, (SC1)–(SC3) imply the following

LEMMA 1.1. *Let D be the open unit disc in $\boldsymbol{R}^2$ and $D_\pm=D\cap\{x\in\boldsymbol{R}^2;\ x_2\gtrless 0\}$. There is a C^∞-diffeomorphism $F_{\partial\Lambda}(\eta, \xi)$ from $\partial\Lambda\times D$ onto an open neighborhood $V_{\partial\Lambda}$ of $\partial\Lambda$ in $\boldsymbol{R}^3$ such that:*

$$F_{\partial\Lambda}(\eta, 0)=\eta \qquad \textit{for every} \quad \eta \quad \textit{in} \quad \partial\Lambda;$$

$$F_{\partial\Lambda}(\partial\Lambda\times[D\cap\{\xi_2=0\}])=\Pi\cap V_{\partial\Lambda};$$

$$F_{\partial\Lambda}(\partial\Lambda\times[D\cap\{\xi_1>0,\ \xi_2=0\}])=\Lambda\cap V_{\partial\Lambda};$$

$$F_{\partial\Lambda}(\partial_1\Sigma\times[D\cap\{\xi_1>0,\ \xi_2=0\}])=\Sigma\cap V_{\partial\Lambda};$$

$$F_{\partial\Lambda}(\{\lambda_i\}\times D)\subset\partial G \quad \textit{and} \quad F_{\partial\Lambda}(\{\lambda_i\}\times D_\pm)\subset\Gamma_\pm,\ i=1, 2.$$

We can construct a spray (see Lang [7; IV, 3]) wich fits into the shape of ∂G and $\partial\Lambda$, and then the usual argument on the existence of tubular neighborhood leads to Lemma 1.1; see e.g. [7; IV, 5]. We may as well start with assuming the existence of such $F_{\partial\Lambda}$. In terms of the curvilinear coordinate $(V_{\partial\Lambda}, F_{\partial\Lambda})$, each edge $\partial_1\Sigma(t)$ is parametrized as follows:

There are a number $T_0>0$ and a family $\{h(\cdot, t)$: $t\in[0, T_0]\}$ of smooth functions on $\partial\Lambda$ such that:

(i) $-1<h(\eta, t)\leqq 0$ and $h(\eta, 0)=0$ for all $\eta\in\partial\Lambda$ and $t\in[0, T_0]$.

(ii) The map h is smooth from $\partial\Lambda\times[0, T_0]$ into $(-1, 0]$.

(iii) $\partial_1\Sigma(t)\cap V_{\partial\Lambda}=\{x|x=F_{\partial\Lambda}(\eta, h(\eta, t), 0), \eta\in\partial_1\Sigma\}$,
$\Sigma(t)\cap V_{\partial\Lambda}=\{x|x=F_{\partial\Lambda}(\eta, \xi_1, 0), \eta\in\partial_1\Sigma,\ h(\eta, t)<\xi_1<1\}$,
$\lambda_i(t)=F_{\partial\Lambda}(\lambda_i, h(\lambda_i, t), 0)$ for each t and $i=1, 2$.

This is shown in essentially the same way as in [12, Lemma 4.3], and so the details are omitted.

For each $x \in V_{\partial A}$, let c_x be a positive number such that, for $0 \leqq t \leqq c_x$,

$$\kappa_t(x) = F_{\partial A}(\eta(x), \xi_1(x) + h(\eta(x), t), \xi_2(x)) \tag{1.1}$$

belongs to $V_{\partial A}$, where $(\eta(x), \xi_1(x), \xi_2(x)) = F_{\partial A}^{-1}(x)$. The parametrized path: $t \to \kappa_t(x)$ defines a vector field X on $V_{\partial A}$ as

$$\begin{aligned} X(x) &= (d/dt)\kappa_t(x)|_{t=0} \\ &= ((\partial/\partial t)h(\eta(x), t)|_{t=0})\,[(\partial/\partial\xi_1)F_{\partial A}]\,(\eta(x), \xi_1(x), \xi_2(x)). \end{aligned} \tag{1.2}$$

1.2. Let $\boldsymbol{u}(x) = \{u_j(x)\}_{j=1}^3$ denote the displacement vector of the elastic body under consideration. The constituent law is formulated as follows:

$$\sigma_{ij}(x) = [\sigma_{ij}(\boldsymbol{u})](x) = a_{ijkl}(x)e_{kl}(x),$$
$$e_{ij}(x) = [e_{ij}(\boldsymbol{u})](x) = (D_j u_i(x) + D_i u_j(x))/2, \quad D_i = \partial/\partial x_i,$$

where the components a_{ijkl} of Hooke's tensor are assumed to be in $C^\infty(\bar{G})$ and satisfy the symmetricity:

$$a_{ijkl} = a_{jilk} = a_{klij}$$

and the uniform ellipticity condition:

$$a_{ijkl}\xi_{ij}\xi_{kl} \geqq \alpha\xi_{ij}\xi_{ij} \ (\xi_{ij} \in \boldsymbol{R}^9) \text{ for some constant } \alpha > 0.$$

(Here and in what follows the summation convention is used.) Now let Γ_0 be a portion of ∂G along which the elastic body cannot move. We assume that Γ_0 is measurable with respect to the surface element of ∂G, has positive surface measure and dist $(\Gamma_0, \partial_2\Sigma(t)) > 0$. Let us denote by $\mathscr{Q}(t)$ the surface force acting on $\Gamma_1(t) = \Gamma(t) - \Gamma_0$, by $\mathscr{P}(t)$ the force acting on $\Sigma(t)$ and by $\mathscr{F}(t)$ the body force prescribed inside $\Omega(t)$. In this paper we assume that they are given by $\mathscr{F}(t) = \mathscr{F}|_{\Omega(t)}$, $\mathscr{P}(t) = \mathscr{P}|_{\Sigma(t)}$ and $\mathscr{Q}(t) = \mathscr{Q}|_{\Gamma_1(t)}$ for some $(\mathscr{F}, \mathscr{P}, \mathscr{Q})$ in $\{L^2(G)\}^3 \times \{H^{1/2}(G^0)\}^3 \times \{H^{1/2}(\partial G)\}^3$. Here, for an open set Q or a surface Q, we denote by $H^m(Q)$ $(m \geqq 0)$ the Sobolev space of order m with the norm $|\cdot|_{m,Q}$. Let

$$V(\Omega(t)) = \{\boldsymbol{v} \in H^1(\Omega(t))\}^3 | \boldsymbol{v}_\Gamma = 0 \text{ on } \Gamma_0\},$$

where $\boldsymbol{v}_\Gamma$ is the trace of $\boldsymbol{v}$ on Γ. We shall denote by $[\![\boldsymbol{v}]\!]$ the jump of $\boldsymbol{v}$ across $\Sigma(t)$, i.e., $[\![\boldsymbol{v}]\!] = \boldsymbol{v}^+ - \boldsymbol{v}^-$, where $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ are the traces of $\boldsymbol{v}$ on $\partial G^+ \cap G^0$ and $\partial G^- \cap G^0$, respectively. dS will denote generically the surface element. Then, under the hypotheses stated above, the quasi-static problem to be discussed is the following:

PROBLEM 1.2. *Find for each $t \in [0, T]$ a displacement vector $\boldsymbol{u}(t) \in V(\Omega(t))$ which minimizes the potential energy functional*

$$E(v;\mathscr{L}) = \int_{\Omega(t)} [\sigma_{ij}(v)e_{ij}(v)]/2dx - \int_{\Omega(t)} \mathscr{F}\cdot v dx$$
$$- \int_{\Gamma_1(t)} \mathscr{Q}\cdot v_\Gamma dS - \int_{\Sigma(t)} \mathscr{P}\cdot [\![v]\!] dS \quad \textit{for } v \in V(\Omega(t)).$$

This problem can be restated in the following form:

PROBLEM 1.2′. *Find* $u(t) \in V(\Omega(t))$ *such that*

$$\text{(1.3)} \quad \int_{\Omega(t)} \sigma_{ij}(u(t))\, e_{ij}(v)dx = \int_{\Omega(t)} \mathscr{F}\cdot v dx + \int_{\Gamma_1(t)} \mathscr{Q}\cdot v_\Gamma dS + \int_{\Sigma(t)} \mathscr{P}\cdot [\![v]\!]\, dS$$

for all $v \in V(\Omega(t))$.

Problem 1.2 is equivalent to the determination of a solution $u(t)$ of

$$\text{(1.4)} \quad \begin{cases} -D_j\sigma_{ij}(u(t)) = \mathscr{F}_i \text{ in } \Omega(t) \text{ in the distribution sense,} \\ (\sigma_{ij}(u(t)))_\Gamma \nu_j = \mathscr{Q}_i \text{ on } \Gamma_1(t), \\ \sigma_{ij}(u(t))^+ \nu_j = \sigma_{ij}(u(t))^- \nu_j = \mathscr{P}_i \text{ on } \Sigma(t) \ (i=1,2,3), \end{cases}$$

where ν_j are components of the outward normal to ∂G or the outward normal to G^0 with respect to ∂G^+.

Since $\Omega(t)$ satisfies the cone property (see e.g. [12, p. 23]), an argument similar to the one used in the proofs of [12, Theorem 2.5. and (4.28)] yields the estimate:

$$\text{(1.5)} \quad \int_{\Omega(t)} \sigma_{ij}(v)e_{ij}(v)dx \geqq C|v|^2_{1,\Omega(t)}$$

for all $v \in V(\Omega(t))$, $0 \leqq t \leqq T$, with $C>0$ independent of v and t. This estimate, together with the Lax-Milgram theorem, implies that Problem 1.2′ has a unique solution $u(t)$ in $V(\Omega(t))$ for each $t \in [0, T]$ and for any $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathscr{Q}) \in \mathscr{M}$;

$$\mathscr{M} = \{L^2(G)\}^3 \times \{H^{1/2}(G^0)\}^3 \times \{H^{1/2}(\partial G)\}^3,$$

and Green's operators

$$\mathscr{T}(t)\colon \mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathscr{Q}) \longrightarrow u(t) = \mathscr{T}(t;\mathscr{L}),\ t \in [0, T]$$

are uniformly bounded linear operators from $\mathscr{M}$ into $V(\Omega(t))$.

1.3. Since the differential operator $u \to D_j\sigma_{ij}(u)$ satisfies the uniform ellipticity condition as shown in Fichera [2, p. 91], a regularity result (see e.g. [2, Lectures 5 and 10]) implies that singular points of the elastic field belong to $(\bar{\Gamma}_0 \cap \bar{\Gamma}_1) \cup \partial\Sigma$. Here the term *singular* is used in the following sense: We call $p \in \bar{\Omega}$ a regular point of the elastic field, if there is an open neighborhood V_p of p such that

$u|_{\Omega\cap V_p}\in H^2(\Omega\cap V_p)$. A point p is called a singular point of the elastic field if p is not a regular point.

In order to state the relationship between the regularity and the direction of the crack, we introduce the space $\mathscr{N}(\Omega)$ of functions v in $L^2(\Omega)$ satisfying the following conditions:

(1.6) $v|_{\mathscr{U}\cap\Omega}\in H^1(\mathscr{U}\cap\Omega)$ for all open sets $\mathscr{U}$ such that $\mathscr{U}\subset G$, $\operatorname{dist}(\mathscr{U}, \partial\Sigma)>0$ and $\operatorname{dist}(\mathscr{U}, \bar{\Gamma}_0\cap\bar{\Gamma}_1)>0$;

(1.7) $((\mathscr{X}\cdot\nabla)v)|_{\mathscr{V}\cap\Omega}$ belongs to $L^2(\mathscr{V}\cap\Omega)$ for all domains $\mathscr{V}$ such that $\mathscr{V}\subset V_{\partial A}$, $\operatorname{dist}(\mathscr{V}, \partial V_{\partial A})>0$ and $\operatorname{dist}(\mathscr{V}, \partial_1\Sigma)>0$, where the vector field $\mathscr{X}$ on $V_{\partial A}$ is defined by

$$\mathscr{X}(x)=[(\partial/\partial\xi_1)]F_{\partial A}(\eta(x), \xi_1(x), \xi_2(x)), \quad x\in V_{\partial A}.$$

The space $\mathscr{N}(\Omega)$ is topologized as follows: A sequence $\{v_m\}$ converges to v in $\mathscr{N}(\Omega)$ if and only if

$$|v_m-v|_{0,\Omega}+|v_m-v|_{1,\mathscr{U}\cap\Omega}+|(\mathscr{X}\cdot\nabla)v_m-(\mathscr{X}\cdot\nabla)v|_{0,\mathscr{V}\cap\Omega}\longrightarrow 0 \quad \text{as} \quad m\to\infty$$

for each $\mathscr{U}$ as in (1.6) and each $\mathscr{V}$ as in (1.7). Notice that $\mathscr{N}(\Omega)$ is a Fréchet space with respect to the above topology.

