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Cuts of ordered fields

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We denote an ordered field by (F, σ) or simply F, where σ is an ordering of a field F. For ordered fields (F, σ) and (K, τ), we say that K/F is *an extension of ordered fields* if K/F is an extension of fields and τ is an extension of σ. In this paper, $F(x)$ always means a simple transcendental extension of F. A pair (C, D) of subsets of F is called a cut of F if $C \cup D = F$ and $c < d$ for any $c \in C$ and $d \in D$. Let $(F(x), \tau)/(F, \sigma)$ be an extension of ordered fields. Then $g(\tau) := (C, D)$, where $C = \{a \in F;\, a < x\}$ and $D = \{a \in F;\, a > x\}$, is a cut of F. If F is a real closed field, then g is a bijective map from the set of all orderings of $F(x)$ to the set of all cuts of F (Theorem 1.2). In [2], we defined the *rank* of an ordered field and we said that an ordered field F is *a maximal ordered field of rank* n if rank $F = n$ and for any proper extension K/F of ordered fields, rank $K > n$.

Let F be a real closed field of finite rank n and let $A_1 \subset \cdots \subset A_n \subset A_{n+1} = F$ be the compatible valuation rings of F. In this paper, we define the subsets W_i, $i = 1, \ldots, n+1$, of the set of all cuts of F (Definition 3.4) and show that for an ordering τ of $F(x)$, the following statements are equivalent (Theorem 3.10):

(1) $g(\tau) \in W_i$.

(2) There exist distinct convex valuation rings B and B' of $F(x)$ with respect to τ such that $B \cap F = B' \cap F = A_i$.

As a corollary of the above assertion, we have the following statement: $\operatorname{rank}(F(x), \tau) = \operatorname{rank} F + 1$ if and only if $g(\tau) \in \bigcup_{i=1}^{n+1} W_i$. In particular, F is a maximal ordered field if and only if any cut of F is contained in $\bigcup_{i=1}^{n+1} W_i$.

§1. Real closed fields and cuts

Let F be an ordered field. If C and D are subsets of F, we write $C < D$ if $c < d$ for all $c \in C$, $d \in D$. If $a \in F$, then we write $C < a$ or $a < D$ instead of $C < \{a\}$ or $\{a\} < D$, respectively. A pair (C, D) of subsets of F is called a cut of F if $F = C \cup D$ and $C < D$. We regard (F, ϕ) and (ϕ, F) as cuts of F. Throughout this paper, we denote by X the set of orderings σ of $F(x)$ where $(F(x), \sigma)/F$ is an extension of ordered fields. Let C_F be the set of all cuts of F. We define the map g_F: $X \to C_F$ by $g_F(\sigma) = (C, D)$, where $C = \{c \in F;\, c < x(\sigma)\}$ and $D = \{d \in F;\, x < d(\sigma)\}$; here we write $a < b(\sigma)$ if $a < b$ with respect to the ordering σ. It is well known that there is an ordering $\sigma \in X$ such that $F < x(\sigma)$ and it is uniquely determined (cf. [1]). In this case, it is clear that $g_F(\sigma) = (F, \phi)$.

The following definition is stated in [2], Definition 2.1.

DEFINITION 1.1. Let (C, D) be a cut of an ordered field F.

(1) We say that (C, D) is *proper* if C and D are non-empty, C has no largest element and D has no smallest element.

(2) We say that (C, D) is *archimedean* if for any $e \in F$, $e > 0$, there exist elements $c \in C$ and $d \in D$ such that $d - c < e$.

Let K/F be an extension of ordered fields. We say that an element $b \in K$ is infinitely large (with respect to F) if $F < b$. If there is no infinitely large element in K, then we say that F is cofinal in K.

The following Theorem 1.2 is stated in [1], Theorem 1, and we give a proof as a preliminary step to §2 and §3.

THEOREM 1.2. *If F is real closed, then the map g_F: $X \to C_F$ is bijective.*

PROOF. First we show that g_F is injective. Let σ and τ be elements of X such that $g_F(\sigma) = g_F(\tau)$. Let $f(x) \in F[x]$ be a polynomial over F. Since F is real closed, we can write $f(x) = a\Pi(x - b_j)\{(x - c_i)^2 + d_i^2\}$. By the fact $g_F(\sigma) = g_F(\tau)$, the signatures of $x - b_j$ with respect to σ and τ coincide. Hence it is clear that $\sigma = \tau$.

Next we show that g_F is surjective. Let (C, D) be any cut of F. We must show that there exists $\sigma \in X$ such that $g_F(\sigma) = (C, D)$.

Case 1. Assume that $(C, D) = (F, \phi)$ (resp. $(C, D) = (\phi, F)$). Let σ be the ordering of $F(x)$ where x (resp. $-x$) is infinitely large. Then it is clear that $g_F(\sigma) = (C, D)$.

Case 2. Assume that there exists $c_0 := \max C$ (resp. $d_0 := \min D$). Put $y = (x - c_0)^{-1}$(resp. $y = (d_0 - x)^{-1}$) and let σ be the ordering of $F(x) = F(y)$ for which y is infinitely large. Then we can readily see that $g_F(\sigma) = (C, D)$.

