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On the asymptotic solutions for a weakly coupled elliptic boundary value problem with a small parameter

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1. Introduction

In this paper we consider the asymptotic solutions of the Dirichlet boundary value problem for a pair of second-order partial differential equations

$$(1.1)\qquad \begin{cases} \varepsilon^2 \Delta u - f(u, v) = 0 \\ \Delta v - g(u, v) = 0 \end{cases}, \qquad x \in \Omega,$$

with the boundary conditions

$$(1.2)\qquad u = \alpha(x, \varepsilon), \quad v = \beta(x, \varepsilon), \quad x \in \Gamma.$$

Here Ω is a bounded domain in $\mathbf{R}^N$ ($N \geq 2$), its boundary Γ is a connected C^∞ hypersurface, and ε is a small positive parameter. Moreover we assume that f and g are infinitely differentiable functions defined in $\mathbf{R}^2$. Such problems arise in the study of steady-state solutions of chemically reacting and diffusing systems (Aris [2] and the references therein). An interesting phenomenon in such systems is the occurrence of boundary and/or interior transition layer. For $N=1$, by using singular perturbation technique, Fife [7] showed that the existence of boundary and/or interior transition layer phenomena for (1.1) and (1.2), and recently Ito [9] modified his results into a more useful version. For $N \geq 2$, as far as we know, these phenomena have not yet been analyzed except numerical simulations. In this paper, we restrict our attention to the problem (1.1) and (1.2) with boundary layer phenomena. Consider the case where the reduced problem

$$\begin{cases} f(u_0, v_0) = 0 \\ \Delta v_0 - g(u_0, v_0) = 0 \end{cases}, \qquad x \in \Omega,$$

and

$$v_0 = \beta_0(x), \quad x \in \Gamma,$$

has a regular solution (u_0, v_0). In this case, we shall show constructively the existence of a solution (u, v) of (1.1) and (1.2) such that

$$\Big\{\ u \to u_0 \text{ uniformly in any compact subset of } \Omega,$$

$$\Big\rangle\ v \to v_0 \text{ uniformly in } \bar{\Omega},$$

as $\varepsilon \to 0$. Note that the above solution exhibits its irregular behavior, as a function of ε, in a neighborhood of the boundary Γ.

The boundary value problem of the type

$$(1.3) \qquad \begin{cases} \varepsilon^2 \Delta u - f(x, u) = 0, & x \in \Omega, \\ u = \alpha(x, \varepsilon), & x \in \Gamma, \end{cases}$$

were studied in Berger et al. [3], De Villiers [5] and Fife [6], and others. They showed the existence of a solution of (1.3) in the following way. First, they constructed a formal asymptotic approximation by using an outer expansion, valid in the region not adjacent to Γ, and a boundary layer expansion, valid in a neighborhood of Γ. And next, they proved that there exists a solution of (1.3) close to the formal approximation by using the contraction mapping principle [3], [5] or the Schauder fixed point theorem [6].

Our main contribution is an extension of their method to the one applicable to the boundary value problems for weakly coupled elliptic systems. In Section 2 we state the assumptions for the nonlinearities of f and g, which are of great importance to construct the formal approximation. In Section 3 we construct the outer expansion. Since this expansion dose not satisfy the correct boundary condition, in Section 4 we construct the boundary layer expansion in a neighborhood of the boundary Γ. And in Section 5, using these expansions we obtain the formal approximation up to order ε^m to the solution of (1.1) and (1.2), where m denotes any nonnegative integer. In Section 6, applying the implicit function theorem we show the validity of this approximation (see e.g., [7], [9]). Finally in Section 7, we give a few comments on the equation (1.1) and (1.2).

2. Preliminaries

By singular perturbation techniques, we shall construct an asymptotic expansion to the solution $(u(x, \varepsilon), v(x, \varepsilon))$ for a weakly coupled elliptic boundary value problem

$$(2.1) \qquad \begin{cases} M[u, v] \equiv \varepsilon^2 \Delta u - f(u, v) = 0, \\ N[u, v] \equiv \Delta v - g(u, v) = 0, \end{cases}$$

in a bounded domain Ω ($\subset \mathbf{R}^N$), satisfying the boundary conditions

$$(2.2) \qquad u = \alpha(x, \varepsilon), \quad v = \beta(x, \varepsilon),$$

on the boundary Γ, where f and g are infinitely differentiable functions of u and v, Γ is a connected C^∞ hypersurface, and ε is a small positive parameter. It is assumed that $\alpha(x, \varepsilon)$ and $\beta(x, \varepsilon)$ are infinitely differentiable functions with respect to $x \in \Gamma$ and have asymptotic expansions

$$\alpha(x, \varepsilon) \sim \sum_{n=0}^{\infty} \alpha_n(x)\varepsilon^n, \quad \beta(x, \varepsilon) \sim \sum_{n=0}^{\infty} \beta_n(x)\varepsilon^n,$$

as $\varepsilon \to 0$ respectively, where $\alpha_n(x)$'s and $\beta_n(x)$'s are infinitely differentiable functions of $x \in \Gamma$. Moreover we make the following assumptions for f and g:

(i) $f(u, v)=0$ has at least one root $u=h(v)$ with h defined on an interval I;

(ii) there exists an interval J ($\subseteq I$) such that $\beta_0(x) \in J$ for $x \in \Gamma$, and $f_u(h(v), v)>0$ and $(d/dv)\,(g(h(v), v))>0$ for $v \in J$;

(iii) for any $\gamma(x) \in C^\infty(\Gamma)$ with range in J, the boundary value problem

$$\text{(2.3)} \qquad \begin{cases} \Delta v - g(h(v), v) = 0, & x \in \Omega, \\ v = \gamma(x), & x \in \Gamma, \end{cases}$$

has a solution $v(x) \in C^\infty(\bar{\Omega})$ with range in J;

(iv) at each point $x \in \Gamma$,

$$\int_{h(\beta_0(x))}^{c} f(s, \beta_0(x))ds > 0$$

for $c \neq h(\beta_0(x))$ in the closed interval between $h(\beta_0(x))$ and $\alpha_0(x)$.

REMARK 1. From (i) and (ii),

$$D(f, g)(h(v), v) \equiv [f_u(u, v)g_v(u, v) - f_v(u, v)g_u(u, v)]_{u=h(v)} > 0$$

for $v \in J$.

REMARK 2. Let $J=(a, b)$. If $g(h(a), a)>0$, $g(h(b), b)<0$ and $a<\gamma(x)<b$ for $x \in \Gamma$, then there exists a solution $v(x) \in C^\infty(\bar{\Omega})$ of the problem (2.3) such that $a<v(x)<b$ for $x \in \bar{\Omega}$ (Sattinger [10]).

First, we introduce several function spaces useful for our purpose. Let $\sigma=(\sigma_1, \sigma_2, \ldots, \sigma_N)$ denote the usual multi-index of order $|\sigma|=\sigma_1+\sigma_2+\cdots+\sigma_N$ with nonnegative integers σ_i, and write $\partial_i=\partial/\partial x_i$ $(1 \le i \le N)$, $\nabla=(\partial_1, \partial_2, \ldots, \partial_N)$ and $\partial^\sigma=\partial_1^{\sigma_1}\partial_2^{\sigma_2}\cdots\partial_N^{\sigma_N}$. Suppose Ω is a bounded domain in $\mathbf{R}^N$.

(a) Let k be a nonnegative integer and $\alpha \in (0, 1)$. By $C^{k+\alpha}(\bar{\Omega})$ we mean the Banach space of all real-valued functions $u \in C^k(\bar{\Omega})$ for which the derivatives $\partial^\sigma u$, with $|\sigma|=k$, are Hölder continuous on $\bar{\Omega}$ with exponent α. The norm is

$$\|u\|_{C^{k+\alpha}(\bar{\Omega})} = \sum_{j=0}^{k} |u|_{j,\bar{\Omega}} + |u|_{k+\alpha,\bar{\Omega}},$$

where

$$|u|_{j,\bar{\Omega}} = \max_{|\sigma|=j} \sup_{x \in \bar{\Omega}} |\partial^\sigma u(x)|,$$

$$|u|_{k+\alpha,\bar{\Omega}} = \max_{|\sigma|=k} \sup_{x,y\in\bar{\Omega}} |\partial^\sigma u(x) - \partial^\sigma u(y)|/|x-y|^\alpha \quad (x \neq y).$$

(b) $C_0^{k+\alpha}(\bar{\Omega})$ is the subspace of $C^{k+\alpha}(\bar{\Omega})$ whose elements are functions vanishing on $\partial\Omega$.

(c) Let k be a nonnegative integer and $1 \leq p < \infty$. We denote by $W^{k,p}(\Omega)$ the Banach space of (equivalent classes of) real-valued functions u, defined on Ω, such that u and all its generalized derivatives up to k, are pth-power integrable over Ω. The norm is

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\sigma|\leq k} \int_\Omega |\partial^\sigma u(x)|^p \, dx\right)^{1/p}.$$

(d) $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

$W^{0,p}(\Omega)$ is usually denoted by $L^p(\Omega)$. Let ε be a positive number.

(e) $C_\varepsilon^{k+\alpha}(\bar{\Omega})$ is the Banach space of all real-valued functions in $C^{k+\alpha}(\bar{\Omega})$, but with the special norm depending on ε:

$$\|u\|_{C_\varepsilon^{k+\alpha}(\bar{\Omega})} = \sum_{j=0}^k \varepsilon^j |u|_{j,\bar{\Omega}} + \varepsilon^{k+\alpha} |u|_{k+\alpha,\bar{\Omega}}.$$

(f) $C_{\varepsilon,0}^{k+\alpha}(\bar{\Omega})$ is the subspace of $C_\varepsilon^{k+\alpha}(\bar{\Omega})$ whose elements are functions vanishing on $\partial\Omega$.

(g) Let X and Y be Banach spaces. $X \to Y$ is the totality of continuous linear operator from X into Y equipped with the usual norm.

