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Stability analysis for free boundary problems in ecology

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Abstract

This paper deals with a free boundary problem for reaction-diffusion equations, which was previously proposed by the authors. The main purpose is to estabilish the stability theory for stationary solutions for the free boundary problem with fairly general nonlinearity. Our argument is based on the notion of ω-limit set and the comparison principle.

§1. Introduction

In the previous paper [3] the authors have proposed the following free boundary problem for a pair of unknown functions $\{u(x, t), s(t)\}$:

$$(1.1) \qquad u_t = d_1 u_{xx} + uf(u), \qquad 0 < x < s(t), \qquad 0 < t < \infty,$$

$$(1.2) \qquad u_t = d_2 u_{xx} + ug(u), \qquad s(t) < x < 1, \qquad 0 < t < \infty,$$

$$(1.3) \qquad u(0, t) = m_1, \qquad 0 < t < \infty,$$

$$(1.4) \qquad u(1, t) = -m_2, \qquad 0 < t < \infty,$$

$$(1.5) \qquad u(s(t), t) = 0,$$

$$(1.6) \qquad \dot{s}(t) = -\mu_1 u_x(s(t)-0, t) + \mu_2 u_x(s(t)+0, t), \qquad 0 < t < \infty,$$

$$(1.7) \qquad u(x, 0) = \varphi(x), \qquad 0 < x < 1,$$

$$(1.8) \qquad s(0) = l,$$

where d_i, m_i and μ_i $(i=1, 2)$ are positive constants, f and g are locally Lipschitz continuous functions, φ and l are given initial data, $\dot{s}(t)$ denotes $ds(t)/dt$ and $u_x(s(t)-0, t)$ (resp. $u_x(s(t)+0, t)$) means the limit of $u_x(x, t)$ at $x=s(t)$ from the left (resp. right). Our problem (1.1)–(1.8), which is simply denoted by (P), stems from regional partition phenomena arising in ecology; we consider the situation where two species, that cannot coexist in the same region, are struggling to get their own halitats on the intermediate boundary $x=s(t)$. $u(x, t)$ (resp. $-u(x, t)$) means the population density in $0<x<s(t)$ (resp. $s(t)<x<1$). For more details, see [3].

In [3], we have discussed (P) under the following assumptions:

(A.1) $f(u)$ is locally Lipschitz continuous on $[0, \infty)$, non-increasing on $[0, 1]$ and satisfies
$f(u)>0$ on $[0, 1), f(1)=0$ and $f(u) \leqq 0$ on $(1, \infty)$.

(A.2) $g(u)$ is locally Lipschitz continuous on $(-\infty, 0]$, non-decreasing on $[-1, 0]$ and satisfies
$g(u)>0$ on $(-1, 0], g(-1)=0$ and $g(u) \leqq 0$ on $(-\infty, -1)$.

(A.3) $0<m_1 \leqq 1$ and $0<m_2 \leqq 1$.

(A.4) $0<l<1$.

(A.5) $\varphi \in H^1(0, 1)$ satisfies $\varphi(0)=m_1, \varphi(l)=0, \varphi(l)=-m_2$ and $(l-x)\varphi(x) \geqq 0$ for $x \in (0, 1)$.

A comparatively realistic growth rate of f satisfying (A.1) is of the Verhulst-Pearl logistic form $f(w)=r(1-w/k)$, where r is the instrinsic growth rate and K is some carrying capacity. However, if the species is attacked by a predator, then the growth rate f is slightly deformed. A realistic expression of $f(u)$ is, due to Holling [2], of the form

$$f(w)=r(1-w/k)-P/(w+D).$$

Here P is a constant attack capacity of the predator and D refers to some given value of the prey population beyond which the predator's attack capability begins to saturate. When $D^2/P<K/r$, it turns out that f loses the monotonicity and, instead of it, has a maximum or "hump".

This ecological interest motivates us to study (P) with (A.1) and (A.2) replaced by the following general (in an ecological sense) assumptions:

(A.1)* f is locally Lipschitz continuous on $[0, \infty)$ and satisfies $f(1)=0$ and $f(u) \leqq 0$ on $(1, \infty)$.

(A.2)* g is locally Lipschitz continuous on $(-\infty, 0]$ and satisfies $g(-1)=0$ and $g(u) \leqq 0$ on $(-\infty, -1)$.

Even if (A.1) and (A.2) are replaced by (A.1)* and (A.2)*, as was stated in [3; Remarks 6.1 and 7.2], global existence of smooth solutions for (P) ([3, Theorem I]) and structure of the ω-limit set associated with each solution ([3, Theorem II]) remain true. Roughly speaking, these results imply that (P) has a unique smooth solution $\{u(\cdot, t), s(t)\}$, which converges (in a suitable topology) as $t \rightarrow \infty$ to one of $\{u^*, s^*\}$ satisfying

$$\text{(SP)} \quad \begin{cases} d_1 u_{xx}^* + u^* f(u^*) = 0, \quad u^* \geqq 0, \quad 0<x<s^*, \\ d_2 u_{xx}^* + u^* g(u^*) = 0, \quad u^* \leqq 0, \quad s^*<x<1, \\ u^*(0)=m_1, \quad u^*(s^*)=0, \quad u^*(1)=-m_2, \\ -\mu_1 u_x^*(s^*-0) + \mu_2 u_x^*(s^*+0) = 0. \end{cases}$$

This problem (SP) is called the *stationary problem* associated with (P) and any pair $\{u^*, s^*\}$ satisfying (SP) is called a *stationary solution* of (P). In the present paper, we will concentrate our efforts on the analysis of (SP) to get information about stability properties of stationary solutions for (P). We would like to emphasize the number of the stationary solutions heavily depends on the nonlinearity of f and g (see Fig. 6 in §5).

The plan of this paper is as follows. In §2, we state some basic results on global existence of smooth solutions for (P), structure of the ω-limit set associated with the solution orbit and comparison principle. In §3, we study auxiliary problems related to the stationary problem (SP) and construct solutions by the phase plane analysis. Here we take f and g to be quadratic polynomials as representative functions satisfying (A.1)* and (A.2)*, so that our arguments will be made transparent and technical complexity will be avoided. §4 is devoted to the investigation of (SP). We seek all stationary solutions with the aid of solutions of the auxiliary problems studied in §3. In §5, stability theory is developed. We make use of the comparison principle, which enables us to decide stability or instability of each stationary solution.

Notation

We summerize some notation used throughout this paper. We set as follows:

$$I = (0, 1), \quad Q = I \times (0, \infty),$$

$$S^- = \{(x, t) \in Q; \quad 0 < x < s(t)\}, \ S^+ = \{(x, t) \in Q; \quad s(t) < x < 1\}.$$

For $\delta > 0$,

$$S_\delta^- = \{(x, t) \in S^-; \quad t \geqq \delta\}, \ S_\delta^+ = \{(x, t) \in S^+; \quad t \geqq \delta\}.$$

Let u_i $(i=1, 2)$ be continuous functions on I and let s_i $(i=1, 2)$ be numbers in I. We simply write

$$\{u_1, s_1\} \geqq \{u_2, s_2\} \quad \text{if} \quad u_1(x) \geqq u_2(x) \quad \text{for} \quad x \in I \quad \text{and} \quad s_1 \geqq s_2,$$

and

$$\{u_1, s_1\} > \{u_2, s_2\} \quad \text{if} \quad u_1(x) > u_2(x) \quad \text{for} \quad x \in I \quad \text{and} \quad s_1 > s_2.$$

§2. Preliminary results

In what follows, we always assume (A.1)*, (A.2)*, (A.3) and the initial data $\{\varphi, l\}$ so as to satisfy (A.4) and (A.5). We will show some results which can be proved almost in the same way as in [3]. The first one is concerned with the existence and uniqueness of solutions for (P).

THEOREM 2.1 ([3; Theorem I]). *There exists a unique pair of functions* $\{u, s\} \in C(\bar{Q}) \times C([0, \infty))$ *with the following properties*:

(i) $s(0) = l$, $\dot{s} \in L^3(0, \infty)$ *and* $b \leqq s(t) \leqq 1 - b$, $t \in [0, \infty)$, *with some constant* $b \in (0, 1)$.

