HIROSHIMA MATH. J.
18 (1987), 219–225

Maximal ordered fields of rank n II

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(Received August 5, 1987)

The notion of maximal ordered fields of finite rank was introduced and the theory was developed in [2] by the first and second authors of this paper.

In §1 we give the definition of *rank* of a valuation ring, which is a slight modification of the definition given by P. Ribenboim [4]; for an ordered field F, we define the *rank* of F as the rank of the finest valuation ring $A(F, \boldsymbol{Q})$ compatible with the ordering of F.

The aim of this paper is to study the theory of maximal ordered fields of any rank. We say that K is a *maximal* ordered field if $\psi_{L/K}$ (for the definition of $\psi_{L/K}$, see §1) is not bijective for any proper extension L/K of ordered fields (Definition 2.1). Let F be any ordered field. We show that there exists an extension K/F of ordered fields such that $\psi_{K/F}$ is bijective and K is a maximal ordered field (Theorem 3.3). In Theorem 3.4, we determine the structure of a maximal ordered field.

§1. The rank of an ordered field

Let F be an ordered field. For a subfield k of F, we put $A(F, k)=\{a \in F; |a|<b \text{ for some } b \in k\}$; then $A(F, k)$ is a convex valuation ring of F containing k. It is well known that $A_0:=A(F, \boldsymbol{Q})$ is the finest convex valuation ring in F; every convex valuation ring in F is a localization of A_0 and conversely. We denote by Δ' the index set of all convex valuation rings in F; namely $\mathscr{C}(F)=\{A_i; i \in \Delta'\}$ is the set of convex valuation rings in F which coincides with the set of localizations of A_0.

Let v_0 be a valuation defined by A_0 and G_0 the value group of v_0. Since there is a one to one correspondence between $\mathscr{C}(F)$ and the set $\mathscr{H}(F)$ of convex subgroups of G_0, $\mathscr{H}(F)$ is also indexed by Δ', i.e. $\mathscr{H}(F)=\{H_i; i \in \Delta'\}$. $\mathscr{C}(F)$ is a totally ordered set under inclusion relations and so is $\mathscr{H}(F)$; the bijection between $\mathscr{C}(F)$ and $\mathscr{H}(F)$ gives an isomorphism as totally ordered sets. $\mathscr{C}(F)$ (resp. $\mathscr{H}(F)$) has the initial element A_0 (resp. $\{0\}$) and the final element F(resp. G_0). We put $\Delta:=\Delta' \setminus \{0\}$. Then $\{A_i; i \in \Delta\}$ is the set of convex valuation rings which contain A_0 properly, and $\{H_i; i \in \Delta\}$ is the set of non-zero convex subgroups of G_0. These observations show that Δ' and Δ are also totally ordered sets; Δ' has both the initial element and the final element and Δ has the final element.

Let A be a convex valuation ring in F, namely $A=A_i$ for some $i\in\Delta'$. We define the rank of A by the order type of the totally ordered set $\{j\in\Delta';\ i<j\}$ and denote it by rank A (in case $A=F$, we understand that rank A is 0).

DEFINITION 1.1. We define the *rank* of the ordered field F by rank $F:=$ rank $A(F, \boldsymbol{Q})$. Note that rank F is the order type of Δ.

Let Γ be any totally ordered set. A subset S of Γ is said to be a segment if $i\leqq j\in S$, then $i\in S$ (S may be an empty subset ϕ). We denote by ${}^S\Gamma$ the set of segments of Γ. Then ${}^S\Gamma$ is a totally ordered set under inclusion relation; ${}^S\Gamma$ has both the inital element ϕ and the final element Γ. Let Γ^P be the set of elements of Γ which have predecessors in Γ (we understand that the initial element of Γ, if it exists, does not belong to Γ^P). It is easy to see that $({}^S\Gamma)^P\simeq\Gamma$ and ${}^S(({}^S\Gamma)^P)\simeq{}^S\Gamma$ as totally ordered sets.