In this paper, the symbol C will denote a positive constant independent of ε.

3. Outer expansion

We begin with constructing an outer expansion $(U(x, \varepsilon), V(x, \varepsilon))$ of the form

$$U(x, \varepsilon) = \sum_{n=0}^\infty u_n(x)\varepsilon^n, \quad V(x, \varepsilon) = \sum_{n=0}^\infty v_n(x)\varepsilon^n, \quad x \in \Omega. \tag{3.1}$$

Substituting (3.1) into (2.1), we have

$$\begin{cases} \sum_{n=0}^\infty \Delta u_n(x)\varepsilon^{n+2} = f(\sum_{n=0}^\infty u_n(x)\varepsilon^n, \ \sum_{n=0}^\infty v_n(x)\varepsilon^n) \\ \sum_{n=0}^\infty \Delta v_n(x)\varepsilon^n = g(\sum_{n=0}^\infty u_n(x)\varepsilon^n, \ \sum_{n=0}^\infty v_n(x)\varepsilon^n) \end{cases}, \quad x \in \Omega. \tag{3.2}$$

The boundary conditions for (3.2) are assumed to be

$$\sum_{n=0}^\infty v_n(x)\varepsilon^n = \sum_{n=0}^\infty \xi_n(x)\varepsilon^n, \quad x \in \Gamma, \tag{3.3}$$

where $\xi_n(x)$'s are infinitely differentiable functions on Γ. We will specify these functions in Section 4.

Equate the coefficients of the like powers of ε on both sides of (3.2) and (3.3). Then for $n=0$, we obtain

$$\begin{cases} 0 = f(u_0, v_0) \\ \Delta v_0 = g(u_0, v_0) \end{cases}, \quad x \in \Omega,$$

and

$$v_0 = \xi_0(x), \quad x \in \Gamma. \tag{3.4}$$

From (i) and (iii), we know that $(u_0(x), v_0(x))$ $(u_0(x) = h(v_0(x)))$ exists and $u_0(x)$, $v_0(x) \in C^\infty(\bar{\Omega})$ whenever $\xi_0(x) \in C^\infty(\Gamma)$ has a range in J. For $n=1$,

$$\begin{cases} 0 = f_u(u_0, v_0)u_1 + f_v(u_0, v_0)v_1 \\ \Delta v_1 = g_u(u_0, v_0)u_1 + g_v(u_0, v_0)v_1 \end{cases}, \quad x \in \Omega, \tag{3.5}$$

and

$$v_1 = \xi_1(x), \quad x \in \Gamma. \tag{3.6}$$

By the assumption (ii), we have

$$u_1 = -f_v(u_0, v_0)v_1 / f_u(u_0, v_0) \tag{3.7}$$

and

$$\Delta v_1 = D(f, g)(u_0, v_0)v_1 / f_u(u_0, v_0). \tag{3.8}$$

Since $D(f, g)(u_0, v_0)/f_u(u_0, v_0) > 0$ for $x \in \bar{\Omega}$, it is easy to see that (3.8) and (3.6) have a unique solution $v_1(x) \in C^\infty(\bar{\Omega})$ if $\xi_j(x) \in C^\infty(\Gamma)$ $(j=0, 1)$ are specified, and thus $u_1(x) \in C^\infty(\bar{\Omega})$ can be obtained from (3.7). For $n \geq 2$, we have

$$\begin{cases} \Delta u_{n-2} = f_u(u_0, v_0)u_n + f_v(u_0, v_0)v_n + P_{n-1} \\ \Delta v_n \quad = g_u(u_0, v_0)u_n + g_v(u_0, v_0)v_n + Q_{n-1} \end{cases}, \quad x \in \Omega, \tag{3.9}$$

and

$$v_n = \xi_n(x), \quad x \in \Gamma, \tag{3.10}$$

where P_{n-1} and Q_{n-1} are functions determined only by $u_0, v_0, \ldots, u_{n-1}, v_{n-1}$. It is analogous to the case of (3.5) and (3.6) that (3.9) and (3.10) have a unique solution $(u_n(x), v_n(x))$ with $u_n(x)$ and $v_n(x) \in C^\infty(\bar{\Omega})$, if $\xi_j(x) \in C^\infty(\Gamma)$ $(j=0, 1, \ldots, n)$ are specified.

Accordingly the outer expansion $(U(x, \varepsilon), V(x, \varepsilon)) = (\sum_{n=0}^\infty u_n(x)\varepsilon^n, \sum_{n=0}^\infty v_n(x)\varepsilon^n)$ can be formally constructed if $\xi_n(x) \in C^\infty(\Gamma)$ $(n=0, 1, 2, \ldots)$ are specified.

4. Boundary layer expansion

It is clear that the outer expanison $(U(x, \varepsilon), V(x, \varepsilon))$ formally satisfies (2.1) in Ω, but does not (2.2) on Γ. Therefore we have to modify this expansion by supplementing $U(x, \varepsilon)$ and $V(x, \varepsilon)$ with boundary layer corrections Φ and $\varepsilon^2\Psi$ in a neighborhood of the boundary Γ, respectively. So $U+\Phi$ and $V+\varepsilon^2\Psi$ will formally satisfy both (2.1) and (2.2), and Φ and Ψ will decay exponentially in regions bounded away from Γ. Namely $(U+\Phi, V+\varepsilon^2\Psi)$ will converge to the outer expansion (U, V) as $\varepsilon \to 0$ away from Γ, while the boundary layer correction $(\Phi, \varepsilon^2\Psi)$ will be significant only in a neighborhood of the boundary.

To study the behavior of the solution in a neighborhood of the boundary Γ, as in Fife [6], we need to rewrite (2.1) and (2.2) in terms of a set of variables adapted to describing boundary layer phenomena. Let Ω be a bounded domain in $\mathbf{R}^N$. Its boundary Γ is called a C^∞ hypersurface if, for each $s \in \Gamma$, there is a neighborhood $\mathscr{V}$ $(\subset \mathbf{R}^N)$ of s and a real-valued function $\phi \in C^\infty(\mathscr{V})$, such that $\nabla\phi$ is nonvanishing on $\Gamma \cap \mathscr{V}$ and $\Gamma \cap \mathscr{V} = \{x \in \mathscr{V}; \phi(x)=0\}$. As Γ is a (compact) connected C^∞ hypersurface, it is orientable, that is, there is a unit vector $\nu(s)$ for each $s \in \Gamma$, varying continuously and perpendicular to Γ at s. $\nu(s)$ will be called the normal to Γ at s. Γ being infinitely smooth, $\nu(s)$ is a C^∞ function. Moreover we always choose the orientation so that ν points into Ω. The following proposition is basic for our object.

PROPOSITION 3. *Let Γ be a compact orientable C^∞ hypersurface. Then there is a positive number d such that the map*

$$F: \quad (t, s) \longrightarrow s + t\nu(s)$$

is a C^∞ homeomorphism of $[-d, d] \times \Gamma$ onto $\bar{\mathscr{V}}_d = \{x \in \mathbf{R}^N; \operatorname{dist}(x, \Gamma) \leq d\}$.

PROOF. The proof can be found in Appendix A.

In this paper we shall restrict F to the domain $[0, d] \times \Gamma$. Then the above map F will be a C^∞ homeomorphism of $[0, d] \times \Gamma$ onto $\bar{\Omega}_d = \{x \in \bar{\Omega}; \operatorname{dist}(x, \Gamma) \leq d\}$. In $\bar{\Omega}_d$, we introduce the stretched variable $\eta = t/\varepsilon$ and throughout this paper we shall use the following notation

$$\partial_t = \partial/\partial t,\ \partial_\eta = \partial/\partial\eta,\ \partial_\varepsilon = \partial/\partial\varepsilon, \quad \text{and}$$
$$u(x, \varepsilon) = \hat{u}(t, s, \varepsilon) = \hat{u}(\varepsilon\eta, s, \varepsilon) = \tilde{u}(\eta, s, \varepsilon).$$

Partial derivatives may be decomposed in the following manner. For $u \in C^1(\bar{\Omega}_d)$ the tangential gradient δu of u on Γ is defined by

$$\delta u = \nabla u - (\nu \cdot \nabla u)\nu \tag{4.1}$$

where $\nu \cdot \nabla u$ is the usual Euclidean inner product of ν and ∇u. For any point $x=s+t\nu(s)$ of $\bar{\Omega}_d$, the tangential gradient $\delta u(x)$ is the projection of the gradient $\nabla u(x)$ onto the tangent plane to Γ at s. It is clear that $\nu \cdot \delta u=0$ and δu only depends on the variable of u on Γ. Let $\delta=(\delta_1, \delta_2,\dots, \delta_N)$, where $\delta_i\ (i=1, 2,\dots, N)$ are the C^∞ first order linear differential operators in s. Higher order linear differential operators in s are sums of compositions of the first order operators in s. By (4.1),

$$\nabla = \nu\partial_t + \delta = \varepsilon^{-1}\nu\partial_\eta + \delta.$$

Therefore in terms of η, s and ε, the Laplacian is

$$\Delta = \nabla \cdot \nabla = \varepsilon^{-2}\partial_\eta^2 + \varepsilon^{-1}\delta \cdot \nu\partial_\eta + \varepsilon^{-1}\nu\partial_\eta \cdot \delta + \delta \cdot \delta. \tag{4.2}$$

Applying (4.2), we can rewrite (2.1) in terms of η, s and ε as follows:

$$\begin{cases} \tilde{M}[\tilde{u}, \tilde{v}] \equiv \partial_\eta^2\tilde{u} + \varepsilon\delta \cdot \nu\partial_\eta\tilde{u} + \varepsilon\nu\partial_\eta \cdot \delta\tilde{u} + \varepsilon^2\delta \cdot \delta\tilde{u} - f(\tilde{u}, \tilde{v}) = 0, \\ \tilde{N}[\tilde{u}, \tilde{v}] \equiv \varepsilon^{-2}\partial_\eta^2\tilde{v} + \varepsilon^{-1}\delta \cdot \nu\partial_\eta\tilde{v} + \varepsilon^{-1}\nu\partial_\eta \cdot \delta\tilde{v} + \delta \cdot \delta\tilde{v} - g(\tilde{u}, \tilde{v}) = 0. \end{cases}$$

We note

REMARK 4. Let $\tilde{U}(\eta, s, \varepsilon)=U(x, \varepsilon)$ and $\tilde{V}(\eta, s, \varepsilon)=V(x, \varepsilon)$ be the outer expansions. Then they satisfy

$$\partial_\varepsilon^n\tilde{M}[\tilde{U}, \tilde{V}](\eta, s, 0) = 0, \quad \partial_\varepsilon^n\tilde{N}[\tilde{U}, \tilde{V}](\eta, s, 0) = 0,$$

for any nonnegative integer n.