Case 3. Assume that (C, D) is a proper cut. For any monic polynomial $f(x)$, we can write $f(x) = \Pi(x - b_j)\{(x - c_i)^2 + d_i^2\}$. Let S be the set of all monic polynomials $f(x) = \Pi(x - b_j)\{(x - c_i)^2 + d_i^2\}$ such that the number of elements in the set $\{j;\ b_j \in D\}$ is even. We put $S_1 = \{af(x);\ a$ is a positive element of F and $f(x) \in S\}$ and $S_2 = \{af(x);\ a$ is a negative element of F and $f(x)$ is a monic polynomial which is not contained in $S\}$. Put $P := \{f_1(x)/f_2(x);\ f_1(x), f_2(x) \in S_1 \cup S_2\}$. It is easy to show that P is a multiplicative subgroup of $\dot{F}(x)$ of index 2. We remark that for a polynomial $f(x)$, the following statements are equivalent:

(1) $f(x) \in S_1 \cup S_2$.

(2) there exists an element $c \in C$ such that $f(c') > 0$ for any $c' \in C$, $c < c'$.

By the above remark, $S_1 \cup S_2$ is additively closed and so is P. Hence there is an ordering $\sigma \in X$ such that the positive cone of σ is P. Now it is clear that $g_F(\sigma) = (C, D)$. Q.E.D.

REMARK 1.3. Even if F is not real closed, g_F is surjective. In fact, let K be a real closure of F. For any cut (C, D) of F, we put $C'=\{b\in K; b\leqq c$ for some $c\in C\}$ and $D'=K\setminus C'$. By Theorem 1.2, there exists an ordering σ of $K(x)$ such that $g_K(\sigma)=(C', D')\in C_K$. It is clear that $g_F(\sigma\mid F)=(C, D)$, where $\sigma\mid F$ is the restriction of σ to F, and so g_F is surjective.

PROPOSITION 1.4. *Let F be a real closed field. Then F is cofinal in $(F(x), \sigma)$ if and only if $g_F(\sigma)$ is a proper cut of F.*

PROOF. First we assume that $g_F(\sigma)$ is not a proper cut of F. By case 1 and case 2 in the proof of Theorem 1.2, it is clear that F is not cofinal in $(F(x), \sigma)$. Next we assume that $g_F(\sigma):=(C, D)$ is a proper cut of F. First we show that for any $f(x)\in S_1\cup S_2$ (S_1, S_2 were defined in the proof of Theorem 1.2), there exist elements $a, b\in F$ such that $0<a<f(x)<b$. Let e be the absolute value of the leading coefficient of $f(x)$. Then $f(x)/e$ is a product of polynomials, $x-c$, $c\in C$, $d-x$, $d\in D$, and $(x-c)^2+d^2$. So we may assume that $f(x)=x-c$, $c\in C$, $f(x)=d-x$, $d\in D$, or $f(x)=(x-c)^2+d^2$.

Case 1. Suppose $f(x)=x-c$, $c\in C$. Let $c_1\in C$ with $c<c_1$ and $c_2\in D$. Then we have $0<c_1-c<f(x)=x-c<c_2-c$.

Case 2. Suppose $f(x)=d-x$, $d\in D$. Let $c_1\in C$ and $c_2\in D$ with $c_2<d$. Then we have $0<d-c_2<f(x)=d-x<d-c_1$.

Case 3. Suppose $f(x)=(x-c)^2+d^2$. By case 1 and case 2, there exists an element c_3 with $(x-c)^2<c_3$. Then $0<d^2<(x-c)^2+d^2<c_3+d^2$.

Now we must show that F is cofinal in $(F(x), \sigma)$. Let α be any positive element of $F(x)$. By the proof of Theorem 1.2, we can see that the positive cone of σ is $\{f_1(x)/f_2(x); f_1(x), f_2(x)\in S_1\cup S_2\}$. So we can write $\alpha=f_1(x)/f_2(x)$ for some $f_1(x), f_2(x)\in S_1\cup S_2$. By the above argument, there exist $a, b\in F$ such that $0<a<f_2(x)$ and $0<f_1(x)<b$ and we have $\alpha=f_1(x)/f_2(x)<b/a$. This shows that F is cofinal in $F(x)$. Q.E.D.

LEMMA 1.5. *Let E and F be subfields of a field L. Let σ and τ be orderings of the composite field EF. Suppose that $E/(E\cap F)$ is an algebraic extension and $\sigma\mid E=\tau\mid E$, $\sigma\mid F=\tau\mid F$. Then we have $\sigma=\tau$.*