(ii) $\{u, s\}$ *satisfies* (1.3), (1.4), (1.5) *and* (1.7) *everywhere and*

$$0 \leqq u(x, t) \leqq \max\{1, \sup_{0 \leqq x \leqq l} \varphi(x)\} \qquad \text{for} \quad (x, t) \in \overline{S^-},$$

$$0 \geqq u(x, t) \geqq \min\{-1, \inf_{l \leqq x \leqq 1} \varphi(x)\} \qquad \text{for} \quad (x, t) \in \overline{S^+}.$$

(iii) *Let* $u^+ = \max\{u, 0\}$ *and* $u^- = -\min\{u, 0\}$. *Then* $u^\pm \in C([0, \infty); H^1(I)) \cap L^\infty(0, \infty; H^1(I))$.

(iv) $u_t \in L^2(S^-) \cap L^2(S^+)$.

(v) $u_t, u_{xx} \in C(S^-) \cap C(S^+)$ *and* $\{u, s\}$ *satisfies* (1.1) *and* (1.2) *everywhere.*

(vi) *For each* $\delta > 0$, u_x *is Hölder continuous in* $(x, t) \in S_\delta^\pm$ *and* $\dot{s}$ *is Hölder continuous in* $t \in [\delta, \infty)$.

(vii) $\{u, s\}$ *satisfies* (1.6) *for every* $t \in (0, \infty)$.

The pair $\{u, s\}$ in Theorem 2.1 is called a *smooth solution* of (P) on $[0, \infty)$. We denote by $\{u(x, t, \varphi, l), s(t; \varphi, l)\}$ the smooth solution of (P) for the initial data $\{\varphi, l\}$. In order to study asymptotic behavior of smooth solutions for (P), it is convenient to introduce the notion of *ω-limit set* associated with the solution orbit $\{\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\}; t \geqq 0\}$:

(2.1) $\omega(\varphi, l) = \{\{u^*, s^*\} \in H^1(I) \times I$; *there exists a sequence* $\{t_n\} \uparrow \infty$ *such that*

$$s(t_n; \varphi, l) \to s^*, \quad \textit{and} \quad u(t_n; \varphi, l) \to u^* \ \textit{in} \ H^1(I) \ \textit{as} \ n \to \infty\}.$$

We say that the sequence $\{\{u(t_n; \varphi, l), s(t_n; \varphi, l)\}\}_{n=1}^\infty$ *converges to* $\{u^*, s^*\}$ *in Ω-topology* if the convergence property in (2.1) holds. (Note that the definition of $\omega(\varphi, l)$ by (2.1) is equivalent to that given in [3].)

Our second result reads as follows.

THEOREM 2.2 ([3, Theorem II]).

(i) $\omega(\varphi, l)$ *is non-empty and connected in Ω-topology.*

(ii) *If* $\{u^*, s^*\} \in \omega(\varphi, l)$, *then it satisfies* (SP).

This theorem gives us very useful information about asymptotic behavior of smooth solutions for (P). For example, if it is shown that the set of stationary solutions consists of isolated elements, then every solution of (P) converges in Ω-topology to one of stationary solutions as $t \to \infty$.

Finally we will give a comparison theorem which will be powerful in the study of stability properties of stationary solutions. Before stating the result, we prepare some terminology. Let $\mathscr{R}$ denote the set of all functions $\{u, s\} \in C(\bar{Q}) \times C(\bar{I})$ satisfying

(i) $u_x \in C(\overline{S_\delta^-}) \cap C(\overline{S_\delta^+})$ for any $\delta > 0$,

(ii) $u_t, u_{xx} \in C(S^-) \cap C(S^+)$,

(iii) $s \in C^1((0, \infty))$.

DEFINITION 2.1. A pair of functions $\{u, s\} \in \mathscr{R}$ is called a *subsolution* of (P) for the initial data $\{\varphi, l\}$ if it satisfies

(i) $u_t \leqq d_1 u_{xx} + uf(u)$ in S^-,

(ii) $u_t \leqq d_2 u_{xx} + ug(u)$ in S^+,

(iii) $u(0, t) \leqq m_1$ in $(0, \infty)$,

(iv) $u(1, t) \leqq -m_2$ in $(0, \infty)$,

(v) $u(s(t), t) = 0$ in $(0, \infty)$,

(vi) $\dot{s}(t) \leqq -\mu_1 u_x(s(t)-0, t) + \mu_2 u_x(s(t)+0, t)$ in $(0, \infty)$,

(vii) $u(x, 0) = \varphi(x)$ in I,

(viii) $s(0) = l$.

A *supersolution* of (P) for the initial data $\{\varphi, l\}$ is defined by reversing the inequality signs in (i), (ii), (iii), (iv) and (vi). If $\{u, s\}$ is a super- and subsolution of (P), it is called a *classical solution* of (P).

Then our comparison theorem is

THEOREM 2.3 ([3, Theorems 5.1 and 6.3]).
(i) *Assume that* $\{u^1, s^1\}$ *(resp.* $\{u^2, s^2\}$*) is a supersolution (resp. subsolution) of* (P) *for the initial data* $\{\varphi^1, l^1\}$ *(resp.* $\{\varphi^2, l^2\}$*). If* $\varphi^1 \geqq \varphi^2$ *in* I *and* $l^1 > l^2$, *then*

$$u^1(x, t) \geqq u^2(x, t) \quad \textit{for} \quad (x, t) \in \bar{Q}$$

and

$$s^1(t) > s^2(t) \quad \textit{for} \quad t \in [0, \infty).$$

(ii) *In addition to the assumptions of* (i), *assume that one of* $\{u^i, s^i\}$ $(i=1, 2)$ *is a classical solution of* (P). *Then*

$$\{u^1(\cdot, t), s^1(t)\} \geqq \{u^2(\cdot, t), s^2(t)\} \quad \textit{for all} \quad t \in [0, \infty),$$

whenever $\{\varphi^i, l^i\}$ *satisfies* $\{\varphi^1, l^1\} \geqq \{\varphi^2, l^2\}$. *Moreover, if* $\varphi^1 \not\equiv \varphi^2$, *then*

$$\{u^1(\cdot, t), s^1(t)\} > \{u^2(\cdot, t), s^2(t)\} \qquad \textit{for all} \quad t \in (0, \infty).$$

§ 3. Analysis of auxiliary problems

In this section we will consider the following auxiliary problems $(\mathrm{AP})_\xi$ before studying (SP):

$$(\mathrm{AP})_\xi \quad \begin{cases} (3.1) & d_1 u_{xx} + uf(u) = 0, \qquad 0 < x < \xi, \\ (3.2) & u \geqq 0, \qquad 0 < x < \xi, \\ (3.3) & d_2 u_{xx} + ug(u) = 0, \qquad \xi < x < 1, \\ (3.4) & u \leqq 0, \qquad \xi < x < 1, \\ (3.5) & u(0) = m_1, \\ (3.6) & u(\xi) = 0, \\ (3.7) & u(1) = -m_2, \end{cases}$$

where ξ is regarded as a parameter moving over I. For the sake of simplicity, we sometimes denote (AP) in place of $(\mathrm{AP})_\xi$.

Let $u(x; \xi)$ be a solution of $(\mathrm{AP})_\xi$. Our strategy for solving (SP) is to seek a pair $\{u(\cdot\,; \xi), \xi\}$ satisfying the last equation of (SP) with $\{u^*, s^*\}$ replaced by $\{u(\cdot\,; \xi), \xi\}$. In order to avoid technical complexity in studying (AP), hereafter, we will specify f and g as quadratic polynomials of the form

$$(3.8) \qquad f(u) = -\nu_1(u-a)(u-1) \quad \text{with} \quad \nu_1 > 0 \quad \text{and} \quad a < 1,$$

$$(3.9) \quad g(u) = -\nu_2(u+b)(u+1) \quad \text{with} \quad \nu_2 > 0 \quad \text{and} \quad b < 1,$$

which cleary satisfy (A.1)* and (A.2)*, respectively. Our subsequent arguments will be valid for general f and g.

Observe that (AP) is composed of two boundary-value problems; that is, (AP) can be treated separately on $[0, \xi]$ (denoted by $(\mathrm{AP}\text{–}1)_\xi$, or simply, (AP–1)) and on $[\xi, 1]$ (denoted by $(\mathrm{AP}\text{–}2)_\xi$, or simply, (AP–2)). The general theory for nonlinear elliptic equations (see, e.g., Sattinger [4]) tells us that both (AP–1) and (AP–2) have a maximal solution and a minimal solution. However, we will employ the standard phase plane analysis to get preciser information about the solutions for (AP–1) and (AP–2). (The results of Smoller-Wasserman [6] and Smoller [5] are relevant here.)