The expansions

$$\Phi(\eta, s, \varepsilon) = \textstyle\sum_{n=0}^\infty \phi_n(\eta, s)\varepsilon^n, \quad \Psi(\eta, s, \varepsilon) = \sum_{n=0}^\infty \psi_n(\eta, s)\varepsilon^n,$$

valid in $\bar{\Omega}_d$, may be constructed by seeking the formal solutions of $\tilde{M}[\tilde{u}, \tilde{v}]=0$ and $\tilde{N}[\tilde{u}, \tilde{v}]=0$ in the form

$$\begin{cases} \tilde{u}(\eta, s, \varepsilon) = \tilde{U}(\eta, s, \varepsilon) + \Phi(\eta, s, \varepsilon), \\ \tilde{v}(\eta, s, \varepsilon) = \tilde{V}(\eta, s, \varepsilon) + \varepsilon^2\Psi(\eta, s, \varepsilon). \end{cases}$$

The terms ϕ_n and ψ_n can be interpreted as $\partial_\varepsilon^n\Phi(\eta, s, 0)/n!$ and $\partial_\varepsilon^n\Psi(\eta, s, 0)/n!$, respectively, and they need to satisfy

$$\begin{cases} \partial_\varepsilon^n\tilde{M}[\tilde{U} + \Phi, \tilde{V} + \varepsilon^2\Psi](\eta, s, 0) = 0, \\ \partial_\varepsilon^n\tilde{N}[\tilde{U} + \Phi, \tilde{V} + \varepsilon^2\Psi](\eta, s, 0) = 0. \end{cases}$$

By Remark 4 these equations are equivalent to

$$(4.3)\qquad \begin{cases} \partial_\varepsilon^n(\tilde{M}[\tilde{U}+\Phi,\ \tilde{V}+\varepsilon^2\Psi]-\tilde{M}[\tilde{U},\ \tilde{V}])(\eta,\ s,\ 0)=0, \\ \partial_\varepsilon^n(\tilde{N}[\tilde{U}+\Phi,\ \tilde{V}+\varepsilon^2\Psi]-\tilde{N}[\tilde{U},\ \tilde{V}])(\eta,\ s,\ 0)=0. \end{cases}$$

We shall see that the functions ϕ_n and ψ_n ($n=0, 1, 2,\dots$) can be determined successively by solving (4.3) with the boundary conditions

$$(4.4)\qquad \begin{cases} \sum_{n=0}^{\infty}\hat{u}_n(0,\ s)\varepsilon^n+\sum_{n=0}^{\infty}\phi_n(0,\ s)\varepsilon^n=\sum_{n=0}^{\infty}\hat{\alpha}_n(0,\ s)\varepsilon^n \\ \sum_{n=0}^{\infty}\hat{v}_n(0,\ s)\varepsilon^n+\sum_{n=0}^{\infty}\psi_n(0,\ s)\varepsilon^{n+2}=\sum_{n=0}^{\infty}\hat{\beta}_n(0,\ s)\varepsilon^n \end{cases},\quad s\in\Gamma$$

and

$$(4.5)\qquad \sum_{n=0}^{\infty}\phi_n(\infty,\ s)\varepsilon^n=0,\quad \sum_{n=0}^{\infty}\psi_n(\infty,\ s)\varepsilon^n=0,\quad s\in\Gamma.$$

Since $\hat{v}_n(0,\ s)=\hat{\xi}_n(0,\ s)$ ($n=0, 1, 2,\dots$), (4.4) and (4.5) can be rewritten in the form

$$(4.6)\qquad \hat{\xi}_n(0,\ s)=\hat{\beta}_n(0,\ s)\quad (n=0,\ 1),$$

$$(4.7)\qquad \phi_n(0,\ s)=\hat{\alpha}_n(0,\ s)-\hat{u}_n(0,\ s),\quad \phi_n(\infty,\ s)=0,\quad (n=0,\ 1,\ 2,\dots),$$

$$(4.8)\qquad \psi_n(0,\ s)=\hat{\beta}_{n+2}(0,\ s)-\hat{\xi}_{n+2}(0,\ s),\quad \psi_n(\infty,\ s)=0\quad (n=0,\ 1,\ 2,\dots).$$

Note that (4.6) determines the boundary value in (3.4) and (3.6). That is, $\xi_0(x)=\beta_0(x)$ and $\xi_1(x)=\beta_1(x)$ ($\in C^\infty(\Gamma)$). At times we shall use the term *a C^∞ linear differential operator in η and s* for the sum of compositions of C^∞ linear differential operators in s with differential operators in η of the form $a_k(\eta,\ s)\partial_\eta^k$, where k is a nonnegative integer and $a_k(\eta,\ s)$ are bounded C^∞ functions defined for $\eta\in[0,\ \infty)$ and $s\in\Gamma$.

DEFINITION 5 (Fife [6]). Let $\mathscr{E}$ be the set of functions $W(\eta,\ s,\ \varepsilon)$ defined on $[0,\ \infty)\times\Gamma\times[0,\ \varepsilon_0)$ with the property that for each C^∞ linear differential operator D of any order in the variables η and s, there exist positive constants γ and K (possibly depending on D and W, but not on η, s or ε) with $|DW|\le Ke^{-\gamma\eta}$.

For $n=0$, noting that $\hat{u}_0(0,\ s)=h(\hat{\beta}_0(0,\ s))$ and $\hat{v}_0(0,\ s)=\hat{\beta}_0(0,\ s)$, we can reduce (4.3) to

$$(4.9)\qquad \partial_\eta^2\phi_0(\eta,\ s)=f(h(\hat{\beta}_0(0,\ s))+\phi_0(\eta,\ s),\ \hat{\beta}_0(0,\ s)),$$

$$(4.10)\qquad \partial_\eta^2\psi_0(\eta,\ s)=g(h(\hat{\beta}_0(0,\ s))+\phi_0(\eta,\ s),\ \hat{\beta}_0(0,\ s))-g(h(\hat{\beta}_0(0,\ s)),\ \hat{\beta}_0(0,\ s)).$$

The boundary conditions (4.7) and (4.8) are

$$(4.11)\qquad \phi_0(0,\ s)=\hat{\alpha}_0(0,\ s)-h(\hat{\beta}_0(0,\ s)),\quad \phi_0(\infty,\ s)=0,$$

$$(4.12)\qquad \psi_0(0,\ s)=\hat{\beta}_2(0,\ s)-\hat{\xi}_2(0,\ s),\quad \psi_0(\infty,\ s)=0.$$

For the problem (4.9) and (4.11), Fife [6] proved

LEMMA 6. *Let $s \in \Gamma$ and assume* (i), (ii) *and* (iv). *Then the boundary value problem* (4.9) *and* (4.11) *has only one monotone solution $\phi_0(\eta, s)$ with respect to η, such that $\phi_0(\eta, s) \in \mathscr{E}$.*

Next rewrite (4.10) as

$$\partial_\eta^2 \psi_0(\eta, s) = p(\eta, s) \tag{4.13}$$

where

$$p(\eta, s) = \int_0^1 g_u(h(\hat{\beta}_0(0, s)) + \theta\phi_0(\eta, s), \hat{\beta}_0(0, s))d\theta\, \phi_0(\eta, s) \in \mathscr{E}, \tag{4.14}$$

and integrate (4.13). Then we obtain

$$\psi_0(\eta, s) = a(s) + b(s)\eta + \int_\eta^\infty d\tau \int_\tau^\infty p(\zeta, s)d\zeta$$

for arbitrary functions $a(s)$ and $b(s)$ of $s \in \Gamma$. Since (4.14) implies

$$\lim_{\eta\to\infty} \int_\eta^\infty d\tau \int_\tau^\infty p(\zeta, s)d\zeta = 0,$$

it follows from the second condition of (4.12) that $a(s) = b(s) = 0$. Thus we have

$$\psi_0(\eta, s) = \int_\eta^\infty d\tau \int_\tau^\infty p(\zeta, s)d\zeta \in \mathscr{E}. \tag{4.15}$$

Setting $\eta = 0$ in (4.15), we can specify the boundary value in (3.10) with $n=2$ as

$$\hat{\xi}_2(0, s) = \hat{\beta}_2(0, s) - \int_0^\infty d\tau \int_\tau^\infty p(\zeta, s)d\zeta \in C^\infty(\Gamma).$$

Thus, we have

LEMMA 7. *There exists a unique function $\hat{\xi}_2(0, s) \in C^\infty(\Gamma)$ for which the problem* (4.10) *and* (4.12) *has a unique solution $\psi_0(\eta, s) \in \mathscr{E}$.*

To treat the case $n \geq 1$, the next lemma is of great importance. For convenience, we put

$$\begin{cases} G_1(\Phi, \Psi) \equiv \tilde{M}[\tilde{U} + \Phi, \tilde{V} + \varepsilon^2\Psi] - \tilde{M}[\tilde{U}, \tilde{V}], \\ G_2(\Phi, \Psi) \equiv \tilde{N}[\tilde{U} + \Phi, \tilde{V} + \varepsilon^2\Psi] - \tilde{N}[\tilde{U}, \tilde{V}]. \end{cases}$$