PROOF. Suppose $\sigma\neq\tau$. Then there exists an element $\alpha\in EF$ such that $\alpha>0(\sigma)$ and $\alpha<0(\tau)$. We may assume that $\alpha\in F(e_1,\dots, e_n)$ for some $e_1,\dots, e_n\in E$. We put $N=(E\cap F)(e_1,\dots, e_n)$. Then $N/(E\cap F)$ is a finite extension and NF contains α. Let σ_1 and τ_1 be the restrictions of σ and τ to NF respectively. The fact $\alpha\in NF$ implies $\sigma_1\neq\tau_1$. These observations show that we may assume $E/(E\cap F)$ is a finite extension. We put $E=(E\cap F)(\theta)$. Let $f(x)$ and $g(x)$ be the minimal polynomials fo θ over $E\cap F$ and F respectively. Let K be a real closure of the ordered field $(F, \sigma\mid F)$ and let K' be the algebraic closure of $E\cap F$ in K. It iw well known that K' is a real closure of $E \cap F$. Let α_1 and α_2 be the roots of $g(x)$ in K such that orderings σ and τ are canonically induced by injections f_i: $F(\theta) \to F(\alpha_i) \subset K$, $f_i(\theta) = \alpha_i$, $i = 1, 2$, respectively (cf. [3], Chapter 3, §2). Then the orderings $\sigma | E$ and $\tau | E$ are canonically induced by the injections h_i: $E = (E \cap F)(\theta) \to (E \cap F)(\alpha_i) \subset K'$, $h_i(\theta) = \alpha_i$, $i = 1, 2$, respectively. So the assumption $\sigma | E = \tau | E$ implies $\alpha_1 = \alpha_2$, and this shows $\sigma = \tau$. Q.E.D.

Let F be an ordered field and $F(x, y)$ be an extension field of F where x, y are variables. Let σ and τ be orderings of $F(x, y)$ which are extensions of the ordering of F such that $F < x(\sigma)$, $F(x) < y(\sigma)$, $F < y(\tau)$ and $F(y) < x(\tau)$. Then $F < x(\sigma | F(x))$ and $F < x(\tau | F(x))$. So we have $\sigma | F(x) = \tau | F(x)$ and similarly we have $\sigma | F(y) = \tau | F(y)$. From the fact that $x < y(\sigma)$ and $y < x(\tau)$, it follows that $\sigma \neq \tau$. So in Lemma 1.5, the assumption that $E/(E \cap F)$ is an algebraic extension is essential.

THEOREM 1.6. *Let K be a real closure of an ordered field F and Y be the set of all orderings of $K(x)$. For $\tau \in Y$, we let $\psi(\tau)$ be the restriction of τ to $F(x)$. Then the map ψ: $Y \to X$ is bijective.*

PROOF. First we show that ψ is surjective. Let σ be any element of X and L be a real closure of $(F(x), \sigma)$. The algebraic closure of F in L is a real closure of F, and so we can identify it with K. It is clear that $x \in L$ is transcendental over K. Let τ be the restriction of the ordering of L to $K(x)$. Then it is easily shown that $\psi(\tau) = \sigma$, and so ψ is surjective. By Lemma 1.5, it is clear that ψ is injective. Q.E.D.

As a corollary of Theorem 1.2 and Theorem 1.6, we have Theorem 5 in [1]. We also have the following corollary.

COROLLARY 1.7. *Let F be a real closed field. Then the following statements hold:*

(1) *Let $(F(x), \sigma)$ and $(F(y), \tau)$ be ordered fields where x and y are variables. If $\{a \in F;\ a < x(\sigma)\} = \{a \in F;\ a < y(\tau)\}$, then the isomorphism h: $F(x) \to F(y)$, defined by $h(x) = y$, is an order preserving isomorphism.*

(2) *Let σ and τ be orderings of $F(x)$. If there exist elements y, z of $F(x)$ so that $F(x) = F(y) = F(z)$ and $\{a \in F;\ a < y(\sigma)\} = \{a \in F;\ a < z(\tau)\}$, then $(F(x), \sigma)$ and $(F(x), \tau)$ are isomorphic as ordered fields.*

§2. Ordered fields of finite rank

In this section, we assume that F is a real closed field of finite rank (cf. [2], Definition 1.1). Take an ordering $\sigma \in X$ and suppose that F is cofinal in $(F(x), \sigma)$ and rank $(F(x), \sigma) = \text{rank}\, F + 1$ (as for the existence of such an ordering, see Remark 2.1). We fix this ordering $\sigma \in X$. Since rank $(F(x), \sigma) = \text{rank}\, F + 1$, there exist convex valuation rings B_1, B_2 of $F(x)$ such that $B_1 \neq B_2$ and $B_1 | F = B_2 | F$ and the valuation rings B_1 and B_2 are overrings of $A(F(x), Q)$ (cf. [2]). So we may assume that $B_1 \subset B_2$. We put $A := B_1 | F = B_2 | F$. We denote the maximal ideals, the groups of units, the valuations and the value groups of A, B_1 and B_2 by (A, M, U, v, G), $(B_1, M_1, U_1, v_1, G_1)$ and $(B_2, M_2, U_2, v_2, G_2)$ respectively. We denote by $h: G_1 \to G_2$ the canonical surjection. $H := \operatorname{Ker} h$ is the convex subgroup $v_1(U_2)$ of G_1 corresponding to the prime ideal M_2 of B_1. There are canonical injections $h_1: G \to G_1$ and $h_2: G \to G_2$. It is clear that $hh_1 = h_2$, and we identify $h_1(G)$ and $h_2(G)$ with G.