First we will treat (AP–1). Rewriting (3.1) as a first-order system for (v, v')

(3.10) $$v_x = v' \quad \text{and} \quad v'_x = -vf(v)/d_1,$$

we consider the phase plane for (3.10). Define

(3.11) $$E_1(v, v') = (v')^2 + F(v),$$
$$\text{with} \quad F(v) \equiv \frac{2}{d_1}\int_0^v uf(u)du = -\frac{v_1 v^2}{6d_1}\{3v^2 - 4(a+1)v + 6a\}.$$

Since $E_1(v, v')$ is constant along orbits of (3.10), the phase portrait for (3.10) is depicted as in Fig. 1. Clearly, a solution $v(x; \xi)$ for $(\text{AP–1})_\xi$ corresponds to an orbit of (3.10) which "begins" on the line $v = m_1$, "ends" on the line $v = 0$ and take "time" ξ to make the journey with $v \geqq 0$.

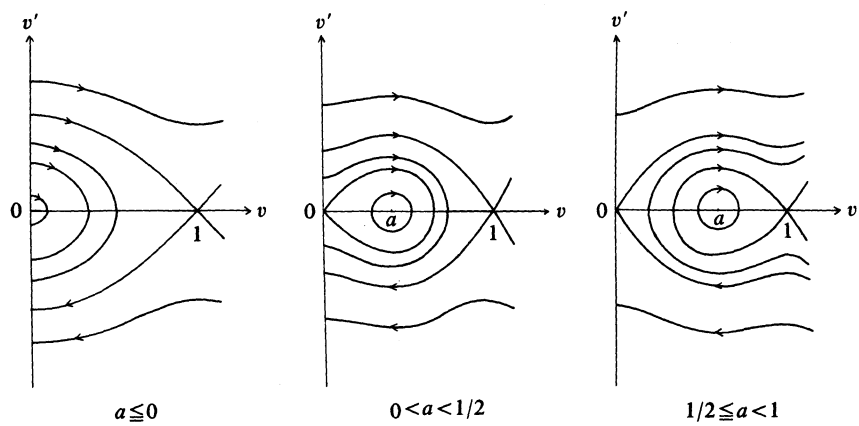


Fig. 1. Phase portraits for (v, v')

Our first result is concerned with an order relation for solutions of $(\text{AP–1})_\xi$ with any fixed $\xi \in I$.

LEMMA 3.1. *For each* $\xi \in (0, 1)$, *let* $v_i(x; \xi)$ $(i = 1, 2)$ *be two solutions of* $(\text{AP–1})_\xi$. *If*

(3.12) $$(v_1)_x(0; \xi) < (v_2)_x(0; \xi),$$

then

(3.13) $$v_1(x; \xi) < v_2(x; \xi) \qquad \textit{for all} \quad x \in (0, \xi).$$

PROOF. Suppose that (3.13) does not hold. Then there exists a point $x_0 \in (0, \xi)$ satisfying $v_1(x; \xi) < v_2(x; \xi)$ for $x \in (0, x_0)$ and $v_1(x_0; \xi) = v_2(x_0; \xi)$. Therefore,

(3.14) $$(v_2)_x(x_0; \xi) < (v_1)_x(x_0; \xi).$$

For the orbits of (3.10) corresponding to $v_i(x;\xi)$ $(i=1, 2)$ one can see from (3.12) and (3.14) that $(v_2)_x(0;\xi)>0$ and $(v_2)_x(x_0;\xi)<0$. Therefore, the orbit corresponding to $v_2(\cdot\,;\xi)$ must be below the orbit corresponding to $v_1(\cdot,\xi)$ for $0\leqq v\leqq v_2(x_0;\xi)$. This fact implies that

$$(3.15)\qquad (v_2)_x(x_1;\xi) < (v_1)_x(x_1;\xi)$$

at any $x_1\in[x_0, \xi]$ where $v_2(x_1;\xi)=v_1(x_1;\xi)$. Since (3.15) holds true for $x_1=x_0$ (by (3.14)) and $x_1=\xi$, there exists a point $x_2\in(x_0, \xi)$ such that $v_1(x_2;\xi)=v_2(x_2;\xi)$ and $(v_1)_x(x_2;\xi)<(v_2)_x(x_2;\xi)$, which contradicts (3.15). Thus the proof is complete. q. e. d.

By Lemma 3.1, the set of solutions for $(\mathrm{AP}\text{–}1)_\xi$ has an obvious order relation for each fixed $\xi\in I$. Clearly, the analogous result is valid for $(\mathrm{AP}\text{–}2)_\xi$.

We will construct solutions of (AP–1) by the shooting method. As in [5] or [6], let $T_1(p)$ denote the "time" that an orbit starting from (m_1, p) takes to arrive at the v'-axis. In other words, $T_1(p)$ possesses the property

$$u_1(x;p)>0 \quad \text{for} \quad x\in[0, T_1(p)) \quad \text{and} \quad u_1(T_1(p);p)=0,$$

where $u_1(x;p)$ is the solution of the initial value problem

$$(3.16)\qquad \begin{cases} d_1u_{xx}+uf(u)=0, & x>0,\\ u(0)=m_1, \quad u_x(0)=p. & \end{cases}$$

Since

$$E_1(u_1(x;p), u_{1,x}(x;p)) = p^2+F(m_1) \qquad \text{for} \quad x\geqq 0,$$

where E_1 is defined by (3.11), one can find that $T_1(p)$ is expressed as follows:

Case I. $a\leqq 0$ or $0<a<1/2$ with $a^*\leqq m_1$, where $a^*=\{2(a+1)-\sqrt{2(2a-1)(a-2)}\}/3$ (a^* is the number determined by $F(a^*)=0$ with $a^*\in(0, 1]$).

$$(3.17)\qquad T_1(p)=\int_0^{m_1}\{p^2+F(m_1)-F(v)\}^{-1/2}dv \qquad \text{for} \quad p<0,$$

and

$$(3.18)\quad T_1(p)=2\int_0^{\alpha(p)}\{p^2+F(m_1)-F(v)\}^{-1/2}dv-\int_0^{m_1}\{p^2+F(m_1)-F(v)\}^{-1/2}dv$$
$$\text{for} \quad 0\leqq p<A_1\equiv\{F(1)-F(m)\}^{1/2},$$

where $\alpha(p)$ is the first point on the line $v'=0$ which meets the orbit passing through $(m_1;p)$; that is

$$(3.19)\qquad p^2+F(m_1)=F(\alpha(p)) \quad \text{with} \quad \alpha(p)\in[m_1, 1).$$

Case II. $0<a<1/2$ and $m_1<a^*<1$. $T_1(p)$ is given by (3.17) for $p<-A_2\equiv -\{-F(m_1)\}^{1/2}$ and by (3.18) for $A_2<p<A_1$.

Case III. $1/2\leqq a<1$. $T_1(p)$ is given by (3.17) only for $p<-A_2$.

In view of the expression (3.17), it is easily seen that $T_1(p)$ is continuous and monotone increasing for $p<0$ lying in the domain of T_1 and that

$$\lim_{p\downarrow-\infty} T_1(p)=0, \tag{3.20}$$

$$\lim_{p\uparrow-A_2} T_1(p)=\infty \qquad \text{for Cases II and III.}$$

For $a\leqq-1$, f satisfies (A.1); so that one can find from the result of [3] (see the proof of Lemma 8.1) that $T_1(p)$ is also monotone increasing on $[0, A_1)$. For $a>-1$, we recall the result of Smoller and Wasserman [6, Theorems 2.1 and 2.2] which says that the integral $\int_0^\alpha \{F(\alpha)-F(v)\}^{-1/2}\,dv$ has exactly one critical point (a minimum) as a function of $\alpha\in(0, 1)$. Since $\alpha(p)$ is a monotone increasing function of $p\geqq0$ (use (3.19)), the above result assures that $T_1(p)$ is monotone increasing on $[A_0, A_1)$ with some A_0. Moreover,

$$\lim_{p\uparrow A_1} T_1(p)=\infty \qquad \text{for Cases I and II} \tag{3.21}$$

and

$$\lim_{p\downarrow A_2} T_1(p)=\infty \qquad \text{for Case II.} \tag{3.22}$$

By virtue of (3.20), (3.21) and (3.22), typical graphs of $T_1(p)$ can be depicted as in Fig. 2.