Then

$$\begin{cases} G_1(\Phi, \Psi) = \partial_\eta^2\Phi + \varepsilon\delta\cdot v\partial_\eta\Phi + \varepsilon v\partial_\eta\cdot\delta\Phi + \varepsilon^2\delta\cdot\delta\Phi \\ \qquad\qquad - f(\tilde{U} + \Phi, \tilde{V} + \varepsilon^2\Psi) + f(\tilde{U}, \tilde{V}), \\ G_2(\Phi, \Psi) = \partial_\eta^2\Psi + \varepsilon\delta\cdot v\partial_\eta\Psi + \varepsilon v\partial_\eta\cdot\delta\Psi + \varepsilon^2\delta\cdot\delta\Psi \\ \qquad\qquad - g(\tilde{U} + \Phi, \tilde{V} + \varepsilon^2\Psi) + g(\tilde{U}, \tilde{V}). \end{cases} \tag{4.16}$$

LEMMA 8. *For given functions $W^{(1)}(\eta, s, \varepsilon)$ and $W^{(2)}(\eta, s, \varepsilon)$, let $W_n^{(1)}(\eta, s, \varepsilon) = \partial_\varepsilon^n W^{(1)}(\eta, s, \varepsilon)/n!$ and $W_n^{(2)}(\eta, s, \varepsilon) = \partial_\varepsilon^n W^{(2)}(\eta, s, \varepsilon)/n!$ $(n=0, 1, 2,\dots)$. Then for $n \geq 1$, we have*

$$\begin{cases} \partial_\varepsilon^n G_1(W^{(1)}, W^{(2)})(\eta, s, \varepsilon) = L_1(\varepsilon)[W_n^{(1)}, W_n^{(2)}](\eta, s, \varepsilon) \\ \qquad - \prod_{n-1}^{(1)}[W_0^{(1)}, W_0^{(2)},\dots, W_{n-1}^{(1)}, W_{n-1}^{(2)}](\eta, s, \varepsilon), \\ \partial_\varepsilon^n G_2(W^{(1)}, W^{(2)})(\eta, s, \varepsilon) = L_2(\varepsilon)[W_n^{(1)}, W_n^{(2)}](\eta, s, \varepsilon) \\ \qquad - \prod_{n-1}^{(2)}[W_0^{(1)}, W_0^{(2)},\dots, W_{n-1}^{(1)}, W_{n-1}^{(2)}](\eta, s, \varepsilon), \end{cases}$$

where L_1 and L_2 are the linear differential operators with

$$\begin{cases} L_1(0)[W_n^{(1)}, W_n^{(2)}](\eta, s, 0) = \partial_\eta^2 W_n^{(1)}(\eta, s, 0) \\ \qquad - f_u(\hat{u}_0(0, s) + W_0^{(1)}(\eta, s, 0), \hat{v}_0(0, s))W_n^{(1)}(\eta, s, 0), \\ L_2(0)[W_n^{(1)}, W_n^{(2)}](\eta, s, 0) = \partial_\eta^2 W_n^{(2)}(\eta, s, 0) \\ \qquad - g_u(\hat{u}_0(0, s) + W_0^{(1)}(\eta, s, 0), \hat{v}_0(0, s))W_n^{(1)}(\eta, s, 0), \end{cases}$$

and $\prod_{n-1}^{(i)}$ $(i=1, 2; n \geq 1)$ are the differential operators (in the variables η and s) applied to the $2n$ functions such that

$$\prod_{n-1}^{(i)}[W_0^{(1)}, W_0^{(2)},\dots, W_{n-1}^{(1)}, W_{n-1}^{(2)}] \in \mathscr{E}$$
$$\textit{whenever } W_0^{(1)}, W_0^{(2)},\dots, W_{n-1}^{(1)}, W_{n-1}^{(2)} \in \mathscr{E}.$$

PROOF. We omit the proof, since it is similar to that of Lemma 3.3 in Fife [6].

Using the above lemma, for $n \geq 1$, we can rewrite the problem (4.3), (4.7) and (4.8) as follows:

$$\partial_\eta^2 \phi_n(\eta, s) - f_u(h(\hat{\beta}_0(0, s)) + \phi_0(\eta, s), \hat{\beta}_0(0, s))\phi_n(\eta, s) = \prod_{n-1}^{(1)}(\eta, s, 0), \tag{4.17}$$

$$\partial_\eta^2 \psi_n(\eta, s) - g_u(h(\hat{\beta}_0(0, s)) + \phi_0(\eta, s), \hat{\beta}_0(0, s))\phi_n(\eta, s) = \prod_{n-1}^{(2)}(\eta, s, 0), \tag{4.18}$$

with the boundary conditions

$$\phi_n(0, s) = \hat{\alpha}_n(0, s) - \hat{u}_n(0, s), \quad \phi_n(\infty, s) = 0, \tag{4.19}$$

$$\psi_n(0, s) = \hat{\beta}_{n+2}(0, s) - \hat{\xi}_{n+2}(0, s), \quad \psi_n(\infty, s) = 0. \tag{4.20}$$

We assume that $\phi_0, \psi_0,\dots, \phi_{n-1}, \psi_{n-1} \in \mathscr{E}$, that is, $\prod_{n-1}^{(i)}(\eta, s, 0) \in \mathscr{E}$ $(i=1, 2)$.

LEMMA 9 (Fife [6]). *For $s \in \Gamma$, consider the boundary value problem*

$$\begin{cases} \partial_\eta^2 w(\eta, s) - f_u(h(\hat{\beta}_0(0, s)) + \phi_0(\eta, s), \hat{\beta}_0(0, s))w(\eta, s) = k(\eta, s), \\ w(0, s) = w_0(s), \ w(\infty, s) = 0, \end{cases}$$

where $w_0(s)$ *is some function of* $C^\infty(\Gamma)$ *and* $k(\eta, s) \in \mathscr{E}$. *Then there exists a unique solution* $w(\eta, s) \in \mathscr{E}$.

Applying Lemma 9 to the problem (4.17) and (4.19), we find a unique solution $\phi_n(\eta, s) \in \mathscr{E}$. Also, we see that in a way similar to the case (4.10) and (4.12), the problem (4.18) and (4.20) has a unique solution

$$\psi_n(\eta, s) = \int_\eta^\infty d\tau \int_\tau^\infty \{g_u(h(\hat{\beta}_0(0, s)) + \phi_0(\zeta, s), \hat{\beta}_0(0, s))\phi_n(\zeta, s) + \Pi_{n-1}^{(2)}(\zeta, s, 0)\} d\zeta \in \mathscr{E}.$$

Further, we have

$$\hat{\xi}_{n+2}(0, s) = \hat{\beta}_{n+2}(0, s) - \int_0^\infty d\tau \int_\tau^\infty \{g_u(h(\hat{\beta}_0(0, s)) + \phi_0(\zeta, s), \hat{\beta}_0(0, s))\phi_n(\zeta, s) + \Pi_{n-1}^{(2)}(\zeta, s, 0)\} d\zeta \in C^\infty(\Gamma).$$

Thus we have obtained

LEMMA 10. *Assume* (i), (ii) *and* (iv). *Then there exists a unique function* $\hat{\xi}_{n+2}(0, s) \in C^\infty(\Gamma)$ *for which the boundary value problem* (4.17), (4.18), (4.19) *and* (4.20) *has a unique solution* $(\phi_n(\eta, s), \psi_n(\eta, s))$ *belonging to* $\mathscr{E} \times \mathscr{E}$.

REMARK 11. The above discussion yields that boundary values of the outer expansion $v_n(x)$ at $x \in \Gamma$ are

$$\begin{aligned}
\hat{v}_n(0, s) &= \hat{\xi}_n(0, s), \\
&= \hat{\beta}_n(0, s) \quad (n = 0, 1), \\
&= \hat{\beta}_n(0, s) - \int_0^\infty d\tau \int_\tau^\infty \int_0^1 g_u(h(\hat{\beta}_0(0, s)) + \theta\phi_0(\zeta, s), \\
&\qquad \hat{\beta}_0(0, s)) d\theta \phi_0(\zeta, s) d\zeta \qquad (n = 2), \\
&= \hat{\beta}_n(0, s) - \int_0^\infty d\tau \int_\tau^\infty \{g_u(h(\hat{\beta}_0(0, s)) + \phi_0(\zeta, s), \\
&\qquad \hat{\beta}_0(0, s))\phi_{n-2}(\zeta, s) + \Pi_{n-3}^{(2)}(\zeta, s, 0)\} d\zeta \qquad (n \geq 3).
\end{aligned}$$

5. Uniform approximation

For an arbitrary nonnegative integer m, define

$$U_m(x, \varepsilon) = \textstyle\sum_{n=0}^m u_n(x)\varepsilon^n, \quad V_m(x, \varepsilon) = \sum_{n=0}^m v_n(x)\varepsilon^n,$$

where u_n and v_n $(1 \leq n \leq m)$ are coefficients of the outer expansions determined in Section 3 by using the boundary values in Remark 11.

LEMMA 12.

$$\|M[U_m, V_m](x, \varepsilon)\|_{C^1(\bar{\Omega})} \leq C\varepsilon^{m+1}, \tag{5.1}$$

$$\|N[U_m, V_m](x, \varepsilon)\|_{C^1(\bar{\Omega})} \leq C\varepsilon^{m+1}. \tag{5.2}$$

PROOF. We shall prove only (5.1) since (5.2) follows from a similar argument. $M[U_m, V_m](x, \varepsilon)$ is an infinitely differentiable function of ε. Clearly

$$M[U_m, V_m](x, \varepsilon) = \frac{\varepsilon^{m+1}}{m!}\int_0^1 (1-\theta)^m \partial_\varepsilon^{m+1} M[U_m, V_m](x, \theta\varepsilon)d\theta \equiv \varepsilon^{m+1}R(x, \varepsilon).$$

Obviously $R(x, \varepsilon)$ and its derivative with respect to x are bounded. Thus we have (5.1).