A convex subgroup of G_0 generated by one element is called a principal convex subgroup. For a non-zero convex subgroup H of G_0, H is principal if and only if there exists a maximal convex subgroup H^* properly contained in H. Therefore the set of non-zero principal convex subgroups of G_0 is indexed by Δ'^P. In Ribenboim [4], A, the definition of rank A_0 is given as the dual order type of Δ'^P. The relations $({}^S\Delta')^P\simeq{}^S(\Delta'^P)\simeq\Delta'$ imply that there is no essential difference between the above two definitions.

Let K/F be an extension of ordered fields. Let $\mathscr{C}(K)=\{B_i;\ i\in\Delta'(K)\}$ be the set of convex valuation rings of K and $\mathscr{C}(F)=\{A_i;\ i\in\Delta'(F)\}$ that of F. The mapping $\psi\colon\mathscr{C}(K)\to\mathscr{C}(F)$ defined by $\psi(B_i)=B_i\cap F$ is a surjection (cf. [2], §1). If it is necessary to specify the fields K and F, we use the symbol $\psi_{K/F}$. If K is algebraic over F, then ψ is a bijection and also if F is dense in K, then so is ψ (cf. [2], §1). We say that ψ is not bijective at $i\in\Delta'(F)$ if there exist convex valuation rings $B\subsetneqq B'$ of K such that $B\cap F=B'\cap F=A_i$ and we put $\psi^{-1}(i):=\{j\in\Delta'(K);\ B_j\cap F=A_i\}$.

PROPOSITION 1.2. *Let K/F be a simple transcendental extension of ordered fields. Then we have the following assertions*:

(1) *ψ is not bijective for at most one $i\in\Delta'(F)$.*

(2) *If ψ is not bijective at i, then $\psi^{-1}(i)$ consists of two elements.*

PROOF. Let v_0 (resp. v_0') be the valuation defined by $A_0(F)=A(F, \boldsymbol{Q})$ (resp. $A_0(K)=A(K, \boldsymbol{Q})$) and G_0 (resp. G_0') be the value group of v_0 (resp. v_0'); here we suppose that (v_0', G_0') is a prolongation of (v_0, G_0). It is well known that rational rank $G_0'/G_0\leqq 1$ (cf. [1], Chapter 6, §10, Corollaire 2); our assertions now follow immediately from the above fact. Q. E. D.

Let (A, M) be a convex valuation ring of F. By Zorn's Lemma, there exists a maximal subfield k contained in A. The maximal ideal M of A is a prime ideal of the finest valuation ring $A_0=A(F, \boldsymbol{Q})$. The residue field $\bar{F}=A/M$ is an ordered field with the ordering canonically induced by that of F.

PROPOSITION 1.3. *The notation being as above, we have*

$$\operatorname{rank} F = \operatorname{rank} \bar{F} + \operatorname{rank} A$$

where the right hand side means a sum of order types and $\operatorname{rank} \bar{F} = \operatorname{rank} k$.

PROOF. We may suppose that $A_0=A(F, \boldsymbol{Q})$ is contained in A properly and so the convex valuation ring (A, M) coincides with (A_i, M_i) for some $i \in \Delta$. By [2], Lemma 1.4, we have $A(\bar{F}, \boldsymbol{Q})=A_0/M$ and so $\mathscr{C}(\bar{F})=\{A_j/M;\ j \leqq i, j \in \Delta'\}$. This implies that $\operatorname{rank} \bar{F}$ is the order type of the segment $S=\{j \leqq i;\ j \in \Delta\}$ of Δ. The complementary set $\Delta \smallsetminus S$ of S in Δ is the index set of the convex valuation rings which contain A properly. The first assertion now follows immediately from these observations. Since $\bar{F}$ is algebraic over k (cf. [2], Proposition 1.5), we can get the second assertion. Q. E. D.