REMARK 2.1. Let $\boldsymbol{R}(x, y)$ be an extension field of $\boldsymbol{R}$, the field of real numbers, where x, y are variables. Let τ be an ordering of $\boldsymbol{R}(x, y)$ such that $\boldsymbol{R} < x(\tau)$ and $\boldsymbol{R}(x) < y(\tau)$. Let L be a real closure of $(\boldsymbol{R}(x, y), \tau)$, K be the algebraic closure of $\boldsymbol{R}(y)$ in L and $\sigma := \tau | K(x)$. Then for any element z of $K(x)$, we have $z < y^n$ for some positive integer n, since the set $\{y^n; n = 1, 2, \ldots\}$ is cofinal in L. This implies K is cofinal in $K(x)$. Next we show that $\operatorname{rank} K(x) = \operatorname{rank} K + 1$. In general, for an ordered field N of finite rank, the following assertions hold (cf. [2], Proposition 1.2):

(1) Let N_1/N be an algebraic extension of ordered fields. Then $\operatorname{rank} N_1 = \operatorname{rank} N$.

(2) $\operatorname{rank} N(x) = \operatorname{rank} N + 1$, where $N(x)/N$ is a simple transcendental extension of ordered fields such that x is infinitely large.

By the above assertions, we have $\operatorname{rank} L = 2$ and $\operatorname{rank} K = 1$. Since $L/K(x)$ is an algebraic extension, $\operatorname{rank} K(x) = 2 = \operatorname{rank} K + 1$.

LEMMA 2.2. *There exists an element $b \in F$ such that $v_1(x - b)$ is not contained in G.*

PROOF. Suppose to the contrary that $v_1(x - b) \in G$ for any $b \in F$. Let $f(x)$ be any monic irreducible polynomial of $F[x]$. Since F is real closed, $\deg f(x) \leqq 2$. If $\deg f(x) = 2$, then we can write $f(x) = (x + a)^2 + b^2$, $b \neq 0$. If $v_1(x + a) \neq v_1(b)$, then $v_1(f(x)) = \min(v_1((x + a)^2), v_1(b^2))$. If $v_1(x + a) = v_1(b)$, then $v_1(f(x)) = v_1((x + a)^2) = v_1(b^2)$ since B_1/M_1 is formally real. So we have $v_1(f(x)) \in G$. This shows that the value of any irreducible polynomial of $F[x]$ is contained in G. This contradicts the fact $G \neq G_1$. Q.E.D.

LEMMA 2.3. *Take an element $b \in F$ so that $e := v_1(x - b) \notin G$. Then $G_1 = G \oplus \boldsymbol{Z}e$.*

PROOF. Since G is divisible ([2], Proposition 1.7), it is clear that $\boldsymbol{Z}e \cap G = \{0\}$. Let α be any polynomial of $F[x]$. We can write $\alpha = a_n(x - b)^n + \cdots + a_1(x - b) + a_0$. Since $\boldsymbol{Z}e \cap G = \{0\}$, the values $v_1(a_i(x - b)^i)$, $i = 0, \ldots, n$, are different from each other. So $v_1(\alpha) = v_1(a_i(x - b)^i)$ for some i and we have $v_1(\alpha) \in G + \boldsymbol{Z}e$.

This implies $G_1 = G \oplus \boldsymbol{Z}e$. Q.E.D.

PROPOSITION 2.4. *$G_2 = G$, $H \cong \boldsymbol{Z}$ and $G_1 = G \oplus H$ (the ordering of $G \oplus H$ is lexicographic).*

PROOF. By Lemma 2.3, G_1/G is isomorphic to $\boldsymbol{Z}$. Since $H \cap G = \{0\}$, H is isomorphic to $(G+H)/G$ which is a subgroup of G_1/G. Hence we have $H \cong \boldsymbol{Z}$. The fact $G_1/G \cong \boldsymbol{Z}$ also shows that $G_1/(G+H) \cong \boldsymbol{Z}/n\boldsymbol{Z}$ for some $n > 0$. Since G_2/G is isomorphic to $G_1/(G+H)$, G_2/G is a torsion group. On the other hand, by [2], Proposition 1.7, G is divisible, and so G_2/G is torsion free. This implies $n = 1$, and $G_2 = G$. Now it is clear that $G_1 = G \oplus H$ and the ordering of $G \oplus H$ is lexicographic. Q.E.D.

The proof of the following Proposition 2.5 is similar to that of Proposition 2.4 and we omit it.

PROPOSITION 2.5. *Let τ be an element of X. Suppose that F is not cofinal in $(F(x), \tau)$. Then $B := \{b \in F(x);\ b < a(\tau)$ for some $a \in F\}$ is a non-trivial valuation ring of $F(x)$ and B is compatible with respect to τ.*

Let v_B be the valuation of B. Then v_B is trivial on F and the value group of v_B is isomorphic to $\boldsymbol{Z}$. Moreover there exists $b \in F$ such that $v_B(x-b)$ is a generator of this value group.

In the situation of Proposition 2.5, there exists an element y of $F(x)$ such that y is a change of variable (i.e. $F(x) = F(y)$) and $v_B(y) = -1$, $y > 0(\tau)$. Then it is clear that $F < y(\tau)$.