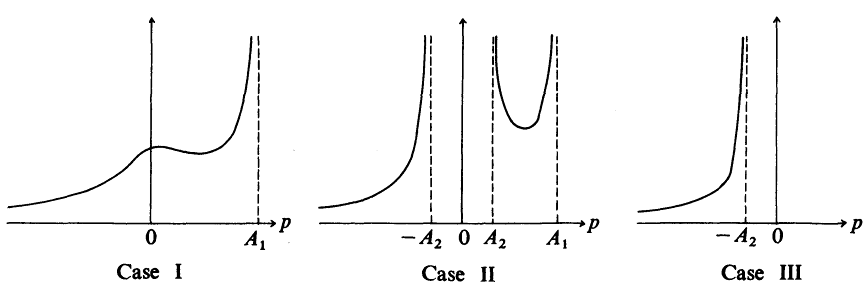


Fig. 2. Graphs of time mapping T_1

REMARK 3.1. In our case ($m_1>0$), it seems considerably difficult to know the exact number of critical points of T_1 as in [6]. The qualitative shape of T_1 heavily depends upon m_1 and a.

Here we will give some properties of T_1 for later use.

LEMMA 3.2. *Let $u_1(x; p)$ be the solution of* (3.16) *and let $p_1 < p_2$ be in the domain of T_1.*

(i) *If $T_1(p_1) < T_1(p_2)$, then $u_1(x; p_1) < u_1(x; p_2)$ for $x \in (0, T_1(p_1))$.*

(ii) *If $T_1(p_1) > T_1(p_2)$, then there exists a point $\xi_0 \in (0, T_1(p_2))$ such that*

$$u_1(x; p_1) < u_1(x; p_2) \quad \text{for} \quad x \in (0, \xi_0),$$

$$u_1(x; p_1) > u_1(x; p_2) \quad \text{for} \quad x \in (\xi_0, T_1(p_2)).$$

PROOF. For the orbits of (3.10) corresponding to $u(x; p_i)$ $(i=1, 2)$, we have only to use arguments similar to those in the proof of Lemma 3.1. q. e. d.

We are ready to state some results for (AP–1).

PROPOSITION 3.3. *Let $\xi \in I$ be fixed. Then*:

(i) (AP–1) *has at least one solution.*

(ii) *The number of solutions for* (AP–1) *is identical with the number of p's satisfying $T_1(p) = \xi$.*

(iii) *Let $K = K(\xi)$ be the number of solutions for* (AP–1) *and let $p_1(\xi) < p_2(\xi) < \cdots < p_K(\xi)$ satisfy $T_1(p_i(\xi)) = \xi$ $(i = 1, 2,\dots, K)$. Then all solutions of* (AP–1) *are given by*

$$v_i(x; \xi) = u_1(x; p_i(\xi)), \quad i = 1, 2,\dots, K, \quad \text{for} \quad x \in [0, \xi], \tag{3.23}$$

where $u_1(\cdot\,; p)$ is the solution of (3.16). *Moreover,*

$$0 < v_1(x; \xi) < v_2(x; \xi) < \cdots < v_K(x; \xi) \quad \text{for} \quad x \in [0, \xi]. \tag{3.24}$$

PROOF. We observe that solutions of (AP–1) correspond to orbits of (3.10) which pass through (m_1, p) and satisfies $T_1(p) = \xi$. In view of (3.20), (3.21) and (3.22), there exists at least one p such that $T_1(p) = \xi$ for every $\xi \in I$ (see Fig. 2). This fact imples (i). Moreover, it is easy to see that (ii) and (3.23) hold. Finally, (3.24) is derived from Lemma 3.1. q. e. d.

REMARK 3.2. From the definition, $T_1(p)$ is an analytic function of p so that $T_1(p) = \xi$ has only a finite number of solutions for each $\xi \in I$. Hence the number of solutions of $(\text{AP–1})_\xi$ is finite for any fixed $\xi \in I$.

REMARK 3.3. Consider the situation where $T_1(p)$ is a monotone increasing function of p. In fact, this situation takes place in the case $1/2 \leqq a < 1$ or $a \leqq -1$. (In the latter case, $f(u)$ satisfies (A.1) and the complete analysis has been carried out in [3].) Since a solution $p(\xi)$ of $T_1(p) = \xi$ is uniquely determined for every $\xi \in I$, $(\text{AP–1})_\xi$ has a unique solution $v(x; \xi) = u_1(x; p(\xi))$.

We can deal with (AP–2) along the same idea as with (AP–1). Define $G(u)$ by

$$G(u) = \frac{2}{d_2}\int_0^u vg(-v)dv.$$

Let $u_2(x; q)$ be the solution of the initial value problem

$$(3.25) \qquad \begin{cases} d_2 u_{xx} + ug(-u) = 0, & x > 0, \\ u(0) = m_2, \quad u_x(0) = q. \end{cases}$$

If we set

$$E_2(w, w') = (w')^2 + G(w),$$

then

$$E_2(u_2(x; q), (u_2)_x(x; q)) = q^2 + G(m_2) \qquad \text{for} \quad x \geqq 0.$$

Therefore, the time mapping $T_2(q)$ for $u_2(\cdot\,; q)$ is defined by (3.17) and (3.18) with F, p, m_1, and $\alpha(p)$ replaced by G, q, m_2 and $\beta(q)$, where $\beta(q) \in [m_2, 1]$ satisfies $q^2 + G(m_2) = G(\beta(q))$. Then $T_2(q)$ has qualitative properties similar to $T_1(p)$.

For (AP–2), the results corresponding to Proposition 3.3 can be stated as follows.

PROPOSITION 3.4. *Let $\xi \in I$ be fixed.*

(i) (AP–2) *has at least one solution.*

(ii) *The number of solutions for* (AP–2) *is identical with the number of q's satisfying $T_2(q) = 1 - \xi$.*

(iii) *Let $L = L(\xi)$ be the number of solutions for* (AP–2) *and let $q_1(\xi) < q_2(\xi) < \cdots < q_L(\xi)$ satisfy $T_2(q_j(\xi)) = 1 - \xi$, $j = 1, 2, \ldots, L$. Then all solutions of* (AP–2) *are given by*

$$(3.26) \quad w_j(x; \xi) = -u_2(1-x; q_j(\xi)), \quad j = 1, 2, \ldots, L \qquad \textit{for} \quad x \in [\xi, 1],$$

where $u_2(\cdot\,; q)$ is the solution of (3.25). *Moreover,*

$$(3.27) \qquad w_L(x; \xi) < \cdots < w_2(x; \xi) < w_1(x; \xi) \qquad \textit{for} \quad x \in (\xi, 1].$$

Here it is convenient to introduce the following classification. For $\xi \in I$, let $p(\xi)$ be any solution of $T_1(p) = \xi$. If $T_1(p)$ is monotone increasing (resp. decreasing) in a neighborhood of $p = p(\xi)$, we say that $p(\xi)$ is in the *stable class* $\mathscr{S}\mathscr{C}_1$ (resp. *unstable class* $\mathscr{U}\mathscr{C}_1$). When $p(\xi)$ is continuous for ξ in an open interval $I_0 \subset I$ and $p(\xi) \in \mathscr{S}\mathscr{C}_1$ (resp. $\mathscr{U}\mathscr{C}_1$) for every $\xi \in I_0$, it is monotone increasing (resp. decreasing) on I_0. If $T_1(p)$ is not monotone in any neighborhood of $p = p(\xi)$ (necessarily, $dT_1/dp(p(\xi)) = 0$), we say that $p(\xi)$ is in the *critical class* $\mathscr{C}\mathscr{C}_1$. Similarly, any solution $q(\xi)$ of $T_2(q) = 1 - \xi(\xi \in I)$ belongs to the stable class $\mathscr{S}\mathscr{C}_2$ (resp. unstable class $\mathscr{U}\mathscr{C}_2$) if $T_2(q)$ is monotone increasing (resp.

decreasing) in a neighborhood of $q=q(\xi)$ and $q(\xi)$ belongs to the critical class $\mathscr{C}\mathscr{C}_2$ if it belongs to neither $\mathscr{S}\mathscr{C}_2$ nor $\mathscr{U}\mathscr{C}_2$.

For $p(\xi)$ (resp. $q(\xi)$), the corresponding solution of $(\text{AP}-1)_\xi$ (resp. $(\text{AP}-2)_\xi$) is given by $v(x; \xi)=u_1(x; p(\xi))$ (resp. $w(x; \xi)=-u_2(1-x; q(\xi))$) (see Propositions 3.3 and 3.4). If we set

$$u(x; \xi) = \begin{cases} v(x; \xi), & 0 \leqq x \leqq \xi, \\ w(x; \xi), & \xi \leqq x \leqq 1, \end{cases} \tag{3.28}$$

then $u(\cdot\,; \xi)$ becomes a solution of $(\text{AP})_\xi$. Conversely, any solution $u(\cdot, \xi)$ of $(\text{AP})_\xi$ is expressed in the form (3.28), where $v(\cdot, \xi)$ (resp. $w(\cdot, \xi)$) is a solution of $(\text{AP}-1)_\xi$ (resp. $(\text{AP}-2)_\xi$).