Finally we give an example of an extension K/F of ordered fields for which $\operatorname{rank} K = \operatorname{rank} F$ but $\psi_{K/F}$ is not bijective.

EXAMPLE 1.4. For a simple transcendental extension $F(x)$ of an ordered field F, there is a unique extension of the ordering of F to $F(x)$, for which x is infinitely large, namely $x>a$ for all $a \in F$. We write this ordering as $x>F$.

For a set of variables $\{x_i;\ i=1, 2,\dots\}$, we put $K=\boldsymbol{Q}(x_1, x_2,\dots)$. There is a unique extension of the ordering of $\boldsymbol{Q}$ ot K such that $x_i>Q_{i-1}$ for $i \geqq 1$, where $Q_0=\boldsymbol{Q}$ and $Q_i=\boldsymbol{Q}(x_1,\dots, x_i)$. Put $F=\boldsymbol{Q}(x_2, x_3,\dots)$. Then F is a subfield of K and $K=F(x_1)$ is a simple transcendental extension of F. Since $K=\cup Q_i$, $i=1, 2,\dots$ and $\operatorname{rank} Q_i=i$, the set $\mathscr{C}(K)$ of convex valuation rings of K is given by $\{A(K, Q_i);\ i=0, 1,\dots\} \cup \{K\}$. Therefore $\operatorname{rank} K=\omega+1$, where ω is the ordinal number of the set of natural numbers. Similarly we have $\operatorname{rank} F=\omega+1$, and so $\operatorname{rank} K=\operatorname{rank} F$. However, since $A(K, \boldsymbol{Q}) \cap F=A(K, \boldsymbol{Q}(x_1)) \cap F=A(F, \boldsymbol{Q})$, $\psi_{K/F}$ is not bijective. In fact, $A(K, \boldsymbol{Q}(x_1)) \cap F$ is a convex valuation ring of F not containing x_2, and it coincides with $A(F, \boldsymbol{Q})$.

§ 2. Maximal ordered fields

DEFINITION 2.1. We say that F is a *maximal* ordered field if for any proper extension K/F of ordered fields, $\psi_{K/F}$ is not bijective.

We say that an ordered field is complete if it has no proper archimedean cuts (cf. [2], Definition 2.1).

PROPOSITION 2.2. *A maximal ordered field F is complete.*

PROOF. Let K be a completion of F (cf. [2], Definition 2.5). Since F is dense in K, $\psi_{K/F}$ is bijective (cf. §1) and so $F=K$. Q.E.D.

PROPOSITION 2.3. *For an ordered field F, the following statements are equivalent:*

(1) *F is a maximal ordered field.*

(2) *F is real closed and $\psi_{F(x)/F}$ is not bijective for any ordering of $F(x)$, where $F(x)/F$ is a simple transcendental extension.*

PROOF. (1)⇒(2): It is clear that $\psi_{F(x)/F}$ is not bijective. Let $\bar{F}$ be a real closure of F. Then, since $\bar{F}/F$ is algebraic, $\psi_{\bar{F}/F}$ is bijective (cf. §1) and so $\bar{F}=F$.

(2)⇒(1): Let K/F be a proper extension of ordered fields. We fix an element $y\in K\setminus F$. Since F is real closed, y is transcendental over F. By the assumption, $\psi_{F(y)/F}$ is not bijective and this implies that $\psi_{K/F}$ is not bijective.

Q.E.D.

For a totally ordered set I and for a family of ordered groups G_i, $i\in I$, we denote by $\vdash\!\dashv G_i$, $i\in I$, the Hahn product of G_i's (cf. [4], A). An element of $\vdash\!\dashv G_i$ is $\alpha=(\alpha_i)$, $\alpha_i\in G_i$, where $\mathrm{supp}(\alpha):=\{i\in I;\ \alpha_i\neq 0\}$ is a well-ordered set.