THEOREM 1.3. *The operator* $\mathscr{L}\to D_j\mathscr{T}(0, \mathscr{L})$ *is continuous from* $\mathscr{M}$ *into* $\mathscr{N}(\Omega)$ *for* $j=1, 2, 3$.

PROOF. Let $\boldsymbol{u}=\mathscr{T}(0, \mathscr{L})$ and $\mathscr{L}\in\mathscr{M}$. For any domain $\mathscr{V}$ as in (1.7), we can prove $(\mathscr{X}\cdot\nabla)\boldsymbol{u}\in\{H^1(\mathscr{V}\cap\Omega)\}^3$ by estimating the difference quotient

$$[\boldsymbol{u}(F_{\partial A}(\eta x), \xi_1(x)+h, \xi_2(x)))-\boldsymbol{u}(x)]/h, \quad x\in\mathscr{V}\cap\Omega, \quad 0<h\leqq h_0,$$

where h_0 is a number such that $F_{\partial A}(\eta(x), \xi_1(x)+h_0, \xi_2(x))\in\Omega\cap V_{\partial A}$ for $x\in\mathscr{V}\cap\Omega$. Using this, we can obtain the assertion by modifying the argument given in [2, Lecture 5]; the details are omitted.

2. Density and trace theorems for $H^1(\Omega)$

Because our domain Ω does not have the local Lipschitz property, the usual density and trace theorems (see e.g. [10, pp. 67, 99 and 103] for local Lipschitz domains) are not applicable. Thus, in this section, we establish corresponding theorems for Ω.

2.1. For a domain Q in $\boldsymbol{R}^3$, we define the distance $\ell_Q(x, y)$ of two points x, y of Q by the infimum of lengths of all broken lines connecting x and y in Q. We notice that $\ell_\Omega(x, y)$ is not equivalent to the Euclidian distance. If Q has the local Lipschitz property (see [10, p. 15]), then ℓ_Q is equivalent to the Euclidian distance.

We can easily show

LEMMA 2.1. *Let Q be a domain with local Lipschitz property. Under a C^2-diffeomorphism of $\boldsymbol{R}^3$ onto $\boldsymbol{R}^3$, the image of Q has also local Lipschitz property.*

LEMMA 2.2. *Let Q and $\mathcal{O}$ be bounded domains in $\boldsymbol{R}^3$ and Φ a mapping from Q into $\mathcal{O}$. Assume that Q is covered by a family $\{Q_i\}_{i=1,\cdots,m}$, of domains with local Lipschitz property in $\boldsymbol{R}^3$ which satisfy the following:*

(2.1) *There is a constant C such that $|\Phi(x)-\Phi(y)|\leqq C|x-y|$ for any x, $y\in Q_i$, $i=1,\cdots,m$.*

(2.2) *The image Q_i^* of Q_i under Φ is a domain with local Lipschitz property for each i.*

Then there is a constant C_0 such that

$$\ell_{\mathcal{O}}(\Phi(x), \Phi(y)) \leqq C_0\ell_Q(x, y) \qquad \textit{for all} \quad x, y\in Q. \tag{2.3}$$

PROOF. Let B be an arbitrary broken line lying inside Q, which connects x, y in Q. By considering a subdivision of B, if necessary, we may assume that B consists of segments $\overline{z_{j-1}z_j}$, $j=1,\cdots,q$, with $z_0=x$, $z_q=y$, where for each j, z_{j-1} and z_j belong to $Q_{k(j)}$ for some $k(j)$, $1\leqq k(j)\leqq m$. By (2.1), (2.2), there is a constant C_1 such that

$$\begin{aligned}\ell_{\mathcal{O}}(\Phi(z_{j-1}), \Phi(z_j)) &\leqq \ell_{Q^*_{k(j)}}(\Phi(z_{j-1}), \Phi(z_j))\\ &\leqq C_1|\Phi(z_{j-1})-\Phi(z_j)| \leqq C_1C|z_{j-1}-z_j|\end{aligned}$$

for each $j=1,\cdots,q$. Hence we have

$$\ell_{\mathcal{O}}(\Phi(x), \Phi(y)) \leqq \textstyle\sum_{j=1}^q \ell_{\mathcal{O}}(\Phi(z_{j-1}), \Phi(z_j)) \leqq C_1C\sum_{j=1}^q |z_{j-1}-z_j|\,.$$

Therefore, we obtain (2.3) with $C_0=C_1C$.

Let us put

$$U = \{F_{\partial A}(\eta, \xi_1, \xi_2);\ \eta\in\partial A,\ |\xi_i|<1/2,\ i=1, 2\}$$

and

$$\Xi(\varepsilon) = \{F_{\partial A}(\eta, \xi_1, \xi_2);\ \eta\in\partial_1\Sigma,\ 0<\xi_1<1/2,\ 0<\xi_2<\varepsilon\xi_1\},$$

for $0\leqq\varepsilon\leqq1$. We consider the family of mappings $\{\Phi_\varepsilon\}_{0\leqq\varepsilon\leqq1}$ defined by

$$\Phi_\varepsilon(x) = \begin{cases} F_{\partial A}(\eta(x), \xi_1(x), [\xi_2(x)+\varepsilon\xi_1(x)]/2) & \text{for} \quad x\in\Xi(\varepsilon),\\ x \qquad \text{for} \quad x\in(U\cap\Omega)-\Xi(\varepsilon), \end{cases}$$

where $(\eta(x), \xi_1(x), \xi_2(x)) = F_{\partial\Lambda}^{-1}(x)$ for $x \in U$.

LEMMA 2.3. *For each* ε, $0 \leqq \varepsilon \leqq 1/2$, Φ_ε *maps* $U \cap \Omega$ *onto* $U(\varepsilon) = U \cap \Omega - \Xi(\varepsilon/2)$ *bijectively, and*

$$C_\varepsilon^{-1} \ell_{U\cap\Omega}(x, y) \leqq |\Phi_\varepsilon(x) - \Phi_\varepsilon(y)| \leqq C_\varepsilon \ell_{U\cap\Omega}(x, y)$$

for all $x, y \in U \cap \Omega$ *with a positive constant* C_ε *independent of* x, y.

PROOF. The bijectivity is clear. Let $Q_\varepsilon = \Xi(2\varepsilon) - \overline{\Xi(\varepsilon/2)}$. By Lemma 2.1, $U(\varepsilon)$, $\Xi(2\varepsilon)$, $U(2\varepsilon)$, Q_ε are domains with local Lipschitz property. By Lemma 2.2 we obtain the inequality of the lemma, since $U \cap \Omega = \Xi(2\varepsilon) \cup U(2\varepsilon)$, $\Phi_\varepsilon(\Xi(2\varepsilon)) = Q_\varepsilon$ and Φ_ε is the identity on $U(2\varepsilon)$.

LEMMA 2.4. *Let* $\zeta \in C_0^\infty(\boldsymbol{R}^3)$ *satisfy*

$$0 \leqq \zeta \leqq 1, \quad \operatorname{supp} \zeta \subset U \quad \textit{and} \quad \zeta = 1 \quad \textit{near} \quad \partial_1 \Sigma. \tag{2.4}$$

Then, for an arbitrary $w \in H^1(\Omega)$ *and* $0 < \varepsilon < 1/2$,

$$C_0^{-1} |\zeta w|_{1,U\cap\Omega} \leqq |(\zeta w) \circ \Phi_\varepsilon^{-1}|_{1,U(\varepsilon)} \leqq C_0 |\zeta w|_{1,U\cap\Omega} \tag{2.5}$$

with a constant C_0 *independent of* ε *and* w.

PROOF. Let $0 < \delta < \varepsilon$. Then Φ_ε^{-1} is a bilipschitz mapping from $U(\varepsilon + \delta)$ onto $U(2\delta)$ with a Lipschitz constant independent of ε and δ. We obtain the estimate (2.5) by the use of [10, Lemma 3.1]. This completes the proof of Lemma 2.4.

2.2. The well-known density theorem (see Meyers and Serrin [9]) shows that the subspace $C^\infty(\Omega) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$. However we cannot replace $C^\infty(\Omega)$ with $C^\infty(\bar{\Omega})$.

DEFINITION. For a bounded domain Q in $\boldsymbol{R}^3$, let $C_*^{0,1}(Q)$ be the set of all Lipschitz continuous functions with respect to ℓ_Q, i.e., for each $f \in C_*^{0,1}(Q)$ there is a constant C such that

$$|f(x) - f(y)| \leqq C \ell_Q(x, y) \qquad \text{for all} \quad x, y \in Q.$$

Using Lemma 2.4 we derive the following

THEOREM 2.5. $C_*^{0,1}(\Omega)$ *is dense in* $H^1(\Omega)$.

PROOF. Let $w \in H^1(\Omega)$ and set $v = \zeta w$ with $\zeta \in C_0^\infty(\boldsymbol{R}^3)$ satisfying (2.4). Since $\operatorname{dist}(\operatorname{supp}(1-\zeta), \partial_1\Sigma) > 0$, we can choose a covering $\{U_j\}$ of $\Omega \cap \operatorname{supp}(1-\zeta)$ such that each U_j is a domain with local Lipschitz property. Then, by the aid of a partition of unity subordinate to $\{U_j\}$ and by the usual density theorem (see e.g.

[10, p. 67]), we can construct a sequence $\{w_j\}_{j=1}^{\infty}$ in $C_*^{0,1}(\Omega)$ which converges to $(1-\zeta)w$ in $H^1(\Omega)$. Let $v^*=v\circ\Phi_{1/4}^{-1}$, which belongs to $H^1(U(1/4))$ by Lemma 2.4. Since $U(1/4)$ has local Lipschitz property, there is a sequence of functions $\{v_j\}_{j=1}^{\infty}$ in $C^{\infty}(\overline{U}(1/4))$ such that $v_j=0$ near $\partial U\cap\partial U(1/4)$ and $|v^*-v_j|_{1,U(1/4)}\to 0$ as $j\to\infty$. We now put $v_j^{\#}=v_j\circ\Phi_{1/4}$. Then the estimate (2.5) yields that

$$|v-v_j^{\#}|_{1,U\cap\Omega}\leqq C_0|v^*-v_j|_{1,U(1/4)}\longrightarrow 0 \quad \text{as} \quad j\longrightarrow\infty.$$

Since $v_j^{\#}=0$ near ∂U, extending $v_j^{\#}$ by 0 on $\Omega-U$, we have $v_j^{\#}\to v$ in $H^1(\Omega)$. Furthermore $v_j^{\#}\in C_*^{0,1}(\Omega)$ for all j by Lemma 2.3. This completes the proof of the theorem.

2.3. For a function v in $C_*^{0,1}(\Omega)$ we put

$$\gamma v=(v^+,\ v^-,\ v_\Gamma) \tag{2.6}$$

where v_Γ is the trace of v on Γ and $v^+(x)=\lim_{y\to x,y\in G^+}v(y)$, $v^-(x)=\lim_{z\to x,z\in G^-}v(z)$ for each $x\in\Sigma$. We shall now extend γ given by (2.6) to $v\in H^1(\Omega)$.

For a surface S in $\boldsymbol{R}^3$ with a distance d_S, we denote by $H^\alpha(S;\ d_S)$, $0<\alpha<1$, the space consisting of functions v in $L^2(S)$ such that

$$|v|_{\alpha,d_S}=\left\{|v|_{0,S}^2+\int_S\int_S|v(x)-v(y)|^2d_S(x,\ y)^{-2(\alpha+1)}dS(x)dS(y)\right\}^{1/2}<\infty.$$

$H^\alpha(S;\ d_S)$ is a Hilbert space with respect to the norm $|\cdot|_{\alpha,d_S}$. We omit d_S in case d_S is induced by the Euclidian distance in $\boldsymbol{R}^3$.

We define a distance $d_*(x,\ y)$ on Γ as follows: $d_*(x,\ y)=\lim_{j\to\infty}\ell_\Omega(x_j,\ y_j)$, where points $x_j\in\Omega$ and $y_j\in\Omega$ approach to x and y, respectively.