We will give the following results for $(\text{AP})_\xi$ when ξ varies over I.

PROPOSITION 3.5. *For each $\xi \in I$, let $p(\xi)$ and $q(\xi)$ be any solutions of $T_1(p)=\xi$ and $T_2(q)=1-\xi$, respectively. Define $u(\cdot\,; \xi)$ by* (3.28) *with $u(x; \xi)= u_1(x; p(\xi))$ and $w(x; \xi)=-u_2(1-x; q(\xi))$. Suppose that both $p(\xi)$ and $q(\xi)$ are continuous in an open interval $I_0 \subset I$.*

(i) *If $p(\xi) \in \mathscr{S}\mathscr{C}_1$ and $q(\xi) \in \mathscr{S}\mathscr{C}_2$ for $\xi \in I_0$, then $u(x; \xi_1)<u(x; \xi_2)$, $x \in I$ for any $\xi_i \in I_0$ with $\xi_1<\xi_2$.*

(ii) *If $p(\xi) \in \mathscr{S}\mathscr{C}_1$ and $q(\xi) \in \mathscr{U}\mathscr{C}_2$ for $\xi \in I_0$, then for any $\xi_i \in I_0$ with $\xi_1<\xi_2$ there eixts a point $x_0 \in (0, 1)$ such that*

$$\begin{aligned} u(x; \xi_1) &< u(x; \xi_2) \quad \textit{for} \quad x \in (0, x_0), \\ u(x; \xi_1) &> u(x; \xi_2) \quad \textit{for} \quad x \in (x_0, 1). \end{aligned} \tag{3.29}$$

(iii) *If $p(\xi) \in \mathscr{U}\mathscr{C}_1$ and $q(\xi) \in \mathscr{S}\mathscr{C}_2$ for $\xi \in I_0$, then the assertion of* (ii) *holds true by reversing the inequality signs in* (3.29).

(iv) *If $p(\xi) \in \mathscr{U}\mathscr{C}_1$ and $q(\xi) \in \mathscr{U}\mathscr{C}_2$ for $\xi \in I_0$, then for any $\xi_i \in I_0$ with $\xi_1<\xi_2$ there exist two points $0<x_0<x_1<1$ such that*

$$\begin{aligned} u(x; \xi_1) &> u(x; \xi_2) \quad \textit{for} \quad x \in (0, x_0) \cup (x_1, 1), \\ u(x; \xi_1) &< u(x; \xi_2) \quad \textit{for} \quad x \in (x_0, x_1). \end{aligned}$$

PROOF. Let $\xi_1<\xi_2$ be in I_0. If $p(\xi) \in \mathscr{S}\mathscr{C}_1$ (resp. $\mathscr{U}\mathscr{C}_1$) for $\xi \in I_0$, then

$$p(\xi_1) < p(\xi_2) \text{ (resp. } p(\xi_1) > p(\xi_2)).$$

The definition of T_1 yields $T_1(p(\xi_i))=\xi_i$ $(i=1, 2)$. Therefore, it follows from Lemma 3.2 that, if $p(\xi) \in \mathscr{S}\mathscr{C}_1$ for $\xi \in I_0$, then $u(x; \xi_1)<u(x; \xi_2)$ for $x \in (0, \xi_1)$ and, if $p(\xi) \in \mathscr{U}\mathscr{C}_1$ for $\xi \in I_0$, then $u(x; \xi_1)>u(x; \xi_2)$ for $x \in (0, \xi_0)$, $u(x; \xi_1)< u(x; \xi_2)$ for $x \in (\xi_0, \xi_1)$ with some $\xi_0 \in (0, \xi_1)$. The analogous results hold for (AP–2) so that the assertins of this theorem are easily derived. q.e.d.

§4. Analysis of stationary problems

For each $\xi \in (0, 1)$, let $p_1(\xi) < p_2(\xi) < \cdots < p_K(\xi)$ be K solutions of $T_1(p) = \xi$ and let $q_1(\xi) < q_2(\xi) < \cdots < q_L(\xi)$ be L solutions of $T_2(q) = 1 - \xi$. By Propositions 3.3 and 3.4, $(\mathrm{AP}\text{–}1)_\xi$ has K solutions $v_i(x; \xi)$, $i = 1, 2, \ldots, K$ given by (3.23) and $(\mathrm{AP}\text{–}2)_\xi$ has L solutions $w_j(x; \xi)$, $j = 1, 2, \ldots, L$ given by (3.26). Our task for solving (SP) is to look for a family $\{v_i(x; \xi), w_j(x; \xi), \xi\}$ such that

$$-\mu_1(v_i)_x(\xi - 0; \xi) + \mu_2(w_j)_x(\xi + 0; \xi) = 0. \tag{4.1}$$

We observe that $\Phi_i(\xi) \equiv -\mu_1(v_i)_x(\xi - 0; \xi)$ $(i = 1, 2, \ldots, K)$ are expressed, in terms of $p_i(\xi)$, as

$$\Phi_i(\xi) = \mu_1\{p_i(\xi)^2 + F(m_1)\}^{1/2}, \tag{4.2}$$

(use the invariance of $E_1(v_i(x; \xi), (v_i)_x(x; \xi))$ with respect to x). Moreover, since v_i $(i = 1, 2, \ldots, K)$ satisfy (3.24), it is easily seen that

$$\Phi_1(\xi) < \Phi_2(\xi) < \cdots < \Phi_K(\xi) \qquad \text{for each} \quad \xi \in (0, 1). \tag{4.3}$$

We define the following set in the (ξ, η)-plane

$$C_1 \equiv \{\{\xi, \Phi_i(\xi)\}; 0 < \xi \leqq 1, \quad i = 1, 2, \ldots, K(\xi)\}.$$

Clearly, C_1 consists of a finite number of continuous curves.

Analogously, $\Psi_j(\xi) \equiv -\mu_2(w_j)_x(\xi + 0; \xi)$ $(j = 1, 2, \ldots, L)$ are expressed as

$$\Psi_j(\xi) = \mu_2\{q_j(\xi)^2 + G(m_2)\}^{1/2}, \tag{4.4}$$

which satisfy the following order relation

$$\Psi_1(\xi) < \Psi_2(\xi) < \cdots < \Psi_L(\xi) \quad \text{for each} \quad \xi \in (0, 1), \tag{4.5}$$

(use (3.27)). We define a set C_2 by

$$C_2 \equiv \{\{\xi, \Psi_j(\xi)\}; 0 \leqq \xi < 1, \quad j = 1, 2, \ldots, L(\xi)\}.$$

In view of (4.1), all solutions of (SP) are obtained by finding points where C_1 and C_2 intersect. Therefore, it is required to study the qualitative shapes of C_1 and C_2.

As in §3, we say that $\Phi_i(\xi)$ belongs to the *stable class* $\mathscr{S}_1$ (resp. *unstable class* $\mathscr{U}_1$ or *critical class* $\mathscr{C}_1$) if $p_i(\xi)$ *corresponding to* $\Phi_i(\xi)$ *belongs to* $\mathscr{S}\mathscr{C}_1$ (resp. $\mathscr{U}\mathscr{C}_1$ or $\mathscr{C}\mathscr{C}_1$). The *stable class* $\mathscr{S}_2$, *unstable class* $\mathscr{U}_2$ or *critical class* $\mathscr{C}_2$ for $\Psi_j(\xi)$ is defined in the same way as for $\Phi_i(\xi)$.