Let τ_1 and τ_2 be elements of X and suppose that F is not cofinal in $F(x)$ with repect to τ_1 and τ_2. Then by the above argument, there exist y_1 and y_2 such that $F(x) = F(y_1) = F(y_2)$ and $F < y_1\ (\tau_1)$, $F < y_2\ (\tau_2)$. So $(F(x), \tau_1)$ and $(F(x), \tau_2)$ are isomorphic as ordered fields by Corollary 1.7.

PROPOSITION 2.6. *For $\tau \in X$, if $g(\tau)$ is proper archimedean, then* rank $(F(x), \tau) =$ rank F.

PROOF. Let F_C be the completion of F (cf. [2], Definition 2.5). By [2] Proposition 1.3, rank $F =$ rank F_C. Since F is real closed, $y := g(\tau) \in F_C$ is transcendental over F. Let μ be the ordering of $F(y)$ induced by the ordering of F_C. Then $\{a \in F;\ a < y(\mu)\} = C$ where $(C, D) = g(\tau)$. By Cororally 1.7, $(F(x), \tau)$ and $(F(y), \mu)$ are isomorphic as ordered fields. So we have rank $(F(x), \tau) =$ rank $(F(y), \mu) =$ rank F. Q.E.D.

§3. Maximal ordered fields and cuts

In this section, we assume that F is a real closed field of rank n. Let

$A(F, Q)=A_1\subset\cdots\subset A_n\subset A_{n+1}=F$ be the convex valuation rings of F and v_i be the valuations of A_i, $i=1,\dots, n$.

DEFINITION 3.1. For a cut (C, D) of F, we put $M(C, D):=\{x\in F;\ \pm x\in C$ or $\pm x\in D\}$ and $\dot M(C, D):=M(C, D)\setminus\{0\}$.

If $C=F^-:=\{a\in F;\ a<0\}$, then $M(C, D)=\{0\}$. For any cut (C, D), it is clear that $0\in M(C, D)$.

PROPOSITION 3.2. *Let (C, D) be a cut of F and v be a compatible valuation of F. Then $g'\leqq g$ for any $g\in v(\dot M(C, D))$ and $g'\in v(\dot F\setminus\dot M(C, D))$. In particular, the set $v(\dot M(C, D))\cap v(\dot F\setminus\dot M(C, D))$ consists of at most one element.*

PROOF. First we remark that if $a\in\dot M(C, D)$ and $0<b<a$ then $-a\in\dot M(C, D)$ and $b\in\dot M(C, D)$. There exist elements $a\in\dot M(C, D)$ and $b\in\dot F\setminus\dot M(C, D)$ such that $v(a)=g$ and $v(b)=g'$. By the above remark, we may assume $0<a\leqq b$. Since v is compatible, $v(b)\leqq v(a)$ and so $g'\leqq g$. Q.E.D.

DEFINITION 3.3. For $i=1,\dots, n$ we put $T_i=\{(C, D)$ a proper cut of F; $v_i(\dot M(C, D))\cap v_i(\dot F\setminus\dot M(C, D))=\phi$ and $\min v_i(\dot M(C, D))$ or $\max v_i(\dot F\setminus\dot M(C, D))$ exists$\}$. If $(C, D)\in T_i$, then we denote $\min v_i(\dot M(C, D))$ or $\max v_i(\dot F\setminus\dot M(C, D))$ by $\alpha(v_i, (C, D))$.

If $(C, D)\in T_i$, then it is clear that $v_i^{-1}(v_i(\dot M(C, D)))=\dot M(C, D)$, and we can show that $M(C, D)$ is a fractional ideal of A_i. For a cut (C, D) and an element $a\in F$, we put $C+a=\{c+a, c\in C\}$ and $D+a=\{d+a, d\in D\}$. It is clear that $(C+a, D+a)$ is a cut of F.

DEFINITION 3.4. For $i=1,\dots, n$, we put $W_i:=\{(C+a, D+a);\ (C, D)\in T_i,\ a\in F\}$ and we let W_{n+1} be the set of all non-proper cuts of F.

PROPOSITION 3.5. *Let (C, D) be a cut of F which belongs to some T_i. For an element $y\in F$, the following statements are equivalent:*

(1) $y\in M(C, D)$.

(2) $(C, D)=(C+y, D+y)$.

PROOF. (1)$\Rightarrow$(2): Let y be any element of $M(C, D)$. First we assume that $0\in C$; in this case $M(C, D)\subset C$. Suppose $C+y\neq C$. There are two cases $C+y\supset C$ and $C+y\subset C$. Replacing y by $-y$ if necessary, we may assume that $C+y\supset C$. Then there exists an element $c\in C$ such that $0<c$ and $c+y\in D$. Since $c\in M(C, D)$ and $M(C, D)$ is additively closed, $c+y\notin D$, a contradiction. As for the case $0\in D$, the assertion can be proved similarly.

(2)$\Rightarrow$(1): Suppose $C+y=C$. If $0\in C$, then y and $-y$ are contained in$C$. So $y\in M(C, D)$. If $0\notin C$, then $0\in D$, and the fact $D+y=D$ also implies $y\in M(C, D)$. Q.E.D.