Let Q be either $\boldsymbol{R}_+^2$ or $\boldsymbol{R}_{++}^2=\{x\in\boldsymbol{R}^2;\ x_1>0,\ x_2>0\}$. We denote by $\mathscr{A}(Q)$ the space of functions w in $H^{1/2}(Q)$ such that

$$|w|_{\mathscr{A}(Q)}=\left\{|w|_{1/2,Q}^2+\int_Q x_2^{-1}|w(x)|^2dx\right\}^{1/2}<+\infty.$$

The space $\mathscr{A}(\boldsymbol{R}_+^2)$ coincides with $H_{00}^{1/2}(\boldsymbol{R}_+^2)$ given in Lions and Magenes [8, p. 66] which is the intermediate space $[H_0^1(\boldsymbol{R}_+^2),\ L^2(\boldsymbol{R}_+^2)]_{1/2}$. Hence by the interpolation theorem [8, Theorem 5.1], the zero-extension of w in $\mathscr{A}(\boldsymbol{R}_+^2)$ (resp. $\mathscr{A}(\boldsymbol{R}_{++}^2)$) is continuous from $\mathscr{A}(\boldsymbol{R}_+^2)$ (resp. $\mathscr{A}(\boldsymbol{R}_{++}^2)$) to $H^{1/2}(\boldsymbol{R}^2)$ (resp. $H^{1/2}(\{x\in\boldsymbol{R}^2;\ x_1>0\})$). The operator D_2 is continuous from $H^{1/2}(Q)$ to the dual $\mathscr{A}'(Q)$ of $\mathscr{A}(Q)$. Now let $\mathscr{A}(\Sigma)$ be the space of functions w in $H^{1/2}(\Sigma)$ such that

$$|w|_{\mathscr{A}(\Sigma)}=\left[|w|_{1/2,\Sigma}^2+\int_\Sigma d(x,\ \partial_1\Sigma)^{-1}|w(x)|^2dS\right]^{1/2}<+\infty,$$

where, for $x \in \Sigma$, $d(x, \partial_1\Sigma)=\inf_{y\in\partial_1\Sigma}|x-y|$. Then the above observation implies, via partition of unity, that the zero extension $\tilde{w}$ of w in $\mathscr{A}(\Sigma)$ to G^0, i.e.,

$$w \longrightarrow \tilde{w} = w \quad \text{on} \quad \Sigma; = 0 \quad \text{on} \quad G^0 - \Sigma,$$

is a continuous map from $\mathscr{A}(\Sigma)$ into $H^{1/2}(G^0)$. Here we cannot replace $\mathscr{A}(\Sigma)$ with $H^{1/2}(\Sigma)$. In the following we shall use the same notation w for the zero extension $\tilde{w}$ of w, when there is no ambiguity.

We shall prove

THEOREM 2.6 (cf. [4, Theorem 1.7.3]). *The mapping γ defined by* (2.6) *is extended to a continuous operator from $H^1(\Omega)$ into $\{H^{1/2}(\Sigma)\}^2 \times H^{1/2}(\Gamma; d_*)$ so that*

$$[\![v]\!] \in \mathscr{A}(\Sigma) \quad \textit{for each} \quad v \in H^1(\Omega), \quad \textit{where} \quad [\![v]\!] = v^+ - v^-,$$

and the mapping $v \to [\![v]\!]$ is also continuous from $H^1(\Omega)$ into $\mathscr{A}(\Sigma)$.

2.4. In order to prove Theorem 2.6, we prepare

LEMMA 2.7. *We set $Q=\boldsymbol{R}^2_+$. For $(\phi_1, \phi_2) \in \{C_0^\infty(Q)\}^2$, we consider the following norms*

$$\left\{\int_Q\int_Q |\phi_1(x)-\phi_2(y)|^2 k(x, y)^{-3}dxdy + \textstyle\sum_{j=1}^2 |\phi_j|^2_{1/2,Q}\right\}^{1/2}, \tag{2.7}$$

$$\left\{\int_Q x_2^{-1}|\langle\phi\rangle(x)|^2 dx + \textstyle\sum_{j=1}^2 |\phi_j|^2_{1/2,Q}\right\}^{1/2}, \tag{2.8}$$

where $k(x, y)=\{(x_1-y_1)^2+x_2^2+y_2^2\}^{1/2}$ and $\langle\phi\rangle(x)=\phi_1(x)-\phi_2(x)$.

Then the norms (2.7) *and* (2.8) *are equivalent.*

PROOF. 1) We have

$$\int_Q\int_Q |\phi_1(x)-\phi_2(y)|^2 k(x, y)^{-3}dxdy \leqq 2(I_1+I_2),$$

where

$$I_1 = \int_Q \left\{\int_Q k(x, y)^{-3}dy\right\}|\langle\phi\rangle(x)|^2 dx = \pi\int_Q x_2^{-1}|\langle\phi\rangle(x)|^2 dx,$$

$$I_2 = \int_Q\int_Q |\phi_2(x)-\phi_2(y)|^2 k(x, y)^{-3}dxdy.$$

Since $k(x, y)^2 \geqq (x_1-y_1)^2+(x_2-y_2)^2$ in Q, $I_2 \leqq |\phi_2|^2_{1/2,Q}$. Hence

$$\int_Q\int_Q |\phi_1(x)-\phi_2(y)|^2 k(x, y)^{-3}dxdy \leqq 2\left\{\int_Q x_2^{-1}|\langle\phi\rangle(x)|dx + |\phi_2|^2_{1/2,Q}\right\}.$$

2) For $(\phi_1, \phi_2) \in \{C_0^\infty(Q)\}^2$, integration by parts yields

$$\int_Q x_2^{-1} |\langle\phi\rangle(x)|^2 dx = \int_0^\infty t^{-2} dt \int_{-\infty}^\infty dx_1 \int_0^t |\langle\phi\rangle(x)|^2 dx_2$$
$$\leqq \int_0^\infty t^{-2} dt \int_{-\infty}^\infty dx_1 \int_0^t t^{-1} [I_1(t, x) + I_2(t, x)] dx_2,$$

where

$$I_1(t, x) = \int_{x_1-t}^{x_1+t} |\phi_1(x_1, x_2) - \phi_2(z_1, x_2+t)|^2 dz_1,$$

$$I_2(t, x) = \int_{x_1-t}^{x_1+t} |\phi_2(z_1, x_2+t) - \phi_2(x_1, x_2)|^2 dz_1.$$

Using Fubini's theorem and setting $z_2 = x_2 + t$, we have

$$\int_0^\infty t^{-2} dt \int_{-\infty}^\infty dx_1 \int_0^t t^{-1} I_1(t, x) dx_2 \tag{2.9}$$
$$= \int_{-\infty}^\infty dx_1 \int_{-\infty}^\infty [I_{11}(x_1, z_1) + I_{12}(x_1, z_1)] dz_1,$$

where

$$I_{11}(x_1, z_1) = \int_0^{|z_1-x_1|} dx_2 \int_{x_2+|z_1-x_1|}^\infty (z_2-x_2)^{-3} |\phi_1(x) - \phi_2(z)|^2 dz_2,$$

$$I_{12}(x_1, z_1) = \int_{|z_1-x_1|}^\infty dx_2 \int_{2x_2}^\infty (z_2-x_2)^{-3} |\phi_1(x) - \phi_2(z)|^2 dz_2.$$

Inequalities $|z_1 - x_1| \geqq x_2 \geqq 0$, $z_2 \geqq x_2 + |z_1 - x_1|$ hold on the domain of integration of I_{11}, so that $(z_2 - x_2)^{-1} \leqq 2^{3/2} k(x, z)^{-1}$. Also the same inequality holds on the domain of integration of I_{12}, since $x_2 \geqq |z_1 - x_1|$, $z_2 \geqq 2x_2$ there. Thus it follows from (2.9) that

$$\int_0^\infty t^{-2} dt \int_{-\infty}^\infty dx_1 \int_0^t t^{-1} I_1(t, x) dx_2 \leqq 2^{9/2} \int_Q \int_Q |\phi_1(x) - \phi_2(z)|^2 k(x, z)^{-3} dx dz.$$

In the same way, we have

$$\int_0^\infty t^{-2} dt \int_{-\infty}^\infty dx_1 \int_0^t t^{-1} I_2(t, x) dx \leqq 2^{9/2} \int_Q \int_Q |\phi_2(x) - \phi_2(z)|^2 |x-z|^{-3}\, dx dz.$$

Collecting terms, we thus have

$$\int_Q x_2^{-1} |\langle\phi\rangle(x)|^2 dx \leqq 2^{9/2} \left\{ \int_Q \int_Q |\phi_1(x) - \phi_2(y)|^2 k(x, y)^{-3} dx dy + |\phi_2|_{1/2,Q}^2 \right\}.$$

This completes the proof of Lemma 2.6.

2.5. PROOF OF THEOREM 2.6. By Theorem 2.5 it suffices to prove that γ and $v \to [\![v]\!]$ are continuous on $C_*^{0,1}(\Omega)$. Let ζ be a cut-off function satisfying (2.4). Since $\operatorname{dist}(\operatorname{supp}(1-\zeta), \partial_1\Sigma) > 0$, by considering a covering of $\Omega \cap \operatorname{supp}(1-\zeta)$ by domains U_j with local Lipschitz property, and the partition of unity subordinate to $\{U_j\}$, we see that the usual trace theorem (see e.g. [10, pp. 99 and 103]) yields the continuity of $v \to \gamma((1-\zeta)v)$ from $H^1(\Omega)$ into $\{H^{1/2}\Sigma)\}^2 \times H^{1/2}(\Gamma; d_*)$. Obviously, $[\![(1-\zeta)v]\!] \in \mathscr{A}(\Sigma)$ and $|[\![(1-\zeta)v]\!]|_{\mathscr{A}(\Sigma)} \leqq C\{|[(1-\zeta)v]^+|_{1/2,\Sigma} + |[(1-\zeta)v]^-|_{1/2,\Sigma}\}$.

Let us set

$$H = \{x | x = F_{\partial A}(\eta, \xi_1, \xi_2), \eta \in \partial_1\Sigma, 0 \leqq \xi_1 \leqq 1/2, \xi_2 = \xi_1/8\},$$

$\Phi = \Phi_{1/4}$ and $Q = U(1/4)$. The restriction of Φ to $G^+ \cap U$ (written by Φ^+) is a bilipschitz mapping from $G^+ \cap U$ onto $G^+ \cap Q$ by Lemma 2.3. We continuously extend Φ_+ to the closure of $G^+ \cap U$. Then $\Phi^+(\Sigma \cap U) = H$ and $\Phi^+(\Gamma^+ \cap U) = \Gamma^+ \cap Q$. We set $k_0(x, y) = \{|x-y|^2 + d(x, \partial_1\Sigma)^2 + d(y, \partial_1\Sigma)^2\}^{1/2}$ for $x, y \in \Sigma$. Using local coordinate systems, we obtain the estimate

$$(2.10) \quad C_1^{-1}k_0(x, y) \leqq |\Phi^+(x) - y| \leqq C_1 k_0(x, y) \qquad \text{for all} \quad x, y \in \Sigma \cap U,$$

with C_1 independent of x, y. We next use Lemma 2.3 and the fact that Q, G^+ and G^- have local Lipschitz property, and obtain

$$(2.11) \qquad C_2^{-1}d_*(x, y) \leqq |\Phi^+(x) - y| \leqq C_2 d_*(x, y)$$

for all $x \in \Gamma^+ \cap U$ and $y \in \Gamma^- \cap U$ with C_2 independent of x, y. We put $w = \zeta v$ for $v \in C_*^{0,1}(\Omega)$ and $w^*(x) = w(\Phi^{-1}(x))$ for $x \in Q$. Since Q has local Lipschitz property, the usual trace theorem and (2.5) yield the estimate

$$(2.12) \qquad |w^*_{\partial Q}|_{1/2,\partial Q} \leqq C_3|w^*|_{1,Q} \leqq C_4|v|_{1,U\cap\Omega}$$

with constants C_3, C_4 independent of v. Decomposing the domain of integration, we can show that

$$|w^*_{\partial Q}|_{1/2,\partial Q} \geqq \{|w^*_H|^2_{1/2,H} + |w^*_{(+)}|^2_{1/,\Gamma_+\cap Q} + |w^*_{(-)}|^2_{1/2,\Gamma_-\cap Q} + |w^*_\Sigma|^2_{1/2,\Sigma\cap U} + I_1 + I_2\}^{1/2},$$

where w^*_H, $w^*_{(+)}$, $w^*_{(-)}$ and w^*_Σ denote the restrictions of $w^*_{\partial Q}$ to H, $\Gamma_+ \cap Q$, $\Gamma_- \cap Q$ and $\Sigma \cap Q$, respectively, and

$$I_1 = \int_{\Gamma_-\cap U}\int_{\Gamma_+\cap Q} |w^*_{(+)}(x) - w^*_{(-)}(y)|^2|x-y|^{-3}dS(x)dS(y),$$

$$I_2 = \int_H\int_{\Sigma\cap U} |w^*_\Sigma(x) - w^*_H(y)|^2|x-y|^{-3}dS(x)dS(y).$$