First we will study C_1. By virtue of the monotonicity of T_1 for $p \leqq 0$ in the domain of T_1, it follows from (3.20) that $p_1(\xi)$ is continuous and monotone increasing on $(0, \xi_1]$ with $\xi_1 = \min\{1, T_1(0)\}$ (we set $T_1(0)=\infty$ in Cases II and III) and that $\lim_{\xi\downarrow 0} p_1(\xi) = -\infty$. Therefore, for $\xi \in (0, \xi_1]$, $\Phi_1(\xi)$ is monotone decreasing, belongs to $\mathscr{S}_1$ and satisfies $\lim_{\xi\to 0}\Phi_1(\xi)=\infty$ (see (4.2)). Moreover, in Case I, one can find that $\Phi_1(\xi)$ is continuous and monotone increasing (therefore, belongs to $\mathscr{S}_1$) on $(\xi_1, 1]$ except for a finite number of points where $\Phi_1(\xi)$ belongs to $\mathscr{C}_1$. Other $\Phi_i(\xi)$ $(i\neq 1)$, which can exist in Cases I and II, has the following property; $\Phi_i(\xi)$ belongs to $\mathscr{S}_1$ (resp. $\mathscr{U}_1$) for every ξ in an interval where $\Phi_i(\xi)$ is monotone increasing (resp. decreasing). Moreover, (3.21) yields $\Phi_i(\xi) \leqq \mu_1\{F(1)\}^{1/2}$ for $i\neq 1$ and $\xi\in(0, 1]$. Typical shapes of C_1 are depicted in Fig. 3. In Cases I and II, if some $\Phi_i(\xi)$ belongs to $\mathscr{U}_1$, then $\Phi_{i+1}(\xi) > \Phi_i(\xi)$ with $\Phi_{i+1}(\xi) \in \mathscr{S}_1$.

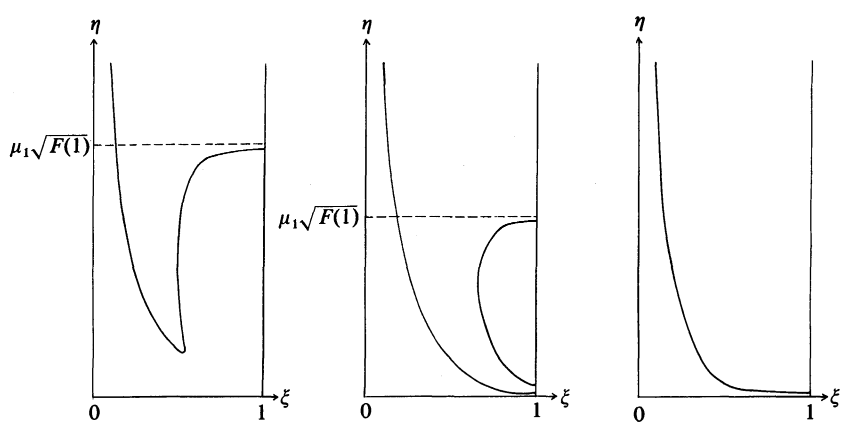


Fig. 3. Graphs of C_1

As for C_2, the analogous results hold. In particular, $\Psi_1(\xi)$ belongs to $\mathscr{S}_2$ except for a finite number of ξ's at which $\Psi_1(\xi)$ is discontinuous and belongs to $\mathscr{C}_2$. Moreover, $\Psi_1(\xi)$ is monotone decreasing on $[0, \xi_2]$ with $\xi_2 = \max\{0, 1-T_2(0)\}$, monotone increasing on $(\xi_2, 1)$ and satisfies $\lim_{\xi\uparrow 1}\Phi_1(\xi)=\infty$. Other $\Psi_j(\xi)$ $(j\neq 1)$, if it exists, belongs to $\mathscr{S}_2$ (resp. $\mathscr{U}_2$) for every ξ in an interval where $\Psi_j(\xi)$ is monotone decreasing (resp. increasing).

We are ready to show the following existence results for (SP).

THEOREM 4.1.

(i) *If* $\{u^*, s^*\}$ *is a solution of* (SP) *then* s^* *satisfies* $\Phi_i(s^*) = \Psi_j(s^*)$ *for some* i, j *and* $u^*(x)$ *is expressed as*

$$u^*(x) = \begin{cases} v_i(x; s^*), & 0 \leqq x \leqq s^*, \\ w_j(x; s^*), & s^* \leqq x \leqq 1, \end{cases} \tag{4.6}$$

where v_i and w_j are given by (3.23) *and* (3.26), *respectively. Conversely, if s^* satisfies $\Phi_i(s^*)=\Psi_j(s^*)$ for some i, j, then $\{u^*, s^*\}$, where u^* is given by* (4.6), *is a solution of* (SP).

(ii) (SP) *has a maximal solution $\{\bar{u}, \bar{s}\}$ and a minimal solution $\{\underline{u}, \underline{s}\}$ in the sense that any solution $\{u^*, s^*\}$ of* (SP) *satisfies*

$$\{\underline{u}, \underline{s}\} \leqq \{u^*, s^*\} \leqq \{\bar{u}, \bar{s}\}.$$

PROOF. From the preceding arguments, it is easy to see (i). Consider the connected component $\mathring{C}_1$ (resp. $\mathring{C}_2$) of C_1 (resp. C_2) containing $\{\{\xi, \Phi_1(\xi)); 0<\xi\leqq 1\}$ (resp. $\{(\xi, \Psi_1(\xi)); 0\leqq\xi<1\}$). Recalling the properties of Φ_1 and Ψ_1, we find that $\mathring{C}_1$ and $\mathring{C}_2$ intersect at some points. This fact implies the existence of solutions for (SP) (the existence result is also derived from Theorem 2.2). Put

$$\bar{s} = \max\{s^* \in I;\ \Phi_i(s^*) = \Psi_j(s^*) \quad \text{for some} \quad i, j\}.$$

Taking account of qualitative shapes of C_1 and C_2, one can show that $\bar{s}$ is given by the maximum among s^*'s satisfying $\Phi_i(s^*)=\Psi_1(s^*)$ for some i. Let $\Psi_1(\bar{s})=\Phi_M(\bar{s})$ and define

$$\bar{u}(x) = \begin{cases} v_M(x;\bar{s}) & \text{for} \quad x \in [0, \bar{s}], \\ w_1(x;\bar{s}) & \text{for} \quad x \in [\bar{s}, 1]. \end{cases}$$

Clearly, $\Phi_M(\bar{s}) \in \mathscr{S}_1$ or $\mathscr{C}_1$. We will show

$$\{u^*, s^*\} \leqq \{\bar{u}, \bar{s}\} \tag{4.7}$$

for any solution $\{u^*, s^*\}$ of (SP). For the sake of simplicity, consider the situation as in Fig. 4, where $\Psi_1(\xi) \in \mathscr{S}_2$ for $\xi \in (s_1, 1)$, $\Phi_M(\xi) \in \mathscr{S}_1$ for $\xi \in (s_1, s_2)$, and $\Phi_{M-1}(\xi) \in \mathscr{U}_1$ for $\xi \in (s_1, s_2)$. By (i) of Proposition 3.5, we have

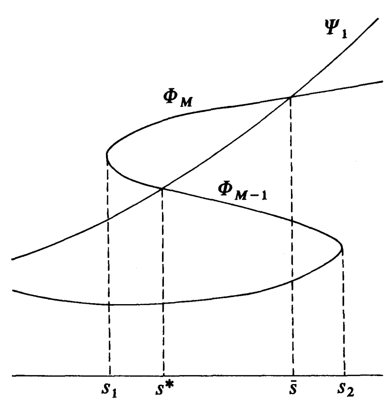


Fig. 4.

$$\{U_1, s^*\} \leqq \{\bar{u}, \bar{s}\},$$

where U_1 is defined by

$$U_1(x) = \begin{cases} v_M(x; s^*) & \text{for} \quad x \in [0, s^*], \\ w_1(x; s^*) & \text{for} \quad x \in [s^*, 1]. \end{cases}$$

Furthermore, Lemma 3.2 gives

$$\{u^*, s^*\} \leqq \{U_1, s^*\},$$

because u^* is expressed as

$$u^*(x) = \begin{cases} v_{M-1}(x; s^*) & \text{for} \quad x \in [0, s^*] \\ w_1(x; s^*) & \text{for} \quad x \in [s^*, 1]. \end{cases}$$

In this way, (4.7) can be proved for any $\{u^*, s^*\}$ by combining Lemma 3.2 and Proposition 3.5.

In order to construct a minimal solution, it suffices to set

$$\underline{s} = \min\{s^* \in I;\ \Phi_i(s^*) = \Psi_j(s^*) \qquad \text{for some} \quad i.j\}$$

and repeat the above procedure with an obvious modification. q.e.d.

REMARK 4.1. Solutions of (SP) other than $\{\bar{u}, \bar{s}\}$ and $(\underline{u}, \underline{s}\}$ are not necessarily ordered. However, if C_1 (resp. C_2) is composed of only Φ_1 (resp. Ψ_1), then the set of solutions for (SP) becomes totally ordered. Indeed, since both $\Phi_1(\xi)$ and $\Psi_1(\xi)$ belong to the stable class for every $\xi \in I$, Proposition 3.5 yields the order relation. See [3].