PROPOSITION 3.6. *$\bigcup_{i=1}^{n} W_i$ is a disjoint union.*

PROOF. Suppose to the contrary that there exists a cut $(C, D) \in W_i \cap W_j$ for some $i \neq j$. We may assume $i<j$. There exist cuts $(C_i, D_i) \in T_i$ and $(C_j, D_j) \in T_j$ such that $(C, D)=(C_i+c_i, D_i+c_i)=(C_j+c_j, D_j+c_j)$ for some $c_i, c_j \in F$. It is clear that $\{a \in F; C_i+a=C_i\}=\{a \in F; C+a=C\}$ and so we have $\{a \in F; C_i+a=C_i\}=\{a \in F; C_j+a=C_j\}$. Let H be the kernel of the canonical surjection $G_i \to G_j$ (cf. §2). We fix an element $0<\beta \in H$. There exist elements s and s' such that $0<s$, $0<s'$ and $v_i(s)=\alpha(v_i, (C_i, D_i))-\beta$, $v_i(s')=\alpha(v_i, (C_i, D_i))+\beta$. Since $v_i(s) \in v_i(\dot{F} \setminus \dot{M}(C_i, D_i))$ and $v_i(s') \in v_i(\dot{M}(C_i, D_i))$, we have $s \notin M(C_i, D_i)$ and $s' \in M(C_i, D_i)$. By Proposition 3.5, $C_i+s'=C_i$ and $C_i+s \neq C_i$ and so by the fact $\{a \in F; C_i+a=C_i\}=\{a \in F; C_j+a=C_j\}$, we have $C_j+s'=C_j$ and $C_j+s \neq C_j$. On the other hand, since $v_j(s)=v_j(s')$, we can see that $s \in M(C_j, D_j)$ if and only if $s' \in M(C_j, D_j)$. Hence by Proposition 3.5, $C_j+s'=C_j$ if and only if $C_j+s=C_j$. This is a contradiction. Q.E.D.

For $\sigma, \tau \in X$, we write $\sigma \sim \tau$ if $(F(x), \sigma)$ is F-isomorphic to $(F(x), \tau)$ as ordered fields. We can easily show that $\sim$ is an equivalence relation in X. We put $X_1=\{\sigma \in X; \operatorname{rank}(F(x), \sigma)=n+1\}$. Then X_1 is a union of equivalence classes. We can define the equivalence relation in C_F which is canonically induced by the bijection $g: X \to C_F$. We denote it by the same symbol $\sim$. By Proposition 1.4 and the argument after Proposition 2.5, W_{n+1} is an equivalence class of C_F.

PROPOSITION 3.7. *Let (C, D) and (C', D') be any cuts of F wihch belong to the set W_i for some $i=1,\dots, n$. Then $(C, D) \sim (C', D')$.*

PROOF. Let σ be the element of X such that $g(\sigma)=(C, D)$. By Corollary 1.7, it is sufficient to show that there exists an element y of $F(x)$ such that $F(x)=F(y)$ and $\{d \in F; d<y(\sigma)\}=C'$. If $C'=C+a$ for some a, then we put $y=x+a$. It is clear that y satisfies the desired condition. So we may assume that (C, D) and (C', D') are contained in T_i. We suppose, for example, that $M(C, D) \subset C$, $M(C', D') \subset C'$ and $\alpha(v_i, (C, D))=\min v_i(\dot{M}(C, D))$, $\alpha(v_i, (C', D'))=\max v_i(\dot{F} \setminus \dot{M}(C, D))$ (in the other cases, the assertions can be proved similarly). Let a and b be elements of F such that $a>0$, $b>0$, $v_i(a)=\alpha(v_i, (C, D))$ and $v_i(b)=\alpha(v_i, (C', D'))$. We put $y=ab/x$. Let d be a positive element of F and suppose $d<y(\sigma)$. Then $ab/d>x$, and so $v_i(ab/d)<\alpha(v_i, (C, D))$. This implies $v_i(d)>\alpha(v_i, (C', D'))$, hence $d \in M(C', D') \subset C'$. These observations show that $\{d \in F; d<y(\sigma)\} \subset C'$. The converse inclusion is proved similarly. Q.E.D.

DEFINITION 3.8. For $\sigma \in X_1$, let $B_1 \subset B_2 \subset \cdots \subset B_{n+2}=F(x)$ be the convex valuation rings of $F(x)$ (with respect to σ). Then there exists a unique number j $(j=1,\dots, n+1)$ such that $B_j \cap F=B_{j+1} \cap F=A_j$. We put $N(\sigma)=j$. It is clear that for $\sigma, \tau \in X_1$, if $\sigma \sim \tau$, then $N(\sigma)=N(\tau)$.

THEOREM 3.9. *The map* $N\colon X_1/\sim\to\{1,\dots,n+1\}$ *is bijective, where* $X_1/\sim$ *means the set of equivalence classes in* X_1.