Since Φ is bilipschitz on $\bar{G}^+ \cap U$, we have by (2.10) and (2.11)

$$I_1 \geqq C_5 \int_{\Gamma_- \cap U} \int_{\Gamma_+ \cap U} |w_\Gamma(x) - w_\Gamma(y)|^2 d_*(x, y)^{-3} dS(x) dS(y),$$

$$I_2 \geqq C_5 \int_{\Sigma \cap U} \int_{\Sigma \cap U} |w^+(x) - w^-(y)|^2 k_0(x, y)^{-3} dS(x) dS(y)$$

with C_5 independent of w. By applying Lemma 2.7 with respect to the local coordinate systems of Σ and using a partition of unity, we obtain

$$\begin{aligned}&\{I_2 + |w^+|_{1/2,\Sigma\cap U}^2 + |w^-|_{1/2,\Sigma\cap U}^2\}\\ &\geqq C_6 \left\{\int_{\Sigma\cap U} d(x, \partial_1\Sigma)^{-1} | [\![w]\!](x)|^2 dS(x) + |w^+|_{1/2,\Sigma\cap U}^2 + |w^-|_{1/2,\Sigma\cap U}^2\right\}\end{aligned}$$

with C_6 independent of w. In the same way as in the proof of Lemma 2.4, we can show that the norms $|w_H^*|_{1/2,\mathrm{H}}^2$, $|w_{(+)}^*|_{1/2,\Gamma_+\cap Q}^2$, $|w_{(-)}^*|_{1/2,\Gamma_-\cap Q}^2$ and $|w_\Sigma^*|_{1/2,\Sigma\cap U}^2$ are equivalent to $|w^+|_{1/2,\Sigma\cap U}$, $|w_\Gamma|_{1/2,\Gamma_+\cap U}$, $|w_\Gamma|_{1/2,\Gamma_-\cap U}$ and $|w^-|_{1/2,\Sigma\cap U}$, respectively. Theorem 2.6 then follows from (2.12).

3. Generalized J-integral

3.1. The Lie derivative L_ξ with respect to a vector field ξ is determined by

$$L_\xi \omega = \lim_{t\to 0} t^{-1}[\alpha_t^* \omega - \omega] \tag{3.1}$$

for an r-form ω ($0 \leqq r \leqq n$), where $\alpha_t^*\omega$ is the pull-back of ω by the flow α_t for ξ (see e.g. [7, pp. 109–127]). If M is a surface in $\boldsymbol{R}^3$, then functions defined on M are 0-forms; for the surface element dS,

$$L_\xi(dS) = \mathrm{div}_M\, \xi dS, \tag{3.2}$$

where $\mathrm{div}_M\, \xi$ is the divergence of ξ with respect to M (see e.g. [7, p. 205]). If, for a submanifold N of M, the restriction $\alpha_t|_N$ is also a flow in N, then $\alpha_t|_N$ gives a Lie derivative, which we denote by $L_\xi|_N$. The local flow $x \to \kappa_t(x)$ on $V_{\partial A}$ given in (1.1) derives the Lie derivative $L_X = X \cdot \nabla$ for functions. We denote by $L_X|_N$ the restriction of L_X to $N \cap V_{\partial A}$, where $N = \partial G$ or G^0.

LEMMA 3.1. *Let ψ be an arbitrary function in $C_0^\infty(\boldsymbol{R}^2)$ such that* $\mathrm{supp}\, \psi \subset V_{\partial A}$. *If $v \in C_*^{0,1}(\Omega)$, then $\psi(L_X|_{\partial G} v_\Gamma) \in L^\infty(\Gamma)$ and $\psi(L_X|_{G^0} [\![v]\!]) \in L^\infty(\Sigma)$. Furthermore, the mapping $v \to \psi(L_X|_{\partial G} v_\Gamma)$* (resp. $v \to \psi(L_X|_{G^0} [\![v]\!])$) *from $C_*^{0,1}(\Omega)$ into $L^\infty(\Gamma)$* (resp. $L^\infty(\Sigma)$) *extends uniquely to a continuous mapping from $H^1(\Omega)$ into $H^{-1/2}(\partial\Omega)$* (resp. $H^{-1/2}(G^0)$). *We use the same notations $\psi(L_X|_{\partial G} v_\Gamma)$ and $\psi(L_X|_{G^0} [\![v]\!])$ for $v \in H^1(\Omega)$. Then*

$$\begin{aligned}\langle f, \psi(L_X|_{\partial G} v_\Gamma)\rangle_{\partial G} &= -\int_\Gamma [(L_X|_{\partial G} + \mathrm{div}_{\partial G} X)\ (\psi f)] v_\Gamma dS\\ \langle g, \psi(L_X|_{G^0} [\![v]\!])\rangle_{G^0} &= -\int_{G^0} [(L_X|_{G^0} + \mathrm{div}_{G^0} X)(\psi g)]\, [\![v]\!]\, dS\end{aligned} \tag{3.3}$$

hold for any element (f, g) *in* $C_L^\infty(\partial G)\times C_0^\infty(G^0)$, *where* $C_L^\infty(\partial G)=\{f\in C^\infty(\partial G);$ $f=0$ *near* $L\}$.

REMARK 3.2. For $v\in C_*^{0,1}(\Omega)$, the element $\{\psi(L_X|_{\partial G}v_\Gamma), \psi(L_X|_{G^0}[\![v]\!])\}$ in $L^\infty(\Gamma)\times L^\infty(\Sigma)$ can be considered as an element of $H^{-1/2}(\partial G)\times H^{-1/2}(G^0)$, i.e.,

$$\langle f, \psi(L_X|_{\partial G}v_\Gamma)\rangle_{\partial G} = \int_{\partial G} f\psi(L_X|_{\partial G}v)\,dS,$$

$$\langle g, \psi(L_X|_{G^0}[v])\rangle_{G^0} = \int_{G^0} g\psi(L_X|_{G^0}[\![v]\!])\,dS,$$

for all $(f, g)\in H^{1/2}(\partial G)\times H^{1/2}(G^0)$.

REMARK 3.3. $C_L^\infty(\partial G)\times C_0^\infty(G^0)$ is dense in $H^{1/2}(\partial G)\times H^{1/2}(G^0)$; see [8, p. 15].

PROOF OF LEMMA 3.1. If $v\in C_*^{0,1}(\Omega)$, the function v_Γ is Lipschitz continuous on Γ with respect to d_*, and $[\![v]\!]\in C^{0,1}(\Sigma)$. Since $\psi(L_X|_{\partial G})$ and $\psi(L_X|_{G^0})$ are differential operators of order 1, $\psi(L_X|_{\partial G})v_\Gamma\in L^\infty(\Gamma)$ and $\psi(L_X|_{G^0})[\![v]\!]\in L^\infty(\Sigma)$. Since $[\![v]\!]=0$ on $\partial_1\Sigma$, the Gauss theorem (see e.g. [7, p. 206]) gives

$$\int_\Gamma f\psi(L_X|_{\partial G}v_\Gamma)dS = -\int_\Gamma [(L_X|_{\partial G}+\mathrm{div}_{\partial G}X)\psi f]v_\Gamma dS,$$

$$\int_\Sigma g\psi(L_X|_{G^0}[\![v]\!])dS = -\int_\Sigma [(L_X|_{G^0}+\mathrm{div}_{G^0}X)\psi g]\,[\![v]\!]\,dS,$$

for $(f, g)\in C_L^\infty(\partial G)\times C_0^\infty(G^0)$, where we used the formula (3.2). Hence we have (3.3) in view of Remark 3.2. By Theorem 2.6, the operators $v\to v_\Gamma$ and $v\to[\![v]\!]$ are continuous from $H^1(\Omega)$ into $H^{1/2}(\Gamma; d_*)$ and from $H^1(\Omega)$ into $\mathscr{A}(\Sigma)$ respectively. Moreover the zero-extension of $w\in\mathscr{A}(\Sigma)$ to G^0 is continuous from $\mathscr{A}(\Sigma)$ into $H^{1/2}(G^0)$. Since the operators $f\to(L_X|_{\partial G}+\mathrm{div}_{\partial G}X)(\psi f)$ and $g\to(L_X|_{G^0}+\mathrm{div}_{G^0}X)(\psi g)$ are continuous from $H^{1/2}(\Gamma_\pm)$ into $H^{-1/2}(\Gamma_\pm)$ and from $H^{1/2}(G^0)$ into $H^{-1/2}(G^0)$, respectively, the right-hand sides of (3.3) are estimated by $|f|_{1/2,\partial G}|v|_{1,\Omega}$ and $|g|_{1/2,G^0}|v|_{1,\Omega}$ respectively. This, together with Remark 3.3 and Theorem 2.5, shows the assertions of the lemma.

3.2. LEMMA 3.4. *Let A be an open neighborhood of $\partial_1\Sigma$ such that the boundary ∂A is smooth and is transversal both to ∂G and to Π. Suppose $\bar{A}\subset V_{\partial A}$,* dist$(\partial A, \partial_1\Sigma)>0$ *and $\partial_2\Sigma\cap\partial A$ consists of two points. Then the trace operator: $g\to g|_{\partial_1 A\cap\Omega}$ from $\mathscr{N}(\Omega)$ into $H^{1/2}_{\mathrm{loc}}(\partial_1 A\cap\Omega)$ is also continuous from $\mathscr{N}(\Omega)$ into $L^2(\partial_1 A\cap\Omega)$.*

PROOF. The transversality between Π and ∂A implies that

$$\langle\mathscr{X}(x), \nu(x)\rangle \leqq -C_0 < 0 \qquad \text{for} \quad x\in\partial_2\Sigma\cap\partial A,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\boldsymbol{R}^3$ and ν the unit outward normal to ∂A. Let ω be a function in $C_0^\infty(V_{\partial A})$ such that $\omega \equiv 1$ near $\partial A \cap \partial_2 \Sigma$, $\partial A \cap \operatorname{supp} \omega = \emptyset$ and $\langle \mathscr{X}, \nu \rangle \leqq -C_0/2$ on $\partial A \cap \operatorname{supp} \omega$. For small $\delta > 0$ we set

$$V(\delta) = \{x \in \boldsymbol{R}^3;\ x = F_{\partial A}(\eta, \xi),\ \eta \in \partial_1 \Sigma,\ \xi \in D - \{\xi_1 \geqq 0,\ |\xi_2| \leqq \delta\}\},$$

$$\partial_2 V(\delta) = \cup_{i=1,2} \{x;\ x = F_{\partial A}(\lambda_i, \xi),\ \xi \in D - \{\xi_1 > 0,\ |\xi_2| \leqq \delta\}\}.$$

Let $f \in \mathscr{N}(\Omega)$; then $\omega f \in H^1(\Omega A(\delta))$, where $\Omega A(\delta) = A \cap V(\delta)$. Since $\Omega A(\delta)$ has the local Lipschitz property, we obtain

$$\int_{\Omega A(\delta)} (\mathscr{X} \cdot \nabla)\ (\omega^2 f^2) dx = \int_{\partial_1 \Omega A(\delta)} \omega^2 f^2 \langle \mathscr{X}, \boldsymbol{n} \rangle dS$$

$$+ \int_{\partial_2 V(\delta) \cap A} \omega^2 f^2 \langle \mathscr{X}, \boldsymbol{n} \rangle\, dS - \int_{\Omega A(\delta)} \omega^2 f^2 \operatorname{div} \mathscr{X} dx$$

for the unit outward normal $\boldsymbol{n}$ to $\partial V(\delta)$. We may write for sufficiently small δ

$$\partial_1 \Omega A(\delta) = (\partial_1 A \cap \overline{V(\delta)}) \cup (A \cap H_+(\delta)) \cup (A \cap H_-(\delta)),$$

where $H_\pm(\delta) = \{x;\ x = F_{\partial A}(\eta, \xi),\ \eta \in \partial_1 \Sigma,\ \xi \in D,\ \xi_1 > 0,\ \xi_2 = \pm \delta\}$. Note that $\langle \mathscr{X}, \boldsymbol{n} \rangle \equiv 0$ on $\partial_2 V(\delta)$ and $\langle \mathscr{X}, \boldsymbol{n} \rangle \equiv 0$ on $A \cap H_\pm(\delta)$, because $\mathscr{X}$ is tangent to $\partial_2 V(\delta)$ and to $A \cap H_\pm(\delta)$. Since $\boldsymbol{n} = \nu$ on $\partial_1 A$, it follows that

$$\int_{\partial_1 A \cap V(\delta)} \omega^2 f^2 \langle \mathscr{X}, \nu \rangle dS = \int_{\Omega A(\delta)} \{(\mathscr{X} \cdot \nabla)(\omega^2 f^2) + \omega^2 f^2 \operatorname{div} \mathscr{X}\} dx.$$