Since the coordinate system (η, s, ε) is defined merely in the strip $\bar{\Omega}_d$, we must extend the domain $\bar{\Omega}_d$ into $\bar{\Omega}$. For this purpose, we take a cutoff function $\zeta(x) \in C^\infty(\bar{\Omega}_d)$ such that

$$\zeta(x) = 1 \text{ on } \bar{\Omega}_{d/3},\ 0 \leq \zeta(x) \leq 1 \text{ on } \bar{\Omega}_{2d/3} - \Omega_{d/3},\ \zeta(x) = 0 \text{ on } \bar{\Omega}_d - \Omega_{2d/3},$$

and define $\Phi_m(x, \varepsilon)$ and $\Psi_m(x, \varepsilon)$ by

$$\Phi_m(x, \varepsilon) = \begin{cases} \zeta(x) \sum_{n=0}^m \phi_n(t(x)/\varepsilon, s(x))\varepsilon^n, & x \in \bar{\Omega}_d, \\ 0, \quad x \in \bar{\Omega} - \Omega_d, \end{cases}$$

and

$$\Psi_0(x, \varepsilon) = \Psi_1(x, \varepsilon) = 0,\ x \in \bar{\Omega},$$

$$\Psi_m(x, \varepsilon) = \begin{cases} \zeta(x) \sum_{n=0}^{m-2} \psi_n(t(x)/\varepsilon, s(x))\varepsilon^n, & x \in \bar{\Omega}_d \\ 0, \quad x \in \bar{\Omega} - \Omega_d \end{cases} \qquad (m \geq 2),$$

respectively. Notice that $\Phi_m(x, \varepsilon)$ and $\Psi_m(x, \varepsilon) \in C^\infty(\bar{\Omega})$.

LEMMA 13.

$$|(\tilde{M}[\tilde{U}_m + \tilde{\Phi}_m, \tilde{V}_m + \varepsilon^2\tilde{\Psi}_m] - \tilde{M}[\tilde{U}_m, \tilde{V}_m])(\eta, s, \varepsilon)|_{0,\bar{\Omega}_{d/3}} \leq C\varepsilon^{m+1}, \tag{5.3}$$

$$|(\tilde{N}[\tilde{U}_m + \tilde{\Phi}_m, \tilde{V}_m + \varepsilon^2\tilde{\Psi}_m] - \tilde{N}[\tilde{U}_m, \tilde{V}_m])(\eta, s, \varepsilon)|_{0,\bar{\Omega}_{d/3}} \leq C\varepsilon^{m+1}, \tag{5.4}$$

and for any C^∞ linear differential operator D in η and s,

$$|D(\tilde{M}[\tilde{U}_m + \tilde{\Phi}_m, \tilde{V}_m + \varepsilon^2\tilde{\Psi}_m] - \tilde{M}[\tilde{U}_m, \tilde{V}_m])(\eta, s, \varepsilon)|_{0,\bar{\Omega}_{d/3}} \leq C\varepsilon^{m+1}, \tag{5.5}$$

$$|D(\tilde{N}[\tilde{U}_m + \tilde{\Phi}_m, \tilde{V}_m + \varepsilon^2\tilde{\Psi}_m] - \tilde{N}[\tilde{U}_m, \tilde{V}_m])(\eta, s, \varepsilon)|_{0,\bar{\Omega}_{d/3}} \leq C\varepsilon^{m+1}. \tag{5.6}$$

PROOF. For $x \in \bar{\Omega}_{d/3}$, we note

$$(\tilde{M}[\tilde{U}_m + \tilde{\Phi}_m, \tilde{V}_m + \varepsilon^2\tilde{\Psi}_m] - \tilde{M}[\tilde{U}_m, \tilde{V}_m])(\eta, s, \varepsilon) = F_1(\eta, s, \varepsilon) - F_2(\eta, s, \varepsilon)$$

where

$$F_1(\eta, s, \varepsilon) = (\tilde{M}[\tilde{U}_m + \tilde{\Phi}_m, \tilde{V}_m + \varepsilon^2\tilde{\Psi}_{m+2}] - \tilde{M}[\tilde{U}_m, \tilde{V}_m])(\eta, s, \varepsilon)$$

and

$$F_2(\eta, s, \varepsilon) = (\tilde{M}[\tilde{U}_m + \tilde{\Phi}_m, \tilde{V}_m + \varepsilon^2\tilde{\Psi}_{m+2}] - \tilde{M}[\tilde{U}_m + \tilde{\Phi}_m, \tilde{V}_m + \varepsilon^2\tilde{\Psi}_m])(\eta, s, \varepsilon).$$

By the scheme in the preceeding section, determining the boundary layer expansion, we easily see that $F_1(\eta, s, \varepsilon)$ is an infinitely differentiable function of ε and

$$F_1(\eta, s, \varepsilon) = \frac{\varepsilon^{m+1}}{m!}\int_0^1 (1-\theta)^m \partial_\varepsilon^{m+1} F_1(\eta, s, \theta\varepsilon)d\theta.$$

Thus by Lemma 8, we have

$$\partial_\varepsilon^{m+1} F_1(\eta, s, \varepsilon) = -\prod\nolimits_m^{(1)} [\phi_0, \psi_0, \dots, \phi_m, \psi_m](\eta, s, \varepsilon) \in \mathscr{E}.$$

Therefore we have

$$|F_1(\eta, s, \varepsilon)|_{0,\bar{\Omega}_{d/3}} \leq C\varepsilon^{m+1}, \quad |DF_1(\eta, s, \varepsilon)|_{0,\bar{\Omega}_{d/3}} \leq C\varepsilon^{m+1}.$$

Next we rewrite F_2 as

$$F_2(\eta, s, \varepsilon) = \int_0^1 \tilde{M}_v[\tilde{U}_m + \tilde{\Phi}_m, \tilde{V}_m + \varepsilon^2\tilde{\Psi}_m + \theta\varepsilon^{m+1}(\psi_{m-1} + \varepsilon\psi_m)]d\theta$$
$$\times \varepsilon^{m+1}(\psi_{m-1} + \varepsilon\psi_m).$$

$\tilde{M}_v$ is an infinitely differentiable function of η and s, whose derivatives are bounded in $\bar{\Omega}_{d/3}$. Since ψ_{m-1} and ψ_m are in $\mathscr{E}$, we obtain

$$|F_2(\eta, s, \varepsilon)|_{0,\bar{\Omega}_{d/3}} \leq C\varepsilon^{m+1}, \quad |DF_2(\eta, s, \varepsilon)|_{0,\bar{\Omega}_{d/3}} \leq C\varepsilon^{m+1}.$$

Thus we have (5.3) and (5.5). In a similar way, we obtain (5.4) and (5.6).

We are now ready to provide the formal approximation to the solution of (2.1) and (2.2). Note that $U_m + \Phi_m$ and $V_m + \varepsilon^2\Psi_m$ satisfy the boundary condition (2.2) approximately up to order ε^m.

THEOREM 14. *Let* $U_m(x, \varepsilon)$, $V_m(x, \varepsilon)$, $\Phi_m(x, \varepsilon)$ *and* $\Psi_m(x, \varepsilon)$ *be as above. Then for any* $\alpha \in (0, 1)$,

(5.7) $$\|M[U_m + \Phi_m, V_m + \varepsilon^2\Psi_m](x, \varepsilon)\|_{C_\varepsilon^\alpha(\bar{\Omega})} \leq C\varepsilon^{m+1}$$

and

(5.8) $$\|N[U_m + \Phi_m, V_m + \varepsilon^2\Psi_m](x, \varepsilon)\|_{C^\alpha(\bar{\Omega})} \leq C\varepsilon^{m+1-\alpha}.$$

PROOF. We only show (5.7) since (5.8) can be analogously obtained. In $\bar{\Omega}-\Omega_{2d/3}$, Lemma 12 yields

$$\|M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m](x, \varepsilon)\|_{C^1(\bar{\Omega}-\Omega_{2d/3})}\leq C\varepsilon^{m+1}.$$

In $\bar{\Omega}_{2d/3}-\Omega_{d/3}$,

$$\begin{aligned} M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m] = {} & M[U_m, V_m]+\partial_\eta^2\tilde{\Phi}_m+\varepsilon\delta\cdot v\partial_\eta\tilde{\Phi}_m+\varepsilon v\partial_\eta\cdot\delta\tilde{\Phi}_m \\ & +\varepsilon^2\delta\cdot\delta\tilde{\Phi}_m-\int_0^1\{f_u(\tilde{U}_m+\theta\tilde{\Phi}_m, \tilde{V}_m+\varepsilon^2\theta\tilde{\Psi}_m)\tilde{\Phi}_m \\ & -f_v(\tilde{U}_m+\theta\tilde{\Phi}_m, \tilde{V}_m+\varepsilon^2\theta\tilde{\Psi}_m)\varepsilon^2\tilde{\Psi}_m\}d\theta. \end{aligned}$$

Lemma 12 and the fact that $\tilde{\Phi}_m$ and $\tilde{\Psi}_m$ are in $\mathscr{E}$ yields

$$\|M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m](x, \varepsilon)\|_{C^1(\bar{\Omega}_{2d/3}-\Omega_{d/3})}\leq C\varepsilon^{m+1}.$$

Finally in $\bar{\Omega}_{d/3}$, write

$$\begin{aligned} M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m] = {} & M[U_m, V_m] \\ & +\{M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m]-M[U_m, V_m]\}. \end{aligned}$$

Applying Lemma 13, we have

$$|M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m]-M[U_m, V_m]|_{0,\bar{\Omega}_{d/3}}\leq C\varepsilon^{m+1},$$
$$|M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m]-M[U_m, V_m]|_{1,\bar{\Omega}_{d/3}}\leq C\varepsilon^{m}.$$

Thus we have

$$|M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m]|_{0,\bar{\Omega}}\leq C\varepsilon^{m+1},\quad |M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m]|_{1,\bar{\Omega}}\leq C\varepsilon^{m}.$$

Therefore by the interpolation inequality, we obtain

$$|M[U_m+\Phi_m, V_m+\varepsilon^2\Psi_m]|_{\alpha,\bar{\Omega}}\leq C\varepsilon^{m+1-\alpha}.$$

Thus we have proved (5.7).