PROOF. First we show that N is surjective. We fix a number j, $j=1,\dots,n+1$. Let $C=F^-\cup A_j$ and $D=F\setminus C$. Then, since A_j is convex, (C, D) is a cut of F. Let σ be the ordering of $F(x)$ such that $g(\sigma)=(C, D)$ and let k_j be a maximal subfield of A_j. We put $B=A((F(x), \sigma), k_j)$. It is clear that B is a convex valuation ring of $F(x)$ with respect to σ. By [2], Proposition 1.5, $A(F, k_j)=A_j$. This implies that $B\cap F=A_j$. We put $B'=\{a\in F(x);\ |a|<x^n(\sigma)$ for some positive integer $n\}$. From the facts $k_j\subset A_j\subset C<x(\sigma)$, and $x\in B'\setminus B$, it follows that $B\subset B'$ and $B\neq B'$. We show that B' is a convex valuation ring of $F(x)$ with respect to σ and $B'\cap F=A_j$. By the definition of B', it is clear that B' is a convex subset of $F(x)$ with respect to σ and B' is multiplicatively closed. Let c and d be any elements of B'. Then there exist positive integers s and t such that $|c|<x^s$ and $|d|<x^t$. We may assume $s\leqq t$. We have $|c+d|\leqq|c|+|d|<x^s+x^t\leqq 2x^t<x^{t+1}$. This shows that $c+d\in B'$. Thus B' is additively closed and so B' is a subring of $F(x)$. Since B' is an overring of B, B' is a valuation ring. Let b be a positive element of $B'\cap F$. Then $b<x^n$ for some n and so $0<b^{1/n}<x$ (note that F is real closed). This implies $b^{1/n}\in F^+\cap C=A_j$, where F^+ is the set of all positive elements of F. Thus $b\in A_j$, and we have $B'\cap F=A_j$. This shows that $\sigma\in X_1$, $N(\sigma)=j$ and therefore N is surjective.

Next we show that N is injective. Let σ, τ be elements of X_1 such that $N(\sigma)=N(\tau)=j$. Let $B_1\subset B_2\subset\cdots\subset B_{n+2}=F(x)$ be the convex valuation rings of $F(x)$ with respect to σ. By Definition 3.8, $B_j\cap F=B_{j+1}\cap F=A_j$. Let v_j, v'_j and v'_{j+1} be the valuations of A_j, B_j and B_{j+1} respectively, and G_j, G'_j and G'_{j+1} be the value groups of v_j, v'_j and v'_{j+1} respectively. By Proposition 2.4, $G'_{j+1}=G_j$ and $G'_j\cong G_j\oplus \boldsymbol{Z}$ (the ordering of $G_j\oplus\boldsymbol{Z}$ is lexicographic). By Lemma 2.2 and Lemma 2.3, there exists an element b of F such that $v'_j(x-b)=(g, \pm 1)$, $g\in G_j$. By a suitable change of variable, $x_1=a/(x-b)$ or $x_1=a(x-b)$, we can find an element x_1 of $F(x)$ such that $x_1>0(\sigma)$, $v'_j(x_1)=(0, -1)$ and $F(x)=F(x_1)$. Let a be an element of F such that $|a|<x_1(\sigma)$. Then $v'_j(a)>v'_j(x_1)$ (note that v'_j is compatible with respect to σ and $v'_j(x_1)=(0, -1)\notin G_j$). This shows that $v_j(a)\geqq 0$, and so $a\in A_j$. Conversely we can prove $\{a\in F;\ |a|<x_1(\sigma)\}\supset A_j$, and so we have $\{a\in F;\ |a|<x_1(\sigma)\}=A_j$. Similarly there exists an element x_2 of $F(x)$ such that $x_2>0(\tau)$, $F(x)=F(x_2)$ and $\{a\in F;\ |a|<x_2(\tau)\}=A_j$. By Corollary 1.7, this shows that the isomorphism $f\colon F(x)\to F(x)$, defined by $f(x_1)=f(x_2)$, gives an order preserving isomorphism between $(F(x), \sigma)$ and $(F(x), \tau)$. Thus we have $\sigma\sim\tau$. Q.E.D.

For $j=1,\dots,n+1$, let Y_j be the equivalence class of X_1 such that $N(Y_j)=j$. It is clear that Y_{n+1} is the set of orderings $\tau\in X$ such that F is not cofinal in $F(x)$ with respect to τ and $g(Y_{n+1})=W_{n+1}$.

THEOREM 3.10. *For any* j, $j=1,\dots,n$, $g(Y_j)=W_j$. *In particular*, W_j, $j=1,\dots,n$, *is an equivalence class of* C_F.

PROOF. We fix a number j, $j=1,\dots,n$. By Proposition 3.7, it is sufficient to show that $g(Y_j)\subset W_j$. Let σ be any element of Y_j. We use the notation in the proof of Theorem 3.9, and we put $C=F^-\cup A_j$, $D=F\setminus C$. Then $C=\{c\in F;\ c<x_1(\sigma)\}$; here the element x_1 satisfies the conditions that $x_1>0(\sigma)$, $v'_j(x_1)=(0,-1)$ and $F(x)=F(x_1)$. It is clear that $M(C, D)=A_j$, and so $v_j(\dot{M}(C, D))=\{g\in G_j;\ 0\leqq g\}$ and $v_j(\dot{F}\setminus\dot{M}(C, D))=\{g\in G_j;\ 0>g\}$.