Using the assumption that $\langle \mathscr{X}, \nu \rangle \leqq -C_0/2$ on $\partial A \cap \operatorname{supp}\, \omega$, together with the Schwarz inequality, we therefore obtain

$$|\omega f|^2_{0, \partial_1 A \cap V(\delta)} \leqq C_1 \{|(\mathscr{X} \cdot \nabla) f|^2_{0, \mathscr{V} \cap \Omega} + |f|^2_{0, \Omega}\}$$

with C_1 independent of δ and f, where $\mathscr{V}$ is an open set in $\boldsymbol{R}^3$ as in (1.7) such that $\operatorname{supp} \omega \subset \mathscr{V}$. Hence, letting $\delta \to 0$, we have

$$|\omega f|^2_{0, \partial_1 A \cap \Omega} \leqq C_1 \{|(\mathscr{X} \cdot \nabla) f|^2_{0, \mathscr{V} \cap \Omega} + |f|^2_{0, \Omega}\}.$$

Since $\operatorname{dist}(\partial_1 A \cap \operatorname{supp}(1-\omega), \partial \Sigma) > 0$, there is an open set $\mathscr{U}$ in $\boldsymbol{R}^3$ as in (1.6) which contains $\partial_1 A \cap \operatorname{supp}(1-\omega)$. Hence by the usual trace theorem,

$$|(1-\omega) f|^2_{0, \partial_1 A \cap \Omega} \leqq C_2 |f|^2_{1, \mathscr{U} \cap \Omega}.$$

This proves Lemma 3.4.

3.3. We are now in a position to give the definition of generalized J-integral for the surface crack problem 1.1.

Definition 3.5. Let A be an open neighborhood of $\partial_1 \Sigma$ as given in Lemma

3.4, and $\varphi \in C_0^\infty(\boldsymbol{R}^3)$ a function such that $\operatorname{supp}\varphi \subset A$ and $\varphi \equiv 1$ near $\partial_1 \Sigma$. For $\mathscr{L}=(\mathscr{F}, \mathscr{P}, \mathscr{Q}) \in \mathscr{M}$, $\boldsymbol{u}=\mathscr{T}(0; \mathscr{L})$ and the vector field X given by (1.2), we define the *generalized J-integral acting on the surface crack* by

$$J_A(\mathscr{L}) = P_A(\mathscr{L}) + R_A^{(1)}(\mathscr{L}) + R_A^{(2)}(\mathscr{L}),$$

where

$$P_A(\mathscr{L}) = \int_{\partial_1 A} \{W(X \cdot \nu) - T \cdot ((X \cdot \nabla)\boldsymbol{u})\} dS;$$

$$R_A^{(1)}(\mathscr{L}) = -\int_{A \cap \Omega} \{((X \cdot \nabla) a_{ijkl}/2) e_{kl} e_{ij} + \mathscr{F} \cdot ((X \cdot \nabla)\boldsymbol{u})\} dx$$

$$+ \int_{A \cap \Omega} \{\sigma_{ij}(D_j X_k)(D_k u_i) - W(\operatorname{div} X)\} dx;$$

$$R_A^{(2)}(\mathscr{L}) = -\langle \mathscr{Q}, \varphi L_X|_{\partial G} \boldsymbol{u}_\Gamma \rangle_{\partial G} - \langle \mathscr{P}, \varphi L_X|_{G^0} [\![\boldsymbol{u}]\!] \rangle_{G^0}$$

$$-\int_{\Gamma \cap A} (1-\varphi)\mathscr{Q} \cdot (L_X|_{\partial G} \boldsymbol{u}_\Gamma) dS - \int_{\Sigma \cap A} (1-\varphi)\mathscr{P} \cdot (L_X|_{G^0} [\![\boldsymbol{u}]\!]) dS.$$

Here $W=\sigma_{ij}e_{ij}/2$ (the strain energy density); $T=(T_i)$ with $T_i=\sigma_{ij}\nu_j$, where ν_j are the components of the unit outward normal ν to ∂A. The term $(X \cdot \nabla)\boldsymbol{u}$ in $P_A(\mathscr{L})$ is the trace of $(X \cdot \nabla)\boldsymbol{u}$ on $\partial_1 A$, and $\langle\ ,\ \rangle_{\partial G}$ (resp. $\langle\ ,\ \rangle_{G^0}$) denotes the duality pairing between $\{H^{1/2}(\partial G)\}^3$ and $\{H^{-1/2}(\partial G)\}^3$ (resp. $\{H^{1/2}(G^0)\}^3$ and $\{H^{-1/2}(G^0)\}^3$). By Lemma 3.1, $R_A^{(2)}(\mathscr{L})$ is well-defined.

As is easily checked, $J_A(\mathscr{L})$ is independent of the choice of the function φ. It will be shown later (see Proposition 3.7), that $J_A(\mathscr{L})$ is also independent of the choice of A.

PROPOSITION 3.6. *$J_A(\mathscr{L})$ defines a continuous functional on $\mathscr{M}$; in fact the mapping $\mathscr{L} \to J_A(\mathscr{L})$ is uniformly continuous on bounded sets in $\mathscr{M}$.*

PROOF. Let $\boldsymbol{u}=\mathscr{T}(0, \mathscr{L})$. By Theorem 1.3, $\sigma_{ij}(\boldsymbol{u})$ and $e_{ij}(\boldsymbol{u})$ all belong to $\mathscr{N}(\Omega)$. Hence by Lemma 3.4 the operators

$$\mathscr{L} \longrightarrow \sigma_{ij}(\boldsymbol{u})|_{\partial_1 A \cap \Omega} \quad \text{and} \quad \mathscr{L} \longrightarrow e_{ij}(\boldsymbol{u})|_{\partial_1 A \cap \Omega}, \quad i, j = 1, 2, 3,$$

are continuous and linear from $\mathscr{M}$ into $L^2(\partial_1 A \cap \Omega)$. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be in $\mathscr{M}$ and $\boldsymbol{u}^k=\mathscr{T}(0; \mathscr{L}_k)$, $\sigma_{ij}^k=\sigma_{ij}(\boldsymbol{u}^k)$, $e_{ij}^k=e_{ij}(\boldsymbol{u}^k)$ and $T_i^k=\sigma_{ij}^k\nu_j$ for $k=1, 2$. Then we have

$$P_A(\mathscr{L}_1) - P_A(\mathscr{L}_2) = 2^{-1}\int_{\partial_1 A \cap \Omega} \{\sigma_{ij}^1(e_{ij}^1 - e_{ij}^2) + e_{ij}^2(\sigma_{ij}^1 - \sigma_{ij}^2)\}(X \cdot \nu) dS$$

$$-\int_{\partial_1 A \cap \Omega} \{T^1 \cdot ((X \cdot \nabla)(\boldsymbol{u}^1 - \boldsymbol{u}^2)) + (T^1 - T^2) \cdot ((X \cdot \nabla)\boldsymbol{u}^2)\} dS.$$

Applying the Schwarz inequality we obtain

$$|P_A(\mathscr{L}_1) - P_A(\mathscr{L}_2)| \leqq C_1(|\mathscr{L}_1|_{\mathscr{M}} + |\mathscr{L}_2|_{\mathscr{M}})|\mathscr{L}_1 - \mathscr{L}_2|_{\mathscr{M}}.$$

with C_1 independent of $\mathscr{L}_k$, $k=1, 2$. Similarly, the continuity property of the Green's operator $\mathscr{T}(0; \mathscr{L})$: $\mathscr{M} \to V(\Omega)$ implies the estimate:

$$|R_A^{(1)}(\mathscr{L}_1) - R_A^{(1)}(\mathscr{L}_2)| \leqq C_2(|\mathscr{L}_1|_{\mathscr{M}} + |\mathscr{L}_2|_{\mathscr{M}})|\mathscr{L}_1 - \mathscr{L}_2|_{\mathscr{M}}.$$

Finally we estimate

$$R_A^{(2)}(\mathscr{L}_1) - R_A^{(2)}(\mathscr{L}_2) = R_A^{21} + R_A^{22},$$

where

$$\begin{aligned} R_A^{21} = &- \langle \mathscr{Q}_1, \varphi L_X|_{\partial G}(\boldsymbol{u}^1 - \boldsymbol{u}^2)_\Gamma \rangle_{\partial G} - \langle (\mathscr{Q}_1 - \mathscr{Q}_2), \varphi L_X|_{\partial G} \boldsymbol{u}_\Gamma^2 \rangle_{\partial G} \\ &- \langle \mathscr{P}_1, \varphi L_X|_{G^0} [\![\boldsymbol{u}^1 - \boldsymbol{u}^2]\!] \rangle_{G^0} - \langle (\mathscr{P}_1 - \mathscr{P}_2), \varphi L_X|_{G^0} [\![\boldsymbol{u}^2]\!] \rangle_{G^0} \end{aligned}$$

and

$$\begin{aligned} R_A^{22} = &- \int_{\Gamma \cap A} (1-\varphi)\mathscr{Q}_1 \cdot (L_X|_{\partial G}(\boldsymbol{u}^1 - \boldsymbol{u}^2)_\Gamma) dS \\ &- \int_{\Gamma \cap A} (1-\varphi)(\mathscr{Q}_1 - \mathscr{Q}_2) \cdot (L_X|_{\partial G} \boldsymbol{u}_\Gamma^2) dS - \int_{\Sigma \cap A} (1-\varphi)\mathscr{P}_1 \cdot (L_X|_{G^0} [\![\boldsymbol{u}^1 - \boldsymbol{u}^2]\!]) dS \\ &- \int_{\Sigma \cap A} (1-\varphi)\ (\mathscr{P}_1 - \mathscr{P}_2) \cdot (L_X|_{G^0} [\![\boldsymbol{u}^2]\!]) dS. \end{aligned}$$

Using Lemma 3.1 we have

$$|R_A^{21}| \leqq C_3(|\mathscr{L}_1|_{\mathscr{M}} + |\mathscr{L}_2|_{\mathscr{M}})|\mathscr{L}_1 - \mathscr{L}_2|_{\mathscr{M}}.$$

with C_3 independent of $\mathscr{L}_k$, $k=1, 2$. Since $\operatorname{dist}(\operatorname{supp}(1-\varphi), \partial_1\Sigma) > 0$ and $\operatorname{dist}(A, \partial V_{\partial A}) > 0$, Theorem 1.3 and the usual trace theorem together imply that the operators $\mathscr{L} \to (1-\varphi)(L_X|_{\partial G}\boldsymbol{u}_\Gamma)|_{\Gamma \cap A}$, $(1-\varphi)(L_X|_{G^0}[\![\boldsymbol{u}]\!])|_{\Sigma \cap A}$ are continuous from $\mathscr{M}$ to $H^{1/2}(\Gamma \cap A)$ and from $\mathscr{M}$ to $H^{1/2}(\Sigma \cap A)$. Hence by the Schwarz inequality we obtain

$$|R_A^{22}| \leqq C_4(|\mathscr{L}_1|_{\mathscr{M}} + |\mathscr{L}_2|_{\mathscr{M}})||\mathscr{L}_1 - \mathscr{L}_2|_{\mathscr{M}}$$

with C_4 independent of $\mathscr{L}_k$, $k=1, 2$. We obtain the desired result by combining the above estimates.