6. Justification

By the use of the outer and boundary layer expansion, we define $(U_0^m(x, \varepsilon), V_0^m(x, \varepsilon))$ on $\bar{\Omega}$ by

$$U_0^m(x, \varepsilon)=\begin{cases} U_m(x, \varepsilon)+\Phi_m(x, \varepsilon)+\zeta(x)\{\hat{\alpha}(0, s, \varepsilon)-\sum_{n=0}^m\hat{\alpha}_n(0, s)\varepsilon^n\}, & x\in\bar{\Omega}_d, \\ U_m(x, \varepsilon)+\Phi_m(x, \varepsilon), \quad x\in\bar{\Omega}-\Omega_d, & \end{cases}$$

$$V_0^m(x,\varepsilon)=\begin{cases} V_m(x,\varepsilon)+\varepsilon^2\Psi_m(x,\varepsilon)+\zeta(x)\{\hat{\beta}(0,s,\varepsilon)-\sum_{n=0}^m\hat{\beta}_n(0,s)\varepsilon^n\}, & x\in\bar{\Omega}_d,\\ V_m(x,\varepsilon)+\varepsilon^2\Psi_m(x,\varepsilon), \quad x\in\bar{\Omega}-\Omega_d. \end{cases}$$

Then $(U_0^m(x,\varepsilon), V_0^m(x,\varepsilon))$ satisfies the boundary condition (2.2) on Γ.

Now we look for a solution $(u(x,\varepsilon), v(x,\varepsilon))$ of (2.1) and (2.2) in the form

$$\text{(6.1)}\qquad \begin{cases} u(x,\varepsilon)=U_0^m(x,\varepsilon)+\varepsilon^m r(x,\varepsilon)+\varepsilon^m h'(v_0(x))s(x,\varepsilon),\\ v(x,\varepsilon)=V_0^m(x,\varepsilon)+\varepsilon^m s(x,\varepsilon), \end{cases}$$

for $0<\varepsilon<\varepsilon_0$, where ε_0 is a small constant. Substituting (6.1) into (2.1), we have

$$\text{(6.2)}\qquad \begin{cases} R(r,s,\varepsilon)\equiv\varepsilon^2\Delta r+\varepsilon^2\Delta(h'(v_0)s)-\varepsilon^{-m}[f(U_0^m+\varepsilon^m r+\varepsilon^m h'(v_0)s,\\ \qquad\qquad V_0^m+\varepsilon^m s)-\varepsilon^2\Delta U_0^m]=0,\\ S(r,s,\varepsilon)\equiv\Delta s-\varepsilon^{-m}[g(U_0^m+\varepsilon^m r+\varepsilon^m h'(v_0)s,V_0^m+\varepsilon^m s)-\Delta V_0^m]=0. \end{cases}$$

Let $t=(r,s)$ and $T(t,\varepsilon)=(R(r,s,\varepsilon),S(r,s,\varepsilon))$. Then T is a continuously differentiable mapping from $X_\varepsilon\times(0,\varepsilon_0)$ into Y_ε where

$$X_\varepsilon=C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})\times C_0^{2+\alpha}(\bar{\Omega})\quad\text{and}\quad Y_\varepsilon=C_\varepsilon^\alpha(\bar{\Omega})\times C^\alpha(\bar{\Omega}).$$

The mapping T has the Fréchet derivative of the form

$$\text{(6.3)}\qquad T_t(t,\varepsilon)=\begin{pmatrix} L_1(t,\varepsilon), & L_2(t,\varepsilon)\\ L_3(t,\varepsilon), & L_4(t,\varepsilon)\end{pmatrix},$$

where

$$\begin{aligned} L_1(t,\varepsilon)&=\varepsilon^2\Delta-f_u(U_0^m+\varepsilon^m r+\varepsilon^m h'(v_0)s,V_0^m+\varepsilon^m s),\\ L_2(t,\varepsilon)&=\varepsilon^2 h'(v_0)\Delta+\varepsilon^2\nabla(h'(v_0))\cdot\nabla+\varepsilon^2\Delta(h'(v_0))\\ &\quad-\Pi(f)(U_0^m+\varepsilon^m r+\varepsilon^m h'(v_0)s,V_0^m+\varepsilon^m s),\\ L_3(t,\varepsilon)&=-g_u(U_0^m+\varepsilon^m r+\varepsilon^m h'(v_0)s,V_0^m+\varepsilon^m s),\\ L_4(t,\varepsilon)&=\Delta-\Pi(g)(U_0^m+\varepsilon^m r+\varepsilon^m h'(v_0)s,V_0^m+\varepsilon^m s),\\ \Pi(f)(z_0,z_1)&=f_u(z_0,z_1)h'(v_0)+f_v(z_0,z_1). \end{aligned}$$

For simplicity we put

$$M_\varepsilon=L_1(0,\varepsilon):\quad C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})\longrightarrow C_\varepsilon^\alpha(\bar{\Omega}).$$

LEMMA 15. *Assume* (ii). *Then there exists a constant* ε_0 *such that* M_ε *has a uniformly bounded inverse for* $0<\varepsilon<\varepsilon_0$.

PROOF. First we consider a linear operator M_ε^0 from $C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})$ into $C_\varepsilon^\alpha(\bar{\Omega})$ such that

$$M_\varepsilon^0 = \varepsilon^2 \Delta - f_u(U_0(x, \varepsilon) + \Phi_0(x, \varepsilon), V_0(x, \varepsilon)).$$

For any $F \in C_\varepsilon^\alpha(\bar{\Omega})$ we obtain, by Appendix B, a unique solution $u(x, \varepsilon) \in C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})$ of

$$M_\varepsilon^0 u = F \tag{6.4}$$

such that

$$\|u(x, \varepsilon)\|_{C^0} \leq C\|F\|_{C^0}. \tag{6.5}$$

We rewrite (6.4) as

$$\Delta u = \varepsilon^{-2}\{f_u(U_0+\Phi_0, V_0)u + F\}.$$

The Schauder estimates imply

$$|u|_{2+\alpha} \leq \varepsilon^{-2}C(\varepsilon^{-\alpha}\|u\|_{C^0} + |u|_\alpha + \|F\|_{C^\alpha}).$$

Further applying the interpolation inequality and (6.5), we obtain

$$\varepsilon^{2+\alpha}|u|_{2+\alpha} \leq C\|F\|_{C_\varepsilon^\alpha}, \quad \varepsilon^2|u|_2 \leq C\|F\|_{C_\varepsilon^\alpha}, \quad \varepsilon|u|_1 \leq C\|F\|_{C_\varepsilon^\alpha}.$$

Thus we have

$$\|u(x, \varepsilon)\|_{C_{\varepsilon,0}^{2+\alpha}} \leq C\|F\|_{C_\varepsilon^\alpha},$$

that is, M_ε^0 is invertible and

$$\|(M_\varepsilon^0)^{-1}\|_{C_\varepsilon^\alpha \to C_{\varepsilon,0}^{2+\alpha}} \leq C.$$

Next we have

$$M_\varepsilon - M_\varepsilon^0 = f_{uu}^*(U_0 + \Phi_0 - U_0^m) + f_{uv}^*(V_0 - V_0^m)$$

where f_{uu}^* and f_{uv}^* are evaluated at $(U_0^m + \theta_1(U_0 + \Phi_0 - U_0^m), V_0^m + \theta_2(V_0 - V_0^m))$ for $0 < \theta_1, \theta_2 < 1$. For any $u \in C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})$,

$$\begin{aligned}
&\|(M_\varepsilon - M_\varepsilon^0)u\|_{C_\varepsilon^\alpha} \\
&\quad \leq \|f_{uu}^*(U_0 + \Phi_0 - U_0^m)u\|_{C_\varepsilon^\alpha} + \|f_{uv}^*(V_0 - V_0^m)u\|_{C_\varepsilon^\alpha}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|f_{uu}^*(U_0 + \Phi_0 - U_0^m)u\|_{C_\varepsilon^\alpha} \\
&\quad \leq (|f_{uu}^*|_0|U_0 + \Phi_0 - U_0^m|_0 + \varepsilon^\alpha|f_{uu}^*|_\alpha|U_0 + \Phi_0 - U_0^m|_0
\end{aligned}$$

$$+ \varepsilon^\alpha |f^*_{uu}|_0 |U_0 + \Phi_0 - U_0^m|_\alpha)\|u\|_{C^{2+\alpha}_{\varepsilon,0}}$$
$$\le \|f^*_{uu}\|_{C^\alpha_\varepsilon} \|U_0 + \Phi_0 - U_0^m\|_{C^\alpha_\varepsilon} \|u\|_{C^{2+\alpha}_{\varepsilon,0}}.$$

In a similar way,

$$\|f^*_{uv}(V_0 - V_0^m)u\|_{C^\alpha_\varepsilon} \le \|f^*_{uv}\|_{C^\alpha_\varepsilon} \|V_0 - V_0^m\|_{C^\alpha_\varepsilon} \|u\|_{C^{2+\alpha}_{\varepsilon,0}}.$$

Thus we have

$$\|M_\varepsilon - M_\varepsilon^0\|_{C^{2+\alpha}_{\varepsilon,0} \to C^\alpha_\varepsilon}$$
$$\le \|f^*_{uu}\|_{C^\alpha_\varepsilon} \|U_0 + \Phi_0 - U_0^m\|_{C^\alpha_\varepsilon} + \|f^*_{uv}\|_{C^\alpha_\varepsilon} \|V_0 - V_0^m\|_{C^\alpha_\varepsilon}.$$