§5. Stability of stationary solutions

Theorem 4.1 gives us a one-to-one correspondence between solutions of (SP) (i.e., stationary solutions) and intersecting points of C_1 and C_2. We will estabilish the criterion to know stability or instability of each stationary solution from geometrical nature of the corresponding intersectiong point of C_1 and C_2.

THEOREM 5.1.

(i) *Let $\{\bar{u}, \bar{s}\}$ be a maximal solution of* (SP). *Then $\{\bar{u}, \bar{s}\}$ is globally asymptotically stable from above in the sense that, for any initial data $\{\varphi, l\}$ such that $\{\varphi, l\} \geqq \{\bar{u}, \bar{s}\}$, the solution $\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\}$ of* (P) *satisfies*

$$\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} \geqq \{\bar{u}, \bar{s}\} \qquad \text{for all} \quad t \geqq 0 \tag{5.1}$$

and

$$(5.2) \qquad \lim_{t\to\infty} \{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} = \{\bar{u}, \bar{s}\} \text{ in } \Omega\text{-}topology.$$

(ii) *Let* $\{\underline{u}, \underline{s}\}$ *be a minimal solution of* (SP). *Then* $\{\underline{u}, \underline{s}\}$ *is globally asymptotically stable from below in the sense that, for any* $\{\varphi, l)$ *such that* $\{\varphi, l\} \leqq \{\underline{u}, \underline{s}\}$, *the solution* $\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\}$ *satisfies*

$$\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} \leqq \{\underline{u}, \underline{s}\} \qquad \textit{for all} \quad t \geqq 0$$

and (5.2) *with* $\{\bar{u}, \bar{s}\}$ *replaced by* $\{\underline{u}, \underline{s}\}$.

PROOF. We will first prove (i). Since $\{\bar{u}, \bar{s}\}$ is also a solution of (P) with initial datum itself, Theorem 2.3 yields the validity of (5.1). Let $\{u^*, s^*\}$ be any element in $\omega(\varphi, l)$. Then (5.2) assures $\{\bar{u}, \bar{s}\} \leqq \{u^*, s^*\}$. However, (ii) of Theorem 2.2, together with the maximal property of $\{\bar{u}, \bar{s}\}$, implies $\{u^*, s^*\} = \{\bar{u}, \bar{s}\}$, from which (5.2) follows.

The proof of (ii) is carried out in a similar manner. q.e.d.

We will continue the stability analysis for general stationary solutions.

Let $\{u^*, s^*\}$ be any stationary solution. Then C_1 and C_2 meet at a point P^* whose (ξ, η)-coordinate is given by $(s^*, -\mu_1 u_x^*(s^*-0))$.

Suppose that a continuous part of C_1 (rest. C_2) is expressed by $\eta = \Phi(\xi)$ (resp. $\eta = \Psi(\xi)$) in a neighborhood of P^*. The stationary solution of $(\text{AP–1})_\xi$ (resp. $(\text{AP–2})_\xi$) associated with $\Phi(\xi)$ (resp. $\Psi(\xi)$) is denoted by $v(x; \xi)$ (resp. $w(x; \xi)$). If we define $u(x; \xi)$ by (3.28), then $u(\cdot\,; \xi)$ is a solution of $(\text{SP})_\xi$ and, in particular, $u(\cdot\,; s^*) = u^*$.

Stability or instability of $\{u^*, s^*\}$ is determined by the following theorem.

THEOREM 5.2. *Set* $V(\xi) \equiv \Phi(\xi) - \Psi(\xi)$.

(i) *Assume*

$$\Phi(\xi) \in \mathscr{S}_1, \ \Psi(\xi) \in \mathscr{S}_2 \quad \textit{and} \quad V(\xi) < 0 \quad (\textit{resp. } V(\xi) > 0)$$

for every $\xi \in (s^*, \xi^+]$ *with some* $\xi^+ > s^*$ (*resp. for every* $\xi \in [\xi^-, s^*)$ *with some* $\xi^- < s^*$). *Then* $\{u^*, s^*\}$ *is asymptotically stable from above* (*resp. below*) *in the sense that for any* $\{\varphi, l\}$ *satisfying*

$$(5.3) \qquad \{u^*, s^*\} \leqq \{\varphi, l\} \leqq \{u(\cdot\,; \xi^+), \xi^+)\}$$
$$(\textit{resp. } \{u(\cdot\,; \xi^-), \xi^-\} \leqq \{\varphi, l\} \leqq \{u^*, s^*\}),$$

the solution $\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\}$ *of* (P) *satisfies*

$$(5.4) \quad \{u^*, s^*\} \leqq \{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} \leqq \{u(\cdot\,; \xi^+), \xi^+)\}$$
$$(\textit{resp. } \{u(\cdot\,; \xi^-), \xi^-\} \leqq \{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} \leqq \{u^*, s^*\})$$

for all $t \leqq 0$ *and*

(5.5) $$\lim_{t\to\infty}\{u(t;\varphi, l), s(t;\varphi, l)\} = \{u^*, s^*\}$$

in Ω-topology.

(ii) *Assume*

$$\Phi(\xi)\in\mathscr{S}_1,\ \Psi(\xi)\in\mathscr{S}_2 \quad\text{and}\quad V(\xi)>0 \quad (\textit{resp. } V(\xi)<0)$$

for every $\xi\in(s^*, \xi^+]$ (resp. for every $\xi\in[\xi^-, s^*)$). Then $\{u^*, s^*\}$ is unstable in the sense that for any $\{\varphi, l\}$ satisfying

(5.6) $\{u^*, s^*\}\leqq\{\varphi, l\}$ (*resp.* $\{\varphi, l\}\leqq\{u^*, s^*\}$) *with* $\varphi\not\equiv u^*$,

the solution $\{u(\cdot, t;\varphi, l), s(t;\varphi, l)\}$ of (P) *converges (in Ω-topology) to a stationary solution other than $\{u^*, s^*\}$ as $t\to\infty$.*

(iii) *Assume that at least one of $\Phi(\xi)$ and $\Psi(\xi)$ is in the unstable calss for $\xi\in(s^*, \xi^+]$ or $\xi\in[\xi^-, s^*)$. Then $\{u^*, s^*\}$ is unstable in the sense that for any neighborhood U^* of $\{u^*, a^*\}$ (in Ω-topology) there exists some $\{\varphi, l\}$ such that the solution $\{u(\cdot, t;\varphi, l), s(t;\varphi, l)\}$ converges (in Ω-topology) to a stationary solution other than $\{u^*, s^*\}$ as $t\to\infty$.*

PROOF. (i) We will prove (i) in the case $V(\xi)<0$ for $\xi\in(s^*, \xi^+]$ (the case $V(\xi)>0$ for $\xi\in[\xi^-, s^*)$ can be treated similarly). First observe that a pair $\{u(\cdot;\xi), \xi)$ $(s^*<\xi\leqq\xi^+)$ is a time-independent supersolution of (P). Moreover, since $\Phi(\xi)$ and $\Psi(\xi)$ are in the stable class for $\xi\in(s^*, \xi^+]$, Proposition 3.5 (i) assures $u^*(x)<u(x;\xi^+)$ for all $x\in I$. Consequently, Theorem 2.3 enables us to conclude that the solution $\{u(t;\varphi, l), s(t;\varphi, l)\}$ fulfills the order relation (5.4) whenever $\{\varphi, l\}$ satisfies the same order relation (5.3). The convergence property (5.5) is an easy consequence of Theorem 2.2 because $\{u^*, s^*\}$ is the unique stationary solution lying between $\{u^*, s^*\}$ and $\{u(\cdot;\xi^+), \xi^+\}$.

(ii) We consider the case $V(\xi)>0$ for $\xi\in(s^*, \xi^+]$, where $\{u(\cdot, \xi), \xi\}$ is a time-independent subsolution of (P) for $\xi\in(s^*, \xi^+]$. If $\{\varphi, l\}$ satisfies (5.6), then Theorem 2.3 assures

$$\{u(\cdot, T;\varphi, \xi), s(T;\varphi, l)\}>\{u^*, s^*\} \qquad \text{for any} \quad T>0.$$

We note that $\{u(\cdot, \xi)\}$, $\xi\in(s^*, \xi^+]$, is an increasing family by Proposition 3.5. Hence it is possible to take $\xi_0\in(s^*, \xi^+]$, depending on T, such that

$$\{u(\cdot, T;\varphi, l), s(T;\varphi, l)\}\geqq\{u(\cdot, \xi_0), \xi_0\}>\{u^*, s^*\}.$$

Since $\{u(\cdot, \xi_0), \xi_0\}$ is a subsolution, application of Theorem 2.3 yields

$$\{u(\cdot, t;\varphi, l), s(t;\varphi, l)\}\geqq\{u(\cdot, \xi_0), \xi_0\} \qquad \text{for all} \quad t\geqq T,$$

which implies the instability of $\{u^*, s^*\}$ in the sense of (ii).