3.4. PROPOSITION 3.7. *The value $J_A(\mathscr{L})$ is independent of the choice of the domain A.*

PROOF. Let A_1 and A_2 be two domains as in Lemma 3.4. Without loss of generality we may assume $\bar{A}_1 \subset A_2$. Let $Q = A_2 - \bar{A}_1$; then as in the proof of Lemma 3.4, we obtain

(3.4) $$\int_{Q\cap\Omega}(X\cdot\nabla)Wdx=\int_{\partial_1 Q\cap\Omega}W(X\cdot\nu)dS-\int_{Q\cap\Omega}W(\operatorname{div}X)dx.$$

On the other hand, since $e_{ij}=e_{ji}$ and $\sigma_{ij}=\sigma_{ji}$, we have

(3.5) $$\int_{Q\cap\Omega}(X\cdot\nabla)Wdx=\int_{Q\cap\Omega}((X\cdot\nabla)a_{ijkl}/2)e_{kl}e_{ij}dx$$
$$+\int_{Q\cap\Omega}\sigma_{ij}D_j((X\cdot\nabla)u_i)dx-\int_{Q\cap\Omega}\sigma_{ij}(D_jX_k)(D_ku_i)dx.$$

As in the proof of Lemma 3.4, it follows that

(3.6) $$\int_{Q\cap\Omega}\sigma_{ij}D_j((X\cdot\nabla)u_i)dx=-\int_{Q\cap\Omega}D_j\sigma_{ij}((X\cdot\nabla)u_i)dx$$
$$+\int_{\partial_1 Q}\sigma_{ij}\nu_j((X\cdot\nabla)u_i)dS+\int_{\Gamma\cap Q}(\sigma_{ij})_\Gamma\nu_j((X\cdot\nabla)(u_i)_\Gamma)dS$$
$$+\int_{\Sigma\cap Q}[\![\sigma_{ij}((X\cdot\nabla)u_i)]\!]\,\nu_jdS.$$

Since $\boldsymbol{u}=\mathscr{T}(0;\mathscr{L})$, we have (1.4) with $t=0$. Hence from (3.5), (3.6) and the fact that dist$(\partial Q, \partial_1\Sigma)>0$, we obtain

(3.7) $$\int_{Q\cap\Omega}(X\cdot\nabla)Wdx=\int_{Q\cap\Omega}\{((X\cdot\nabla)a_{ijkl}/2)e_{kl}e_{ij}+\mathscr{F}\cdot((X\cdot\nabla)\boldsymbol{u})\}dx$$
$$+\int_{\partial_1 Q}T\cdot((X\cdot\nabla)\boldsymbol{u})dS+\int_{\Gamma\cap Q}\mathscr{Q}\cdot((X\cdot\nabla)\boldsymbol{u}_\Gamma)dS$$
$$+\int_{\Sigma\cap Q}\mathscr{P}\cdot((X\cdot\nabla)[\![\boldsymbol{u}]\!])dS-\int_{Q\cap\Omega}\sigma_{ij}(D_jX_k)(D_ku_i)dx.$$

Equalities (3.4) and (3.7) yield $J_{A_1}(\mathscr{L})-J_{A_2}(\mathscr{L})=0$, which shows the assertion

In what follows, the subscript A of $J_A(\mathscr{L})$ will be dropped. If $\boldsymbol{u}=\mathscr{T}(0;\mathscr{L})$ is smooth up to the edge of the crack, then by letting $Q=A$ in the proof of Proposition 3.7, we have

PROPOSITION 3.8. *If $\boldsymbol{u}=\mathscr{T}(0;\mathscr{L})$ belongs to H^2 near $\partial_1\Sigma\cup\{\lambda_1, \lambda_2\}$, then $J(\mathscr{L})=0$.*

This, together with Theorem 4.1 below, indicates that the crack extension is caused by the singularities at $\partial_1\Sigma\cup\{\lambda_1, \lambda_2\}$, while the singularities at $\partial_2\Sigma-\{\lambda_1, \lambda_2\}$ have no effect on crack extension.

4. Calculation of the energy release rate

We now state our main result.

THEOREM 4.1. *Suppose a load $\mathscr{L}\in\mathscr{M}$ and a smooth crack extension $\{\Sigma(t)\}$*

as defined in section 1 *are given. Define the energy release rate by*

$$\mathscr{G}(\mathscr{L};\ \{\Sigma(t)\}) = \lim_{t\to 0} |E(\boldsymbol{u}(0);\ \mathscr{L}) - E(\boldsymbol{u}(t);\ \mathscr{L})| / |\Sigma(t) - \Sigma(0)|,$$

where $\boldsymbol{u}(t) = \mathscr{T}(t;\ \mathscr{L})$. *Then*

$$\mathscr{G}(\mathscr{L};\ \{\Sigma(t)\}) = |\delta\Sigma(t)|^{-1} J(\mathscr{L}),$$

where $|\delta\Sigma(t)| = \lim_{t\to 0} t^{-1}|\Sigma(t) - \Sigma(0)|$.

Before the proof of this theorem, we prepare some lemmas.

4.1. Let $\ell(t)$ be a non-increasing smooth function on $[0, \infty)$ such that $|\ell(t)| \leqq C_1 t$ with $C_1 > 0$ and $\ell(0) = 0$. We set

$$N(t) = \{x \in \boldsymbol{R}^3;\ x = (x_1, 0),\ \ell(t) \leqq x_1 < \infty\}, \quad S(t) = \boldsymbol{R}^2 - N(t).$$

Let $d(S(0)) = d(S(0);\ x, y)$ be the distance function on $S(0)$ defined as the infimum of the lengths of all broken lines inside $S(0)$ which connect the points x and y in $S(0)$. We set

$$\Delta_1^t w(x) = [w(x_1 - \ell(t), x_2) - w(x_1, x_2)] t^{-1}, \quad x \in S(1), \quad 0 < t < 1,$$

for a function w on $S(0)$.

LEMMA 4.2. *For* $f \in H^{1/2}(\boldsymbol{R}^2)$ *and* $w \in H^{1/2}(S(0);\ d(S(0)))$ *we have the estimate*

$$\left| \int_{S(1)} f \Delta_1^t w \, dx \right| \leqq C_0 |f|_{1/2, \boldsymbol{R}^2} |w|_{1/2, d(S(0))}$$

with C_0 *independent of* f, w *and* t. (See subsection 2.3 for $H^{1/2}(S(0);\ d(S(0)))$ and $|\cdot|_{1/2, d(S(0))}$.)

PROOF. Let $f \in C_0^\infty(R^2)$. Obviously

$$\int_{S(1)} f \Delta_1^t w \, dx = \int_{\boldsymbol{R}_+^2} f \Delta_1^t w \, dx + \int_{\boldsymbol{R}_+^2} f \Delta_1^t w \, dx. .$$

Since the path: $t \to (x_1 - \ell(t), x_2)$ is parallel to the line $x_2 = 0$, we have

$$\int_{\boldsymbol{R}_+^2} f \Delta_1^t w \, dx = \int_{\boldsymbol{R}_+^2} (\Delta_1^{-t} f) w \, dx,$$

where

$$\Delta_1^{-t} f(x) = [f(x_1 + \ell(t), x_2) - f(x_1, x_2)] (-t)^{-1} = (-t)^{-1} \int_0^{\ell(t)} D_1 f(x_1 + \xi, x_2) d\xi.$$

Since D_1 si continuous from $H^{1/2}(\boldsymbol{R}_+^2)$ to $H^{-1/2}(\boldsymbol{R}_+^2)$, we obtain, letting $f_\xi(x) = f(x_1 + \xi, x_2)$,

$$\left|\int_{R_+^2} f\Delta_1^t\, w dx\right| \leqq C_2 \sup_{0\leqq|\xi|\leqq C_1} |f_\xi|_{1/2,R_+^2}|w|_{1/2,R_+^2} \leqq C_2|f|_{1/2,R^2}|w|_{1/2,R_+^2}.$$

Noting that $d(S(0); x, y)=|x-y|$ for $x, y\in R_+^2$, we obtain

$$\left|\int_{R_+^2} f\Delta_1^t w dx\right| \leqq C_2|f|_{1/2,R^2}|w|_{1/2,d(S(0))}.$$

The estimate for the integral over R_-^2 is similar. This completes the proof of Lemma 4.2.

We next consider an analogous difference quotient in the direction x_2. Let $k\in C_0^\infty(\bar{R}_+^2)$ be a function such that $k(y, t)\leqq 0$ for $(y, t)\in\bar{R}_+^2$, $k(y, 0)\equiv 0$ and $k(y, t)\geqq k(y, t')$ if $t<t'$, for $y\in R^1$. We put

$$\Delta_2^t w(x) = [w(x_1, x_2-k(x_1, t))-w(x_1, x_2)]t^{-1}, \quad 0<t<1,$$

for a function w on R^2, and

$$K(t) = \{(x_1, x_2)\in R^2;\ x_2>k(x_1, t),\ x_1\in R^1\},$$

$$K_+(t) = K(t)\cap\{x\in R^2;\ x_1>0\}.$$

Then, recalling that D_2 is continuous from $H^{1/2}(R_+^2)$ into $\mathscr{A}(R_+^2)'$, by an argument similar to the above proof we obtain

LEMMA 4.3. *Let $Q(t)=K(t)$ or $Q(t)=K_+(t)$. Let $f\in H^{1/2}(R^2)$ and w be a function in $C^{0,1}(Q(0))\cap\mathscr{A}(Q(0))$ such that $w(x)=0$ on $x_2=0$. Then by considering the zero extension of w to $Q(1)$, we have*

$$\left|\int_{Q(t)} f\Delta_2^t w dx\right| \leqq C_0|f|_{1/2,R^2}|w|_{\mathscr{A}(Q(0))}$$

with C_0 independent of f and w.

4.2. Let A be an open neighborhood of $\partial\Sigma$ in R^3 as in Lemma 3.4 such that

$$\partial_1\Sigma(t)\subset A \qquad \text{for all}\quad t\in[0, T_0].$$

Take a function β in $C_0^\infty(R^3)$ such that supp $\beta\subset V_{\partial A}$ and $\beta\equiv 1$ on A, and set

$$\alpha_t(x) = \begin{cases} F_{\partial A}(\eta(x), \xi_1(x)-\beta(x)h(\eta(x), t), \xi_2(x)) & \text{for}\quad x\in V_{\partial A}, \\ x \quad \text{for}\quad R^3 - V_{\partial A}. \end{cases}$$

Applying arguments similar to those in the proofs of [12, Lemmas 3.8, 3.9], we have the following

LEMMA 4.4. *There is a positive number $T_1<T_0$ such that*

(i) *α_t: $R^3\to R^3$ is a C^∞-diffeomorphism for each $t\in[0, T_1]$,* *(ii)* $\alpha_t(\Sigma(t))=\Sigma$ *for each* $t\in[0, T_1]$,

(iii) *the map* $v\to\alpha_t^*v=v\circ\alpha_t$ *is a linear isomorphism of* $H^1(\Omega)$ *onto* $H^1(\Omega(t))$ *as well as of* $V(\Omega)$ *onto* $V(\Omega(t))$, *for each* $t\in[0, T_1]$.

Let us put

$$\Delta^t w = [\alpha_t^* w - w]t^{-1}, \quad \Delta^{-t} w = [(\alpha_t^{-1})^* w - w](-t)^{-1}$$

for a function w and $0<t<T_1$. Then we have

LEMMA 4.5. *If* $v\in H^1(\Omega)$, *then* $\{\Delta^t v_\Gamma, \Delta^t [\![v]\!]\}$ *converges weakly to* $\{-\beta(L_X|_{\partial G}v_\Gamma), -\beta(L_X|_{G^0}[\![v]\!])\}$ *given by* (3.3) *in* $H^{-1/2}(\partial G)\times H^{-1/2}(G^0)$ *as* $t\to 0$.

PROOF. By Lemma 3.1 we need only to show (see e.g. Yosida [15, p. 125]) that (i) $\{\{\Delta^t v_\Gamma, \Delta^t[\![v]\!]\}; 0<t<T_1\}$ is bounded in $H^{-1/2}(\partial G)\times H^{-1/2}(G^0)$, and (ii) $\langle f, \Delta^t v_\Gamma\rangle_{\partial G}$ (resp. $\langle g, \Delta^t[\![v]\!]\rangle_{G^0}$) tends to $\langle f, -\beta(L_X|_{\partial G}v_\Gamma)\rangle_{\partial G}$ (resp. $\langle g, -\beta(L_X|_{G^0}[\![v]\!])\rangle_{G^0}$) as $t\to 0$ for any $f\in C_L^\infty(\partial G)$ (resp. $g\in C_0^\infty(G^0)$).