In what follows, we understand that for a totally ordered set Γ, Γ^* stands for the dual ordered set of Γ. We also denote by $\vdash\!\dashv(\Gamma)$ the Hahn product $\vdash\!\dashv R_i$, $i\in\Gamma$ where R_i's are the copies of $\boldsymbol{R}$.

PROPOSITION 2.4. *Let F be a maximal ordered field. Let M_0 and v_0 be the maximal ideal and the valuation of A_0 respectively. Then the following statements hold:*

(1) $A_0/M_0=\boldsymbol{R}$.

(2) $v_0(\dot{F})\simeq\vdash\!\dashv((\Delta'(F)^P)^*)$.

PROOF. The proof of (1) is quite similar to that of Proposition 2.10, [2]. We show (2). We put $G_0:=v_0(\dot{F})$ and let (R_i'), $i\in(\Delta'(F)^P)^*$, be the skeleton of G_0; then R_i' is isomorphic to a subgroup of $\boldsymbol{R}$ for any i (cf. [4], A). Therefore we may suppose that $\vdash\!\dashv R_i'$, $i\in(\Delta'(F)^P)^*$, is a subgroup of $\vdash\!\dashv((\Delta'(F)^P)^*)$. Since F is real closed, G_0 is divisible by [2], Proposition 1.7. By [4], A, Théorème 2, we see that G_0 can be identified canonically with a subgroup of $\vdash\!\dashv R_i'$, $i\in(\Delta'(F)^P)^*$. We must show $G_0=\vdash\!\dashv((\Delta'(F)^P)^*)$. Suppose to the contrary that G_0 is a proper subgroup of $\vdash\!\dashv((\Delta'(F)^P)^*)$. Take an element $\xi\in\vdash\!\dashv((\Delta'(F)^P)^*)\setminus G_0$. We have $\boldsymbol{Z}\xi\cap G_0=\{0\}$ since G_0 is divisible. Let $F(x)$ be a simple transcendental extension of F. By [1], Chapter 6, §10, Proposition 1, there is a valuation v' of $F(x)$, which is an extension of v_0, such that the residue field of v' coincides with that of v_0 and the value group of v' is $\boldsymbol{Z}\xi+G_0$. Let σ be the ordering of F and $\bar{\sigma}$

be the ordering of the residue field of v_0 induced by σ. By [2], Remark 2.12, there exists an ordering τ of $F(x)$ such that τ is compatible with v' and $\bar{\tau}=\bar{\sigma}$. Since G_0 is divisible, it follows from [2], Proposition 2.11 that τ is an extension of σ. Since $\bar{\tau}$ is archimedean, the valuation ring of v' is $A(F(x), \boldsymbol{Q})$ by Proposition 1.3.

Let L and L' be convex subgroups of $\boldsymbol{Z}\xi+G_0$ such that $L\subsetneqq L'$. We take an element $a\in L'\setminus L$ with $a>0$. We write $a=(a_i)$. Let $j\in(\Delta'(F)^P)^*$ be the initial element of $\operatorname{supp}(a)$. We take a positive generator $b=(b_i)$ of H_j where $R'_j=H_j/H_j^*$ (cf. [1], A). Since $b_j<na_j$ for some integer $n>0$, we have $b<na$. Similarly we have $a<mb$ for some integer $m>0$. This shows that b is an element of $L'\setminus L$, and therefore $L\cap G_0\neq L'\cap G_0$. These arguments show that $\psi_{F(x)/F}$ is bijective, and this contradicts the fact that F is a maximal ordered field. Q. E. D.

§3. Main Theorem

Let F be an ordered field. In this section, we show that there exists a maximal ordered field K such that K is an extension of F and $\psi_{K/F}$ is bijective.