Noting that

$$\|f^*_{uu}\|_{C^\alpha_\varepsilon} = O(1), \quad \|U_0 + \Phi_0 - U_0^m\|_{C^\alpha_\varepsilon} = O(\varepsilon),$$
$$\|f^*_{uv}\|_{C^\alpha_\varepsilon} = O(1), \quad \|V_0 - V_0^m\|_{C^\alpha_\varepsilon} = O(\varepsilon),$$

as $\varepsilon \to 0$, we obtain

$$\|M_\varepsilon - M_\varepsilon^0\|_{C^{2+\alpha}_{\varepsilon,0} \to C^\alpha_\varepsilon} = O(\varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Therefore with sufficiently small ε_0, we have

$$\|(M_\varepsilon^0)^{-1}(M_\varepsilon^0 - M_\varepsilon)\|_{C^{2+\alpha}_{\varepsilon,0} \to C^{2+\alpha}_{\varepsilon,0}} < 1$$

for $0<\varepsilon<\varepsilon_0$. Since $M_\varepsilon = M_\varepsilon^0(I - (M_\varepsilon^0)^{-1}(M_\varepsilon^0 - M_\varepsilon))$, M_ε is invertible and

$$\|(M_\varepsilon)^{-1}\|_{C^\alpha_\varepsilon \to C^{2+\alpha}_{\varepsilon,0}} \le C$$

for $0<\varepsilon<\varepsilon_0$. This proves the lemma.

Put

$$N_\varepsilon = L_4(0, \varepsilon) \quad : \quad C_0^{2+\alpha}(\bar{\Omega}) \longrightarrow C^\alpha(\bar{\Omega}),$$
$$H = \Delta - \Pi(g)(U_0, V_0) : C_0^{2+\alpha}(\bar{\Omega}) \longrightarrow C^\alpha(\bar{\Omega}).$$

If (ii) is satisfied, clearly we have

$$\|H^{-1}\|_{C^\alpha(\bar{\Omega}) \to C_0^{2+\alpha}(\bar{\Omega})} \le C. \tag{6.6}$$

Moreover by Theorem 2 in Chicco [4], for any $F \in L^p(\Omega)$ $Hv = F$ has a unique solution $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\|v\|_{W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)} \le C\|F\|_{L^p(\Omega)}.$$

Setting $p = N$ and applying the Sobolev imbedding theorem, we have

$$\|v\|_{C^\alpha(\bar{\Omega})} \leq C\|F\|_{L^p(\Omega)}.$$

Hence, noting $v=H^{-1}F$, we obtain

$$\|H^{-1}F\|_{C^\alpha(\bar{\Omega})} \leq C\|F\|_{L^p(\Omega)}. \tag{6.7}$$

LEMMA 16 (Ito [9]). *Suppose that* (ii) *is satisfied. Then there exists a constant* ε_0 *such that* N_ε *has a uniformly bounded inverse for* $0<\varepsilon<\varepsilon_0$.

PROOF. We rewrite N_ε in the form

$$N_\varepsilon = H - K, \quad \text{where} \quad K = \Pi(g)(U_0^m, V_0^m) - \Pi(g)(U_0, V_0).$$

For $v \in C_0^{2+\alpha}(\bar{\Omega})$,

$$\begin{aligned} Kv = [&\Pi(g)(U_0^m, V_0^m) - \Pi(g)(U_0 + \Phi_0, V_0) \\ &+ \Pi(g)(U_0 + \Phi_0, V_0) - \Pi(g)(U_0, V_0)]v \end{aligned}$$

and

$$|Kv| \leq C(\varepsilon + |\Phi_0|)\,\|v\|_{C_0^{2+\alpha}}.$$

Here using (6.7), we have

$$\|H^{-1}K\|_{C_0^{2+\alpha}\to C^\alpha} \leq C(\varepsilon + \|\Phi_0\|_{L^p}) \leq C\varepsilon^{1/p}. \tag{6.8}$$

For any $G \in C^\alpha(\bar{\Omega})$, we consider

$$N_\varepsilon v \equiv Hv - Kv = G.$$

For sufficiently small $\varepsilon>0$, by (6.6) and (6.8), we have

$$\begin{aligned} \|v\|_{C_0^{2+\alpha}} &= \|(I-H^{-1}K)^{-1}H^{-1}G\|_{C_0^{2+\alpha}} \leq C\|H^{-1}\|_{C^\alpha\to C_0^{2+\alpha}}\|G\|_{C^\alpha}, \\ &\leq C\|G\|_{C^\alpha}. \end{aligned}$$

This establishes the lemma.

LEMMA 17 (Ito [9]). *There exists a constant* ε_0 *such that*

$$\|L_2(0, \varepsilon)\|_{C_0^{2+\alpha}(\bar{\Omega})\to C_\varepsilon^\alpha(\bar{\Omega})} \leq C\varepsilon, \tag{6.9}$$

$$\|L_3(0, \varepsilon)\|_{C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})\to C^\alpha(\bar{\Omega})} \leq C\varepsilon^{-\alpha}, \tag{6.10}$$

for $0<\varepsilon<\varepsilon_0$.

PROOF. For any $v \in C_0^{2+\alpha}(\bar{\Omega})$,

$$\|L_2(0, \varepsilon)v\|_{C_\varepsilon^\alpha} \leq C(\varepsilon\|v\|_{C_0^{2+\alpha}} + \|\Phi_0 v\|_{C_\varepsilon^\alpha}).$$

Making use of the fact that $v(x)=0$ at $x \in \Gamma$, we obtain

$$|\Phi_0 v|_0 \leq C\varepsilon \|v\|_{C_0^{2+\alpha}} \quad \text{and} \quad |\Phi_0 v|_1 \leq C\|v\|_{C_0^{2+\alpha}}.$$

Applying the interpolation inequality,

$$|\Phi_0 v|_\alpha \leq \varepsilon^{1-\alpha}|\Phi_0 v|_1 + C\varepsilon^{-\alpha}|\Phi_0 v|_0 \leq C\varepsilon^{1-\alpha}\|v\|_{C_0^{2+\alpha}}.$$

Thus we have

$$\|\Phi_0 v\|_{C_\varepsilon^\alpha} \leq C\varepsilon\|v\|_{C_0^{2+\alpha}},$$

which shows (6.9). Next for any $v \in C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})$,

$$\begin{aligned}
\|L_3(0, \varepsilon)v\|_{C^\alpha} &= \|g_u(U_0^m, V_0^m)v\|_{C^\alpha} \\
&\leq |g_u(U_0^m, V_0^m)|_0|v|_0 + |g_u(U_0^m, V_0^m)|_0|v|_\alpha + |g_u(U_0^m, V_0^m)|_\alpha|v|_0 \\
&\leq (\varepsilon^{-\alpha}|g_u(U_0^m, V_0^m)|_0 + |g_u(U_0^m, V_0^m)|_\alpha)\|v\|_{C_{\varepsilon,0}^{2+\alpha}} \\
&\leq C\varepsilon^{-\alpha}\|v\|_{C_{\varepsilon,0}^{2+\alpha}}.
\end{aligned}$$

This implies (6.10).

By Lemmas 15–17, we have the following

LEMMA 18 (Hosono et al. [8]). *Assume* (ii) *is satisfied. Then there exists a constant* ε_0 *such that* $T_t(0, \varepsilon)$ *has a uniformly bounded inverse for* $0<\varepsilon<\varepsilon_0$.

PROOF. We consider

$$T_t(0, \varepsilon)t = F \tag{6.11}$$

for $F=(f_r, f_s) \in Y_\varepsilon$. By (6.3), Lemmas 15 and 16, the operator equation (6.11) is reduced to

$$\begin{cases} r = M_\varepsilon^{-1}(-L_2(0, \varepsilon)s + f_r), \\ s = N_\varepsilon^{-1}(-L_3(0, \varepsilon)r + f_s), \end{cases}$$

so that

$$s = N_\varepsilon^{-1}L_3(0, \varepsilon)M_\varepsilon^{-1}L_2(0, \varepsilon)s - N_\varepsilon^{-1}L_3(0, \varepsilon)M_\varepsilon^{-1}f_r + N_\varepsilon^{-1}f_s.$$

Thus from Lemma 17, we may choose ε_0 such that

$$\|N_\varepsilon^{-1}\|_{C^\alpha \to C_0^{2+\alpha}}\|L_3(0, \varepsilon)\|_{C_{\varepsilon,0}^{2+\alpha} \to C^\alpha}\ \|M_\varepsilon^{-1}\|_{C_\varepsilon^\alpha \to C_{\varepsilon,0}^{2+\alpha}}\|L_2(0, \varepsilon)\|_{C_0^{2+\alpha} \to C_\varepsilon^\alpha} < 1$$

for $0<\varepsilon<\varepsilon_0$, and so we can find a unique solution $t \in X_\varepsilon$, which satisfies

$$\|r\|_{C_{\varepsilon,0}^{2+\alpha}} \leq C(\|f_r\|_{C_\varepsilon^\alpha} + \|f_s\|_{C^\alpha}) \tag{6.12}$$

and

(6.13) $$\|s\|_{C_0^{2+\alpha}} \leq C(\|f_r\|_{C_\varepsilon^\alpha} + \|f_s\|_{C^\alpha}).$$

Therefore from (6.12) and (6.13), we obtain

$$\|(T_t(0, \varepsilon))^{-1}\|_{Y_\varepsilon \to X_\varepsilon} \leq C$$

for $0<\varepsilon<\varepsilon_0$. This completes the proof of the lemma.

LEMMA 19.

$$\|T(0, \varepsilon)\|_{Y_\varepsilon} \leq C\varepsilon^{1-\alpha}.$$

PROOF. This follows immediately from Theorem 14.