Assertion (i) will be obtained if we show the estimate:

$$\left|\int_{\Gamma(t)} f\Delta^t v_\Gamma dS + \int_{\Sigma(t)} g\Delta^t[\![v]\!]dS\right| \leqq C_0\{|f|_{1/2,\partial G}+|g|_{1/2,G^0}\}|v|_{1,\Omega} \tag{4.1}$$

for $(f, g)\in C_L^\infty(\partial G)\times C_0^\infty(G^0)$, where C_0 is independent of v, f, g and t. In case $v\in C_*^{0,1}(\Omega)$, (4.1) follows from Lemmas 4.2 and 4.3 and Theorem 2.6 by the use of an appropriate partition of unity. Since $C_*^{0,1}(\Omega)$ is dense in $H^1(\Omega)$, we have (4.1) for $v\in H^1(\Omega)$. Next we prove (ii). Using the change of variables $y=\alpha_t(x)$, we obtain, for $(f, g)\in C_L^\infty(\partial G)\times C_0^\infty(G^0)$

$$\int_{\Gamma(t)} f\Delta^t v_\Gamma dS = -\int_\Gamma (\Delta^{-t}f)v_\Gamma dS - \int_{\Gamma(t)} f(\alpha_t^* v_\Gamma)(\alpha_t^*(dS)-dS)/t, \tag{4.2}$$

$$\int_{\Sigma(t)} g\Delta^t[\![v]\!]dS = -\int_{G^0}(\Delta^{-t}g)\,[\![v]\!]\,dS - \int_{G^0} g(\alpha_t^*[\![v]\!])(\alpha_t^*(dS)-dS)/t, \tag{4.3}$$

where $[\![v]\!]$ in the right-hand side of (4.3) is the zero extension to G^0. By (3.1) and (3.2) the last term of the right-hand side of (4.2) tends to

$$\int_\Gamma f v_\Gamma \operatorname{div}_{\partial G}(-\beta X)dS = -\int_\Gamma (\operatorname{div}_{\partial G} X)(\beta f)v_\Gamma dS - \int_\Gamma (X\cdot\Delta\beta)f v_\Gamma dS \tag{4.4}$$

as $t\to 0$. We can easily show that $\Delta^{-t}f$ converges to $(L_X|_{\partial G}+\operatorname{div}_{\partial G}X)(-\beta f)-\operatorname{div}_{\partial G}(-\beta X)f$ as $t\to 0$ uniformly on ∂G. Hence from (3.3) and (4.4), $\langle f, \Delta^t v_\Gamma\rangle_{\partial G}$ converges to $\langle f, -\beta(L_X|_{\partial G})v_\Gamma\rangle_{\partial G}$ as $t\to 0$. Similarly we can prove that $\langle g, \Delta^t[\![v]\!]\rangle_{G^0}\to\langle g, -\beta(L_X|_{G^0}[\![v]\!])\rangle_{G^0}$ as $t\to 0$. This completes the proof of Lemma 4.5.

LEMMA 4.6. *There is a constant* $C>0$ *such that*

$$|\alpha_t^* \boldsymbol{u} - \boldsymbol{u}(t)|_{1,\Omega(t)} \leqq Ct|\mathscr{L}|_{\mathscr{M}} \qquad \textit{for all} \quad t \in [0, T_1],$$

where $\boldsymbol{u}(t) = \mathscr{T}(t, \mathscr{L})$ *and* $\boldsymbol{u} = \boldsymbol{u}(0)$.

PROOF. Given a load $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathscr{Q}) \in \mathscr{M}$, we define $\boldsymbol{v}(t)$ to be the solution of the problem

$$\int_{\Omega(t)} \sigma_{ij}(\boldsymbol{v}(t)) e_{ij}(\boldsymbol{w}) dx = \int_{\Omega(t)} \mathscr{F} \cdot \boldsymbol{w} dx + \int_{\Gamma_1(t)} \mathscr{Q} \cdot \boldsymbol{w}_{\Gamma(t)} dS + \int_{\Sigma(t)} \alpha_t^* \mathscr{P} \cdot [\![\boldsymbol{w}]\!] dS$$

for all $\boldsymbol{w} \in V(\Omega(t))$. From (1.3) we have

$$\int_{\Omega(t)} \sigma_{ij}(\boldsymbol{u}(t) - \boldsymbol{v}(t)) e_{ij}(\boldsymbol{w}) dx = \int_{\Sigma(t)} (\mathscr{P} - \alpha_t^* \mathscr{P}) [\![\boldsymbol{w}]\!] dS. \tag{4.5}$$

By considering the zero extension of $[\![\boldsymbol{w}]\!]$, and a partition of unity, the argument in the proof of Lemma 4.3 yields the estimate

$$\left| \int_{\Sigma(t)} (\mathscr{P} - \alpha_t^* \mathscr{P}) [\![\boldsymbol{w}]\!] dS \right| \leqq C_1 t |\mathscr{P}|_{1/2, G^0} |[\![\boldsymbol{w}]\!]|_{\mathscr{A}(\Sigma)}$$

with C_1 independent of $\mathscr{P}$, w and t. Then in view of (4.5) and (1.5), the Lax-Milgram theorem (see e.g. [2]) and Theorem 2.7 imply

$$|\boldsymbol{u}(t) - \boldsymbol{v}(t)|_{1,\Omega(t)} \leqq C_2 t |\mathscr{P}|_{1/2, G^0} \tag{4.6}$$

with C_2 independent of $\mathscr{P}$ and t. Now let

$$a_t(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Omega(t)} \sigma_{ij}(\boldsymbol{v}) e_{ij}(\boldsymbol{w}) dx \qquad \text{for} \quad \boldsymbol{v}, \boldsymbol{w} \in V(\Omega(t)),$$

and $\ell_t(\boldsymbol{u}, \boldsymbol{w}) = a_t(\alpha_t^* \boldsymbol{u}, \alpha_t^* \boldsymbol{w}) - a_0(\boldsymbol{u}, \boldsymbol{w})$ for $\boldsymbol{w} \in V(\Omega)$. Then, by an argument as in the proof of [12, Lemma 4.10] we have

$$|\ell_t(\boldsymbol{u}, \boldsymbol{w})| \leqq C_3 t |\boldsymbol{u}|_{1,\Omega} |\boldsymbol{w}|_{1,\Omega} \qquad \text{for} \quad \boldsymbol{w} \in V(\Omega). \tag{4.7}$$

From the identity (1.3) we have

$$\begin{aligned}
a_t(\alpha_t^* \boldsymbol{u}, \alpha_t^* \boldsymbol{w}) &= \int_{\Omega} \mathscr{F} \cdot \boldsymbol{w} dx + \int_{\Gamma_1} \mathscr{Q} \cdot \boldsymbol{w}_{\Gamma} dS + \int_{\Sigma} \mathscr{P} \cdot [\![\boldsymbol{w}]\!] dS + \ell_t(\boldsymbol{u}, \boldsymbol{w}) \\
&= \int_{\Omega(t)} \mathscr{F} \cdot (\alpha_t^* \boldsymbol{w}) dx + \int_{\Gamma_1(t)} \mathscr{Q} \cdot (\alpha_t^* \boldsymbol{w})_{\Gamma(t)} dS + \int_{\Sigma(t)} \alpha_t^* \mathscr{P} \cdot [\![\alpha_t^* \boldsymbol{w}]\!] dS \\
&\quad + \int_{\Omega(t)} \mathscr{F} \cdot (\boldsymbol{w} - \alpha_t^* \boldsymbol{w}) dx + \int_{\Gamma_1(t)} \mathscr{Q} \cdot (\boldsymbol{w} - \alpha_t^* \boldsymbol{w})_{\Gamma(t)} dS \\
&\quad + \int_{G^0} \alpha_t^* \mathscr{P} \cdot [\![\alpha_t^* \boldsymbol{w}]\!] (\alpha_t^*(dS) - dS) + \ell_t(\boldsymbol{u}, \boldsymbol{w}).
\end{aligned}$$

Notice that here we have used the fact that the Lebesgue measure of $\Omega - \Omega(t)$ and

$\Gamma_1 - \Gamma_1(t)$ are zero. Using the definition of $v(t)$, we therefore obtain

$$a_t(\alpha_t^* u - v(t), \alpha_t^* w) = \int_{\Omega(t)} \mathscr{F}\cdot(w - \alpha_t^* w)dx + \int_{\Gamma_1(t)} \mathscr{Q}\cdot(w - \alpha_t^* w)_{\Gamma(t)} dS$$
$$+ \int_{G^0} \alpha_t^* \mathscr{P}\cdot [\![\alpha_t^* w]\!] (\alpha_t^* (dS) - dS) + \ell_t(u, w).$$

As in [12, p. 41], we see that

$$\left|\int_{\Omega(t)} \mathscr{F}\cdot(w - \alpha_t^* w)dS\right| \leqq C_4 t|\mathscr{F}|_{0,G}|w|_{1,\Omega} \leqq C_4 t|\mathscr{L}|_{\mathscr{M}}|w|_{1,\Omega}.$$

Using a partition of unity, Lemma 4.2 and Theorem 2.6, we obtain

$$\left|\int_{\Gamma_1(t)} \mathscr{Q}\cdot(w - \alpha_t^* w)_{\Gamma(t)} dS\right| \leqq C_5 t|\mathscr{Q}|_{1/2,\partial G}|w_\Gamma|_{1/2,d*} \leqq C_6 t|\mathscr{L}|_{\mathscr{M}}|w|_{1,\Omega}.$$

Since $(\alpha_t^*(dS) - dS)/t = \omega_t dS$ with uniformly bounded functions ω_t on G^0,

$$\left|\int_{G_0} (\alpha_t^* \mathscr{P})\cdot [\![\alpha_t^* w]\!] (\alpha_t^*(dS) - dS)\right| \leqq C_7 t|\mathscr{P}|_{0,\Sigma}|[\![w]\!]|_{0,\Sigma} \leqq C_8 t|\mathscr{L}|_{\mathscr{M}}|w|_{1,\Omega}.$$

Thus, together with (4.7), we have

$$|a_t(\alpha_t^* u - v(t), \alpha_t^* w)| \leqq C_9 t|\mathscr{L}|_{\mathscr{M}}|w|_{1,\Omega}$$

for all $w \in V(\Omega)$. Since $\alpha_t^* u - v(t) \in V(\Omega(t))$, this gives

$$|a_t(\alpha_t^* u - v(t), \alpha_t^* u - v(t))| \leqq C_{10} t|\mathscr{L}|_{\mathscr{M}}|\alpha_t^* u - v(t)|_{1,\Omega(t)}.$$

Hence, by (1.5), we finally obtain

$$|\alpha_t^* u - v(t)|_{1,\Omega(t)} \leqq C_{11} t|\mathscr{L}|_{\mathscr{M}}. \tag{4.8}$$

The desired estimate now follows from (4.6) and (4.8).