Let G be an ordered group. We let $F((x))^G$ stand for the formal power series field with coefficients in F and exponents in G; an element of $F((x))^G$ is $s=\sum s_g x^g$, $g\in G$, where $\operatorname{supp}(s)=\{g\in G;\ s_g\neq 0\}$ is a well-ordered set. Let $o(s)$ be the initial element of $\operatorname{supp}(s)$ and let v be the valuation of $F((x))^G$ which is defined by $v(s)=o(s)$. The value group and the residue field of v are G and F respectively. We say that v is the canonical valuation of $F((x))^G$. Let σ be the ordering of F. There is an ordering τ of $F((x))^G$ such that the canonical valuation v is compatible with τ and σ is the restriction of τ; it is well known that such an ordering τ is uniquely determined if the value group G is two divisible.

PROPOSITION 3.1. *Let ρ be an order type. Then there exists a cardinal number $c(\rho)$ such that $|F|<c(\rho)$ for any ordered field F of rank ρ.*

PROOF. Let F be an ordered field of rank ρ. We may assume that F is real closed. Let v_0 be the finest valuation of F with the valuation ring $A_0=A(F, \boldsymbol{Q})$ and let $\bar{F}$ (resp. G_0) be the residue field (resp. the value group) of v_0. It is well known that $|F|\leqq|S|$ where $S=\bar{F}((x))^{G_0}$ (cf. [4], D, Lemme 1). Let Γ be the dual ordered set of $\Delta'(F)^P$. We put $T=\boldsymbol{R}((x))^{\mathrm{H}(\Gamma)}$. Since F is real closed, G_0 is divisible, and so G_0 is isomorphic to a subgroup of $\mathrm{H}(\Gamma)$ (cf. [4], A, Théorème 2); moreover, since the residue field $\bar{F}$ is archimedean by [2], Lemma 1.4, $\bar{F}$ may be identified with a subfield of $\boldsymbol{R}$. Therefore we have $|F|\leqq|T|$ and the assertion is proved. Q. E. D.

PROPOSITION 3.2. *Let Γ be a totally ordered set and put $F:=\boldsymbol{R}((x))^{\mathrm{H}(\Gamma)}$. Then F is a maximal ordered field and $\Delta'(F)^P\simeq\Gamma^*$ as totally ordered sets.*

PROOF. Suppose that K/F is an extension of ordered fields for which $\psi_{K/F}$ is bijective. Let v_0 be the canonical valuation of $F=\boldsymbol{R}((x))^{\mathrm{H}(\Gamma)}$ and v_0' be the finest valuation of K with the valuation ring $A(K, \boldsymbol{Q})$. Since the valuation ring of v_0 is $A(F, \boldsymbol{Q})$, we may suppose that v_0' is an extension of v_0. By [2], Lemma 1.4, the residue field of v_0' is archimedean and this implies that the residue field of v_0' coincides with that of v_0. Let G_0' be the value group of v_0'. By the fact that $\psi_{K/F}$ is bijective and the skeleton of $\mathrm{H}(\Gamma)$ is (R_i), $i\in\Gamma$, $R_i\simeq\boldsymbol{R}$, we can easily show that G_0' is an immediate extension of $\mathrm{H}(\Gamma)$ in the terminology of [4], A. Hence we have $\mathrm{H}(\Gamma)=G_0'$ by [4], A, Théorème 3. Therefore we see that K is an immediate extension of F, and so $K=F$ (cf. [3] or [2], Proposition 3.2). Q. E. D.

THEOREM 3.3. *For any ordered field F, there exists an extension K/F of ordered fields such that $\psi_{K/F}$ is bijective and K is a maximal ordered field.*

PROOF. Let L be an algebraically closed field which is an extension of F and $|L|=c$ (rank F) (the cardinal number c (rank F) is defined in Proposition 3.1). Let S be the set of ordered fields T, $F\subseteq T\subset L$, such that T/F is an extension of ordered fields and $\psi_{T/F}$ is bijective. For T and U in S, we write $T\leqq U$ if $T\subseteq U$ and U/T is an extension of ordered fields. Then S is an inductive set. Let K be a maximal element of S. It is clear that K is real closed. Since $|K|<c$ (rank F) by Proposition 3.1, there exists a simple transcendental extension $K(x)$ over K in L. By the maximality of K, $\psi_{K(x)/K}$ is not bijective for any ordering of $K(x)$. It follows from Proposition 2.3 that K is a maximal ordered field.