LEMMA 20 (Fife [7]). *Let X_ε and Y_ε be Banach spaces and $T(t, \varepsilon)$ be a continuously differentiable mapping from X_ε into Y_ε, defined in $0<\varepsilon<\varepsilon_0$. Assume that T satisfies*:

(1) *for any t_1 and $t_2 \in X_\varepsilon$,*

$$\|T_t(t_1, \varepsilon) - T_t(t_2, \varepsilon)\|_{X_\varepsilon \to Y_\varepsilon} \leq C\|t_1 - t_2\|_{X_\varepsilon}$$

where T_t is the Fréchet derivative of T with respect to t;

(2) *$T_t(0, \varepsilon)$ has a uniformly bounded inverse for $0<\varepsilon<\varepsilon_0$;*

(3) $$\|T(0, \varepsilon)\|_{Y_\varepsilon} \leq C\varepsilon^{1-\alpha}.$$

Then there exists a unique function $t(\varepsilon) \in X_\varepsilon$ which is defined for $0<\varepsilon<\varepsilon_0$, satisfying

$$T(t(\varepsilon), \varepsilon) = 0 \quad \textit{and} \quad \|t(\varepsilon)\|_{X_\varepsilon} = O(\varepsilon^{1-\alpha}) \quad \textit{as} \quad \varepsilon \longrightarrow 0.$$

Now we apply Lemma 20 to our operator equation (6.2). Then (1) is obvious, while (2) and (3) follow from Lemmas 18 and 19, respectively. Therefore we have the following

THEOREM 21. *Under the assumptions* (i), (ii), (iii) *and* (iv), *there exists a constant ε_0 such that for $0<\varepsilon<\varepsilon_0$,* (2.1) *and* (2.2) *have a solution* $(u(x, \varepsilon), v(x, \varepsilon))$ *satisfying*

$$\|u(x, \varepsilon) - U_0^m(x, \varepsilon)\|_{C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})} \leq C\varepsilon^{m+1-\alpha},$$

$$\|v(x, \varepsilon) - V_0^m(x, \varepsilon)\|_{C_0^{2+\alpha}(\bar{\Omega})} \leq C\varepsilon^{m+1-\alpha}.$$

COROLLARY.

$\lim_{\varepsilon \to 0} u(x, \varepsilon) = h(v_0(x))$ *uniformly in each closed set in Ω,*

$\lim_{\varepsilon \to 0} v(x, \varepsilon) = v_0(x)$ *uniformly in $\bar{\Omega}$,*

$$\lim_{\varepsilon\to 0} \varepsilon \frac{\partial u}{\partial \nu}(x, \varepsilon) = \sigma\left(2\int_{h(\beta_0(x))}^{\alpha_0(x)} f(s, \beta_0(x))ds\right)^{1/2} \textit{ uniformly in } \Gamma,$$

$$\lim_{\varepsilon\to 0} \frac{\partial v}{\partial \nu}(x, \varepsilon) = \frac{\partial v_0}{\partial \nu}(x) \textit{ uniformly in } \Gamma,$$

where $\sigma = \mathrm{sign}\,(h(\beta_0(x)) - \alpha_0(x))$.

7. Concluding remarks

We have considered a pair of second-order partial differential equations (2.1) and (2.2) and have shown the existence of asymptotic solutions. First we note here that uniqueness is certainly not to be expected. In fact, many different asymptotic solutions for (2.1) and (2.2) may be possible when $f(u, v) = 0$ or (2.3) has many solutions. Secondly, though we have assumed Γ is connected for simplicity and clarity, we can also treat similarly when Γ are the sum of connected C^∞ hypersurfaces. Finally, we remark that our assertions can be easily generalized to any uniformly second-order elliptic equations in divergence form

$$\left\{\begin{array}{l} \varepsilon^2 \sum_{i,j=1}^N \partial_i(a_{ij}(x, \varepsilon)\partial_j u) - f(u, v) = 0 \\ \sum_{i,j=1}^N \partial_i(b_{ij}(x, \varepsilon)\partial_j v) - g(u, v) = 0 \end{array}\right., \quad x \in \Omega$$

with the same boundary conditions (2.2).

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Appendix A. Proof of Proposition 3

For any positive number d, certainly F is a C^∞ map from $[-d, d] \times \Gamma$ onto $\bar{\mathscr{V}}_d$. Moreover for each $s \in \Gamma$, its Jacobian matrix (with respect to local coordinates on $[-d, d] \times \Gamma$) at $(0, s)$ is nonsingular since $\nu(s)$ is normal to Γ. By the compactness of Γ, there exists a positive number d_0 such that the Jacobian matrix is nonsingular in $[-d_0, d_0] \times \Gamma$. We show that, for some positive number d $(\leq d_0)$, F is injective in $[-d, d] \times \Gamma$. Then by the inverse mapping theorem, we shall obtain the assertion. Suppose, on the contrary, that for any d such that $0 < d \leq d_0$, F is not injective in $[-d, d] \times \Gamma$. Then we may choose a positive integer N, and for all $n \geq N$, two distinct points (t_n, s_n) and (t'_n, s'_n) of $[-1/n, 1/n] \times \Gamma$ such that $F(t_n, s_n) = F(t'_n, s'_n)$. Since Γ is compact, there exist subsequences $\{s_{n(k)}\}$ and $(s'_{n(k)}\}$ which converge to some points s and s' of Γ. On the other hand, both $\{t_{n(k)}\}$ and $\{t'_{n(k)}\}$ converge to zero. By the continuity of F we obtain

$F(0, s)=F(0, s')$, so that $s=s'$. By the inverse mapping theorem, there exists a neighborhood $\mathscr{U}$ of $(0, s)$ such that F is injective in $\mathscr{U}$. Thus $F(t_{n(k)}, s_{n(k)})=F(t'_{n(k)}, s'_{n(k)})$ implies $(t_{n(k)}, s_{n(k)})=(t'_{n(k)}, s'_{n(k)})$ for large k. This contradicts the choice of the points (t_n, s_n) and (t'_n, s'_n). With this, the proof is completed.

Appendix B

First we show that we can choose a constant ε_0 sufficiently small, such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a function $\chi(x, \varepsilon) \in C_\varepsilon^{2+\alpha}(\bar{\Omega})$ with the property that

$$0<\delta_1 \leq \chi \leq \delta_2, \tag{B.1}$$

$$M_\varepsilon^0 \chi \leq -\delta_3<0 \tag{B.2}$$

where δ_1, δ_2 and δ_3 are some constants independent of ε, and

$$M_\varepsilon^0=\varepsilon^2 \Delta-f_u(U_0+\Phi_0, V_0).$$

We put

$$\chi(x, \varepsilon)= \begin{cases}1+\zeta(x) \chi_0(t(x) / \varepsilon, s(x)), & x \in \bar{\Omega}_d, \\ 1, \quad x \in \bar{\Omega}-\Omega_d,\end{cases}$$

where $\chi_0(\eta, s)$ is a solution of the following problem

$$\begin{cases}\partial_\eta^2 \chi_0(\eta, s)-f_u(h(\hat{\beta}_0(0, s))+\phi_0(\eta, s), \hat{\beta}_0(0, s)) \chi_0(\eta, s) \\ \quad =f_u(h(\hat{\beta}_0(0, s))+\phi_0(\eta, s), \hat{\beta}_0(0, s))-f_u(h(\hat{\beta}_0(0, s)), \hat{\beta}_0(0, s)), \\ \chi_0(0, s)=W, \\ \chi_0(\infty, s)=0.\end{cases}$$

Observe that by Lemma 9, $\chi_0(\eta, s) \in \mathscr{E}$. The constant W can be chosen sufficiently large so that

$$\chi_0(\eta, s) \geq -1/2 \qquad \text{for} \quad (\eta, s) \in[0, \infty) \times \Gamma.$$

Then we have

$$M_\varepsilon^0 \chi=-f_u(h(v_0), v_0)+O(\varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

By (ii), if we choose a constant ε_0 sufficiently small, there exist constants δ_1, δ_2 and δ_3 such that for any $\varepsilon \in (0, \varepsilon_0)$ (B.1) and (B.2) are satisfied.

Next we shall prove the following lemma as in van Harten [11].

Lemma B. *Let $\varepsilon \in (0, \varepsilon_0)$. Then for each $F \in C_\varepsilon^\alpha(\bar{\Omega})$ there exists a unique solution $u(x, \varepsilon) \in C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})$ of*

(B.3) $$M_\varepsilon^0 u = F$$

which satisfies

(B.4) $$\|u\|_{C^0(\bar{\Omega})} \leq (\delta_2/\delta_3)\|F\|_{C^0(\bar{\Omega})},$$

where δ_2 and δ_3 are as in (B.1) *and* (B.2).

PROOF. With the aid of Theorem 12.7 in Agmon et al. [1], the existence of a solution of (B.3) follows from the uniqueness. Hence it suffices to show that (B.4) holds for any $u \in C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})$ satisfying $M_\varepsilon^0 u = F$. The function $w = u\chi^{-1} \in C_{\varepsilon,0}^{2+\alpha}(\bar{\Omega})$ satisfies

$$\bar{M}_\varepsilon^0 w = F$$

where

$$\bar{M}_\varepsilon^0 = \varepsilon^2 \chi \Delta + 2\varepsilon^2 \textstyle\sum_{i=1}^N \partial_i \chi \partial_i + M_\varepsilon^0 \chi.$$

Since $M_\varepsilon^0 \chi \leq -\delta_3 < 0$, applying the maximum principle for elliptic equations, we have

$$-\delta_3^{-1}\|F\|_{C^0(\bar{\Omega})} \leq w \leq \delta_3^{-1}\|F\|_{C^0(\bar{\Omega})}.$$

So

$$|u| \leq \delta_3^{-1}\chi\|F\|_{C^0(\bar{\Omega})}$$

from which (B.4) follows.

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