4.3. Proof of Theorem 4.1. Since mapping $\mathscr{L} \to J(\mathscr{L})$ is uniformly continuous on bounded sets of $\mathscr{M}$ (Proposition 3.6), and since

$$\tilde{\mathscr{M}} = \{C_0^\infty(G)\}^3 \times \{C_0^\infty(G^0)\}^3 \times \{C_L^\infty(\partial G)\}^3$$

is dense in $\mathscr{M}$(see [8, Chapter 1, (11.6)]), it suffices to consider the case $\mathscr{L} \in \tilde{\mathscr{M}}$. By Lemma 4.6, we have

$$\lim_{t\to 0} t^{-1}|\alpha_t^* u - u(t)|_{1,\Omega(t)}^2 = 0. \tag{4.9}$$

Let $a_t(v, w)$ be as in the proof of the previous lemma. By (1.3) with $v = u(t) - \alpha_t^* u$, we see that

$$\begin{aligned}|E(\boldsymbol{u}(t);\mathscr{L})-E(\alpha_t^*\boldsymbol{u};\mathscr{L})| &= |\{a_t(\boldsymbol{u}(t),\boldsymbol{u}(t))-a_t(\alpha_t^*\boldsymbol{u},\alpha_t^*\boldsymbol{u})\}/2-a_t(\boldsymbol{u}(t),\boldsymbol{u}(t)-\alpha_t^*\boldsymbol{u})| \\ &= a_t(\boldsymbol{u}(t)-\alpha_t^*\boldsymbol{u},\boldsymbol{u}(t)-\alpha_t^*\boldsymbol{u})/2 \leqq C_1|\boldsymbol{u}(t)-\alpha_t^*\boldsymbol{u}|_{1,\Omega(t)}^2.\end{aligned}$$

Hence, by (4.9),

$$E(\boldsymbol{u}(t);\mathscr{L}) = E(\alpha_t^*\boldsymbol{u};\mathscr{L}) + o(t) \quad (t\to 0).$$

By the definition of the energy release rate we have

$$\begin{aligned}|\delta\Sigma(t)|\mathscr{G}(\mathscr{L};\{\Sigma(t)\}) &= \lim_{t\to 0} t^{-1}\{a_0(\boldsymbol{u},\boldsymbol{u})-a_t(\alpha_t^*\boldsymbol{u},\alpha_t^*\boldsymbol{u})\}/2 \\ &+ \lim_{t\to 0} t^{-1}\int_{\Omega(t)}\mathscr{F}\cdot(\alpha_t^*\boldsymbol{u}-\boldsymbol{u})\,dx + \lim_{t\to 0} t^{-1}\left\{\int_{\Gamma_1(t)}\mathscr{Q}\cdot(\alpha_t^*\boldsymbol{u}-\boldsymbol{u})_{\Gamma(t)}dS\right. \\ &\left.+\int_{\Sigma(t)}\mathscr{P}\cdot[\![\alpha_t^*\boldsymbol{u}-\boldsymbol{u}]\!]\,dS\right\}.\end{aligned}$$

Now we introduce cut-off functions ω, $\zeta\in C_0^\infty(\boldsymbol{R}^3)$ such that

$$\begin{aligned}&0\leqq\omega,\ \zeta\leqq 1;\ A\supset V_{\partial A}\cap\operatorname{supp}(1-\omega),\ \omega\equiv 0 \text{ near } \partial_1\Sigma; \\ &\qquad\operatorname{supp}\zeta\subset V_{\partial A},\ \zeta\equiv 1 \text{ on } \operatorname{supp}\omega \text{ and } \zeta\equiv 0 \text{ near } \partial_1\Sigma.\end{aligned}$$

By calculations similar to those in [12, pp. 44–46], we have

$$\begin{aligned}(4.10)\quad |\delta\Sigma(t)|\mathscr{G}(\mathscr{L};\{\Sigma(t)\}) &= \int_\Omega \sigma_{ij}D_j(\beta X_k)D_k u_i dx \\ &+\int_\Omega((X\cdot\nabla)(\zeta W))\beta dx - \int_\Omega(1-\zeta)W\mathscr{J}'\beta dx \\ &-\int_\Omega\{((X\cdot\nabla)a_{ijkl}/2)e_{kl}e_{ij}+\mathscr{F}\cdot((X\cdot\nabla)\boldsymbol{u})\beta\}dx + \lim_{t\to 0}t^{-1}I_t,\end{aligned}$$

where $\mathscr{J}'=(d/dt)|\det(\nabla\alpha_t)|^{-1}|_{t=0}$ and

$$I_t = \int_{\partial G}\mathscr{Q}\cdot(\alpha_t^*\boldsymbol{u}-\boldsymbol{u})_{\Gamma(t)}dS + \int_{G^0}\mathscr{P}\cdot[\![\alpha_t^*\boldsymbol{u}-\boldsymbol{u}]\!]\,dS.$$

From Lemma 4.5 it follows that

$$(4.11)\qquad \lim_{t\to 0}t^{-1}I_t = -\langle\mathscr{Q},\beta L_X|_{\partial G}\boldsymbol{u}_\Gamma\rangle_{\partial G} - \langle\mathscr{P},\beta L_X|_{G^0}[\![\boldsymbol{u}]\!]\rangle_{G^0}.$$

Let $\{\beta_m\}$ be a sequence of functions in $C_0^\infty(\boldsymbol{R}^3)$ such that $0\leqq\beta_m\leqq 1$, $\operatorname{supp}\beta_m\subset V_{\partial A}$, $\beta_m\equiv 1$ on A, and $\beta_m\to\chi_A$ a.e. as $m\to\infty$. Here χ_A is the characteristic function of A. Since $\beta_m\equiv 1$ on $\operatorname{supp}(1-\omega)$, substituting $\beta=\beta_m$ into (4.11) and letting $m\to\infty$, we see that the right-hand side of (4.11) tends to

$$\begin{aligned}(4.12)\quad &-\langle\mathscr{Q},(1-\omega)L_X|_{\partial G}\boldsymbol{u}_\Gamma\rangle_{\partial G} - \int_{\Gamma\cap A}\omega\mathscr{Q}\cdot((X\cdot\nabla)\boldsymbol{u})dS \\ &\qquad -\langle\mathscr{P},(1-\omega)L_X|_{G^0}[\![\boldsymbol{u}]\!]\rangle_{G^0} - \int_{\Sigma\cap A}\omega\mathscr{P}\cdot(X\cdot\nabla)[\![\boldsymbol{u}]\!])dS.\end{aligned}$$

To treat the other terms on the right-hand side of (4.10), we first observe the following: Since $D_j\beta \equiv 0$ on $\operatorname{supp}(1-\omega)$ and $\zeta \equiv 1$ on $\operatorname{supp}\omega$,

$$(4.13) \qquad \int_\Omega \sigma_{ij}(D_j\beta)((X\cdot\nabla)u_i)dx = \int_\Omega \omega\sigma_{ij}(D_j\beta)((X\cdot\nabla)(\zeta u_i))dx.$$

On the other hand, since there is $\mathscr{V}$ as in (1.7), $(X\cdot\nabla)(\zeta \boldsymbol{u}) \in H^1(\Omega)$ by Theorem 1.3. Hence by Green's formula and (1.4),

$$(4.14) \quad \int_\Omega D_j(\omega\sigma_{ij})\beta((X\cdot\nabla)(\zeta u_i))dx = \int_\Gamma \omega\mathscr{Q}_i((X\cdot\nabla)(\zeta u_i))_\Gamma\beta dS + \int_\Sigma \omega\mathscr{P}_i((X\cdot\nabla)[\![\zeta u_i]\!])\beta dS - \int_\Omega \omega\sigma_{ij}D_j\{\beta((X\cdot\nabla)(\zeta u_i))\}dx.$$

Combining (4.13) and (4.14), we obtain

$$(4.15) \quad \int_\Omega \sigma_{ij}(D_j\beta)((X\cdot\nabla)u_i)dx = -\int_\Omega D_j[(\omega\sigma_{ij})((X\cdot\nabla)(\zeta u_i))]\beta dx + \int_\Gamma \omega\mathscr{Q}_i((X\cdot\nabla)(\zeta u_i))_\Gamma\beta dS + \int_\Sigma \omega\mathscr{P}_i((X\cdot\nabla)\,[\![\zeta u_i]\!])\beta dS.$$

We now substitute $\beta=\beta_m$ in the right-hand side of (4.10), and then let $m\to\infty$. Then taking (4.12) and (4.15) into account, we find

$$(4.16) \quad |\delta\Sigma(t)|\mathscr{G}(\mathscr{L};\{\Sigma(t)\})$$
$$= -\int_{\Omega\cap A} D_j[(\omega\sigma_{ij})((X\cdot\nabla)(\zeta u_i))]dx + \int_{A\cap\Omega} \sigma_{ij}(D_jX_k)(D_ku_i)dx + \int_{\Omega\cap A}((X\cdot\nabla)(\zeta W))dx - \int_{\Omega\cap A}(1-\zeta)W\mathscr{I}'dx$$
$$-\int_{\Omega\cap A}\{((X\cdot\nabla)a_{ijkl}/2)e_{kl}e_{ij} + \mathscr{F}\cdot((X\cdot\nabla)\boldsymbol{u})\}dx - \langle\mathscr{Q}, (1-\omega)L_X|_{\partial G}\boldsymbol{u}_\Gamma\rangle_{\partial G} - \langle\mathscr{P}, (1-\omega)L_X|_{G^0}[\![\boldsymbol{u}]\!]\rangle_{G^0}.$$

Since $\omega\sigma_{ij}$, $\zeta e_{ij}\in\mathscr{N}(\Omega)$ and $(X\cdot\nabla)(\zeta u_i)\in H^1(A\cap\Omega)$, a slight modification of the argument in the proof of Lemma 3.4 yields

$$(4.17) \quad \int_{A\cap\Omega}(X\cdot\nabla)(\zeta W)dx = \int_{\partial_1 A} W(X\cdot\nu)dS - \int_{A\cap\Omega} W(\operatorname{div}X)dx - \int_{A\cap\Omega}(1-\zeta)W(\operatorname{div}X)dx.$$

Furthermore, as in the proof of (4.14), we have

$$(4.18) \quad \int_{A\cap\Omega} D_j\{\omega\sigma_{ij}((X\cdot\nabla)(\zeta u_i))\}dx = \int_{\partial_1 A}\sigma_{ij}\nu_j((X\cdot\nabla)(u_i))dS$$

$$+\int_{A\cap\Sigma}\omega\mathscr{P}\cdot((X\cdot\nabla)[\![\zeta\boldsymbol{u}]\!])dS+\int_{A\cap\Gamma}\omega\mathscr{Q}\cdot((X\cdot\nabla)(\zeta\boldsymbol{u}))dS.$$

Here we have used the fact that $\omega\equiv\zeta\equiv 1$ on $\partial_1 A$ and $\omega\equiv\zeta\equiv 0$ on $\partial_1\Sigma$. Collecting terms in (4.16)–(4.18), we therefore obtain

$$(4.19)\quad |\delta\Sigma(t)|\mathscr{G}(\mathscr{L};\{\Sigma(t)\}) = J(\mathscr{L}) + \int_{\Omega\cap A}(1-\zeta)\{W(\operatorname{div}X)-W\mathscr{I}'\}dx.$$

Now take a sequence $\{\zeta_m\}$ of functions in $C_0^\infty(\boldsymbol{R}^3)$ such that $\zeta_m\to 1$ a.e. in A as $m\to\infty$. Replacing ζ in (4.19) by ζ_m and then letting $m\to\infty$, we conclude that

$$|\delta\Sigma(t)|\mathscr{G}(\mathscr{L};\{\Sigma(t)\}) = J(\mathscr{L}),$$

which completes the proof of Theorem 4.1.

References

[1] P. P. Destuynder and Le Chesnay M. Djaoua, Sur une interprétation mathématique de l'intégrale de Rice en théorie de la rupture fragile, Math. Mech. Appl. Sci. **3** (1981), 70–87.

[2] G. Fichera, Linear elliptic differential systems and eigenvalue problems, Lecture Notes in Math. 8, Springer-Verlag, Berlin·Heidelberg·New York, 1965.

[3] W-H. Chen and C-W. Wu, On the J-integral for a pressurized crack in bounded materials, Int. J. Fracture **16** (1980), R47–R51.

[4] P. Grisvard, Elliptic problems in nonsmooth domains, Pitman Advanced Publishing Program, Boston·London·Melbourne, 1985.

[5] A. Karlsson and J. Bäcklund, J-integral at loaded crack surfaces, Int. J. Fracture **14** (1978), R311–R314.

[6] M. Kikuchi, H. Miyamoto and Y. Sakaguchi, Evaluation of three-dimensional J-integral of semi-elliptical surface crack in pressure vessel, Trans. Int. Conf. Struct. Mech. React. Technol. 5, G7/2, 1–9, 1979.

[7] S. Lang, Differential manifolds, Addison-Wesley, Reading, 1972.

[8] J. L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications I, Springer-Verlag, Berlin·Heidelberg·New York, 1972.

[9] N. G. Meyers and J. Serrin, $H=W$, Proc. Nat. Acad. Sci. USA **51** (1964), 1055–1056.

[10] J. Nečas, Les méthodes directes en théorie des équations elliptiques, Masson et C^{ie}, Paris, 1967.

[11] K. Ohtsuka, J-integral in two dimensional fracture mechanics, RIMS Kokyuroku 386, Kyoto Univ., 231–248, 1980.

[12] K. Ohtsuka, Generalized J-integral and three dimensional fracture mechanics I, Hiroshima Math. J. **11** (1981), 21–52.

[13] K. Ohtsuka, Generalized J-integral and its application — An interpretation of Hadama's variational formula — RIMS Kokyuroku 462, Kyoto Univ., 149–165, 1982.

[14] K. Ohtsuka, Generalized J-integral and its applications I —Basic Theory—, Japan J. Appl. Math. 2 (1985), 329–350.

[15] K. Yosida, Functional Analysis, Springer-Verlag, Berlin·Heidelberg·New York, 1968.

Hiroshima-Denki Institute of Technology