Q. E. D.

THEOREM 3.4. *A maximal ordered field F is isomorphic to* $\boldsymbol{R}((x))^{\mathrm{H}(\Gamma)}$ *where $\Gamma^*=\Delta'(F)^P$.*

PROOF. Let M_0 and v_0 be the maximal ideal and the valuation of $A_0=A(F, \boldsymbol{Q})$ respectively. By Proposition 2.4, $A_0/M_0=\boldsymbol{R}$ and $v_0(\dot{F})=\mathrm{H}(\Gamma)$. As for the isomorphism, the proof can be done quite similarly to that of [2], Theorem 3.7. Q. E. D.

COROLLARY 3.5. *Let F be a maximal ordered field and let K be an ordered field such that* rank $K=$ rank F. *Then there is an order preserving isomorphism of K with a subfield of F.*

PROPOSITION 3.6. *Let F be a maximal ordered field. For a subfield k of F, the following statements are equivalent*:

(1) *k is a maximal subfield of some convex valuation ring of F.*

(2) *k is a maximal ordered field and, for the segment $\mathscr{C}_1=\{A_i\in\mathscr{C}(F);$ $A_i\subseteq A(F, k)\}$ of $\mathscr{C}(F)$, $\psi_{F/k}$ induces a bijection: $\mathscr{C}_1\simeq\mathscr{C}(k)$.*

PROOF. (1)$\Rightarrow$(2): Suppose that k is a maximal subfield of a convex valuation ring A; then A is the valuation ring $A(F, k)$ (cf. [2], Proposition 1.5). By [2], Proposition 1.7, k is real closed and the canonical injection $k \to A/M$ is an order preserving isomorphism where M is the maximal ideal of A. Let v be the valuation of A and let x, t be indeterminates. By [1], Chapter 6, §10, Proposition 2, there is a valuation v' of $F(x)$ which is an extension of v such that the residue field of v' is $k(t)$ and the value group of v' coincides with that of v. Let τ' be any ordering of $k(t)$ and let τ be an ordering of $F(x)$ such that τ is compatible with v' and $\bar{\tau}=\tau'$ (cf. [2], Proposition 2.11). Since F is a maximal ordered filed, $\psi_{F(x)/F}$ is not bijective. Let (A', M') be the valuation ring of v'. Since the value group of v' coincides with that of v, $\psi_{F(x)/F}$ induces a bijection:

$$\mathscr{C}'_2 := \{A'_i \in \mathscr{C}(F(x));\ A'_i \supseteq A'\} \xrightarrow{\sim} \mathscr{C}_2 := \{A_i \in \mathscr{C}(F);\ A_i \supseteq A\}.$$

Therefore the canonical map

$$\mathscr{C}'_1 : = \mathscr{C}(F(x)) \setminus \mathscr{C}'_2 \longrightarrow \mathscr{C}_1 : = \mathscr{C}(F) \setminus \mathscr{C}_2$$

is not bijective. Hence $\psi_{k(t)/k}$ is not bijective (cf. the proof of Proposition 1.3). By virtue of Proposition 2.3, this implies that k is a maximal ordered field. It follows from Proposition 1.3 that the map $\mathscr{C}_1 \to \mathscr{C}(k)$ is bijective.

(2)$\Rightarrow$(1): Let k' be a maximal subfield of $A(F, k)$ containing k. Since the composite map of the canonical surjections $\mathscr{C}_1 \to \mathscr{C}(k') \to \mathscr{C}(k)$ is bijective by the assumption, the map $\psi_{k'/k}$: $\mathscr{C}(k') \to \mathscr{C}(k)$ is also bijective. Thus we have $k=k'$ since k is a maximal ordered field. Q.E.D.

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