We can write $x_1=a/(x-b)$ or $x_1=a(x-b)$ (see the proof of Theorem 3.9). Hence $x=a/x_1+b$ or $x=x_1/a+b$. We put $y=x-b$ ($y=a/x_1$ or $y=x_1/a$), and $C'=\{c\in F;\ c<y(\sigma)\}$, $D'=F\setminus C'$. Then we can easily show that if $y=x_1/a$, then $v_j(\dot{M}(C', D'))=\{g\in G_j;\ -v(a)\leqq g\}$, $v_j(\dot{F}\setminus\dot{M}(C', D'))=\{g\in G_j;\ -v(a)>g\}$, and if $y=a/x_1$, then $v_j(\dot{M}(C', D'))=\{g\in G_j;\ v(a)<g\}$, $v_j(\dot{F}\setminus\dot{M}(C', D'))=\{g\in G_j;\ v(a)\geqq g\}$. This shows $(C', D')\in T_j$, and so $g(\sigma)=(C'+b, D'+b)\in W_j$.

Q.E.D.

By Proposition 2.6, any proper archimedean cut is not contained in $\bigcup_{j=1}^{n+1} W_j$.

THEOREM 3.11. *For a real closed field* F *of rank* n, *the following statements are equivalent*:

(1) *F is a maximal ordered field of rank n.*

(2) $C_F=\bigcup_{j=1}^{n+1} W_j$.

PROOF. (1)⇒(2): Let (C, D) be any proper cut of F and $\sigma\in X$ be an ordering such that $g(\sigma)=(C, D)$. Since F is a maximal ordered field of rank n, we have $\operatorname{rank}(F(x), \sigma)=n+1$. So by Theorem 3.10, $g(\sigma)\in W_j$ for some $j=1,\dots,n$.

(2)⇒(1): By [2], Proposition 2.10, it is sufficient to show that $\operatorname{rank}(F(x), \sigma)=n+1$ for any $\sigma\in X$. If $g(\sigma)$ is not proper, then F is not cofinal in $(F(x), \sigma)$ by Proposition 1.4. This shows that $A(F(x), F)$ is a proper valuation ring of $F(x)$, and so $\operatorname{rank}(F(x), \sigma)=n+1$. If $g(\sigma)$ is proper, then $g(\sigma)\in W_j$, for some $j=1,\dots,n$. By Theorem 3.10, $\sigma\in Y_j\subset X_1$, and so $\operatorname{rank}(F(x), \sigma)=n+1$. Q.E.D.

EXAMPLE 3.12. Let $(\boldsymbol{R}(x), \sigma)$ be the ordered field such that $\boldsymbol{R}<x$. Let F be a real closure of $(\boldsymbol{R}(x), \sigma)$. Since rank $\boldsymbol{R}(x)=1$ and $F/\boldsymbol{R}(x)$ is an algebraic extension, we have rank $F=1$. Therefore there exists a unique compatible valuation v of F. Let v' be the restriction of v to $\boldsymbol{R}(x)$ and G, G' be the value groups of v and v' respectively. By Proposition 2.5, G' is isomorphic to $\boldsymbol{Z}$. Since $F/\boldsymbol{R}(x)$ is an algebraic extension, G/G' is a torsion group and by [2], Proposition 1.7, G is divisible. So G is isomorphic to $\boldsymbol{Q}$, the field of rational numbers. Let $C=F^-\cup\{a\in F;\ v(a)>2^{1/2}\}$ and $D=F\setminus C$. Then (C, D) is a cut of F and $v(\dot{M}(C, D))=\{r\in\boldsymbol{Q};\ 2^{1/2}<r\}$, $v(\dot{F}\setminus\dot{M}(C, D))=\{r\in\boldsymbol{Q};\ 2^{1/2}>r\}$. Choose $e\in$

F, $e>0$ such that $v(e)=2$. Then for any $d\in D$ and $c\in C$, $c>0$, we have $v(d-c)=v(d)<v(e)=2$ (note that $c\in M(C, D)$ and so $v(c)>v(d)$). This shows that $d-c>e$; therefore (C, D) is not archimedean. We show that (C, D) is not contained in $W_1 \cup W_2$. It is clear that (C, D) is proper, and so $(C, D)\notin W_2$. It is sufficient to show that $(C+b, D+b)$ is not contained in T_1 for any $b\in F$. If $v(b)>2^{1/2}$, then for any $d\in D$, $v(d+b)=v(d)<2^{1/2}$ and for any $0<c\in C$, $v(c+b)>2^{1/2}$. This shows that $(C+b, D+b)=(C, D)\notin T_1$, since neither $\min v(\dot{M}(C, D))$ nor $\max v(\dot{F}\setminus\dot{M}(C, D))$ exists. If $v(b)<2^{1/2}$, then we can easily show that $\min v(\dot{M}(C+b, D+b))=\max v(\dot{F}\setminus\dot{M}(C+b, D+b))=v(b)$. This shows that $(C+b, D+b)\notin T_1$. Hence in general, there exists a proper cut of a real closed field F which is not archimedean and not contained in W_j, $j=1,\dots, n+1$.

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