(iii) Suppose that at least one of $\Phi(\xi)$ and $\Psi(\xi)$ is in the unstable class.

Then it follows from Proposition 3.5 that $u(\cdot\,;\xi)$ always meets u^* at a point other than $x=0, 1$ when $\xi \neq s^*$ is very near s^*. Moreover, $\{u(\cdot, \xi), \xi)$ becomes a subsolution or supersolution according to the sign of $V(\xi)$. Therefore, the proof can be accomplished with the use of the comparison technique based on Theorem 2.3, which is now a routine work. q. e. d.

REMARK 5.1. In general, by virtue of Theorem 5.1, the geometrical feature of an intersecting point of C_1 and C_2 gives us information on stability or instability of the corresponding stationary solution. However, we cannot apply Theorem 5.1 to the case where the intersecting point P^* lies in the critical classes of C_1 and C_2 and C_1 and C_2 are separated in a neighborhood of P^* by a line passing through P^*. In this case more careful analysis will yield the instability of the stationary solution (associated with P^*) with the aid of Theorem 2.3.

REMARK 5.2. Aronson, Crandall and Peletier [1] have studied the stability of stationary solutions for a certain class of nonlinear degenerate diffusion equations. Their method is based on the notion of ω-limit set and the comparison principle. Our arguments for stability analysis are quite anologous to those used by them.

Suppose that the graphs of C_1 and C_2 are depicted as in Fig. 5. Here a solid line means a curve in the stable class and a dotted line means a curve in the unstable class. Then Theorems 5.1 and 5.2 tell us that the stationary solutions corresponding to P_1, P_3 and P_7 are asymptotically stable (from above and below) and that those corresponding to P_2, P_4, P_5 and P_6 are unstable.

Finally we will show some numerical experiments to know the number of stationary solutions of (SP). We take

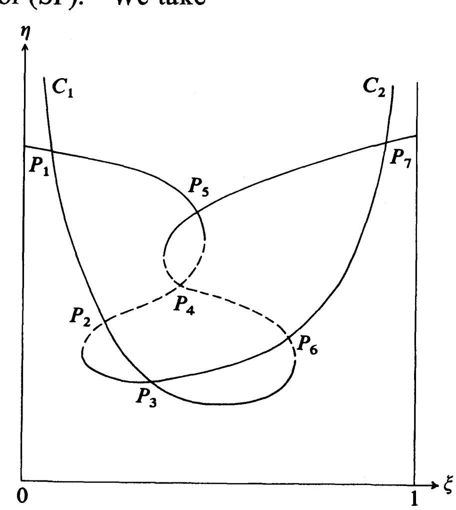
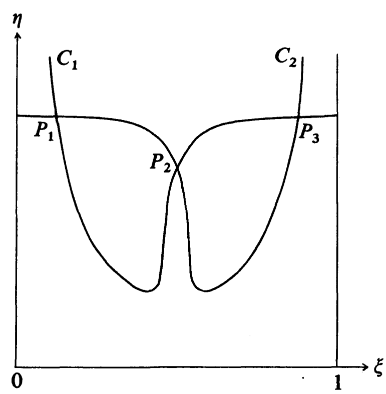
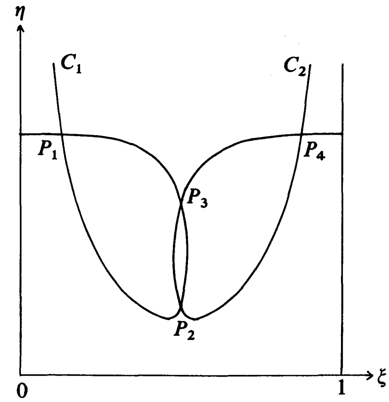


Fig. 5.



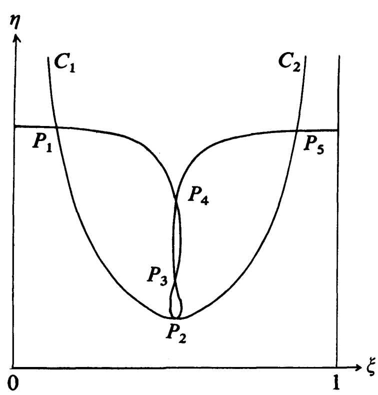
$a=0.27$

Fig. 6A.



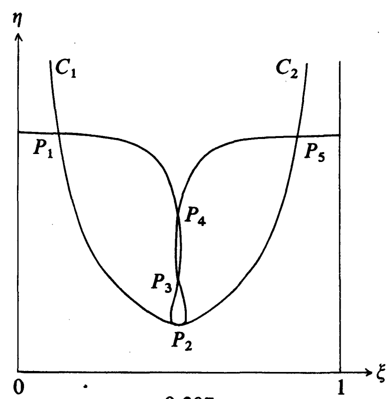
$a=0.29$

Fig. 6B.



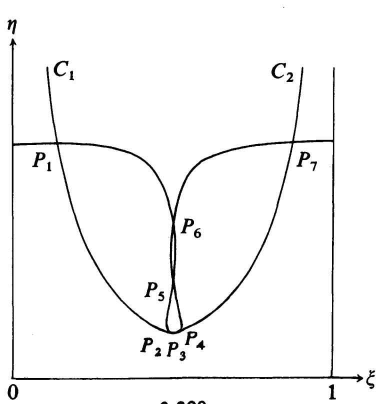
$a=0.295$

Fig. 6D.



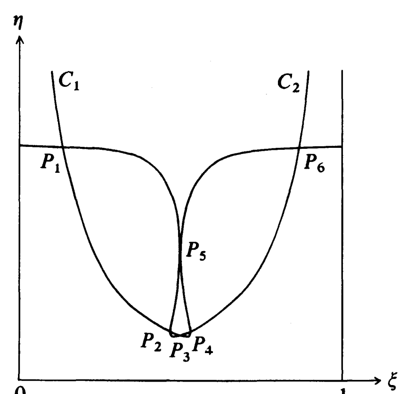
$a=0.297$

Fig. 6C.



$a=0.298$

Fig. 6E.



$a=0.299$

Fig. 6F.

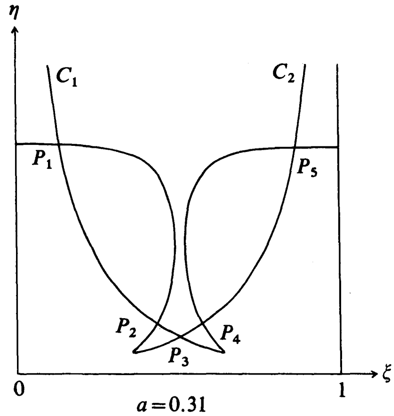


Fig. 6G.

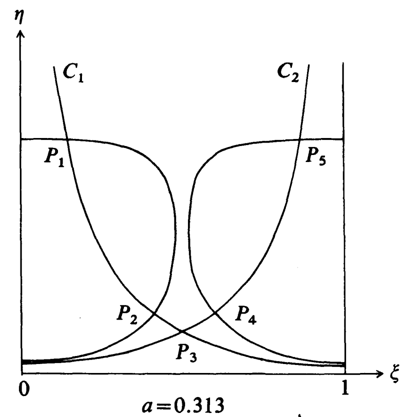


Fig. 6H.

Fig. 6. Graphs of C_1 and C_2, where each intersecting point P_i corresponds to a stationary solution of (SP).

$$d_1 = d_2 = 1, \qquad m_1 = m_2 = 0.5, \qquad \mu_1 = \mu_2 = 1,$$

$$f(u) = g(-u) = -\nu(u-a)(u-1) \quad \text{with} \quad \nu = 200,$$

and regard a as a parameter. Then we meet with interesting bifurcation phenomena as in Fig. 6, which exhibits that the number of stationary solutions varies depending on the value of a. Stability or instability of each stationary solution can be studied in view of the geometrical feature of the corresponding intersecting point of C_1 and C_2.

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